Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 2, (August 2014), pp. 291–304
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Cubic soft sets with applications in BCK/BCI-algebras

G. Muhiuddin, Abdullah M. Al-roqi

Received 22 December 2013; Accepted 2 February 2014

Abstract. The concepts of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set are introduced, and several related properties are investigated. We apply the notion of cubic soft sets to BCK/BCI-algebras, and introduce the notion of cubic soft BCK/BCI-algebras. A characterization of cubic soft BCK/BCI-algebras is provided, and we prove that the R-intersection of two cubic soft BCK/BCI-algebras is also a cubic soft BCK/BCI-algebra.

2010 AMS Classification: 06F35, 03G25, 06D72

Keywords: (Internal, external) cubic soft sets, P-cubic soft subsets, R-cubic soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, Complement of a cubic soft set, Cubic soft BCK/BCI-algebra.

Corresponding Author: G. Muhiuddin (chishtygm@gmail.com)

1. Introduction

Zadeh [7] made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [2] introduced a new notion, called a (internal, external) cubic set, and investigated several properties. They dealt with P-union, P-intersection, R-union and R-intersection of cubic sets, and investigated several related properties. To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [6]. Maji et al. [4] and Molodtsov [6] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [6] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [4] described the application of soft set theory to a decision making problem. Maji et al. [5] also studied several operations on the theory of soft sets. Jun et al. [1, 3] applied the notion of soft sets to BCK/BCI-algberas and d-algebras.

In this paper, we introduce the notions of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set. We investigate several related properties. We apply the notion of cubic soft sets to BCK/BCI-algebras, and introduce the notion of cubic soft BCK/BCI-algebras. We provide a characterization of cubic soft BCK/BCI-algebras. We show that the R-intersection of two cubic soft BCK/BCI-algebras is also a cubic soft BCK/BCI-algebra.

2. Preliminary

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to I$ where $I = [0, 1]$. Denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\},$$

$$(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\},$$

respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by

$$(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x)).$$

For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\Big(\bigvee_{i\in\Lambda} \lambda_i\Big)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\},$$

$$\Big(\bigwedge_{i\in\Lambda} \lambda_i\Big)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\},$$

respectively, for all $x \in X$.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by $\mathbf{a}$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* and *refined maximun* (briefly, rmin and rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned} &\text{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\text{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\ &\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+, \end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\underset{i\in\Lambda}{\text{rinf}}\, \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \underset{i\in\Lambda}{\text{rsup}}\, \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [I]$, its *complement*, denoted by $\tilde{a}^c$, is defined be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^X$, we define

$$A \subseteq B \Leftrightarrow A(x) \preceq B(x) \text{ for all } x \in X,$$

and

$$A = B \Leftrightarrow A(x) = B(x) \text{ for all } x \in X.$$

The complement A^c of $A \in [I]^X$ is defined as follows: $A^c(x) = A(x)^c$ for all $x \in X$, that is,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)] \text{ for all } x \in X.$$

For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets in X where Λ is an index set, the *union* $G = \bigcup_{i\in\Lambda} A_i$ and the *intersection* $F = \bigcap_{i\in\Lambda} A_i$ are defined as follows:

$$G(x) = \left(\bigcup_{i\in\Lambda} A_i\right)(x) = \underset{i\in\Lambda}{\text{rsup}}\, A_i(x)$$

and

$$F(x) = \left(\bigcap_{i\in\Lambda} A_i\right)(x) = \underset{i\in\Lambda}{\text{rinf}}\, A_i(x)$$

for all $x \in X$, respectively.

Molodtsov [6] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 2.1 ([6])**.** A pair (F, A) is called a *soft set* over U, where F is a mapping given by

$$F : A \to \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [6].

3. Cubic soft sets

Definition 3.1 ([2])**.** Let U be a universe. By a *cubic set* in U we mean a structure

$$\mathscr{A} = \{\langle x, \bar{\mu}_A(x), \lambda_A(x)\rangle \mid x \in U\}$$

in which $\bar{\mu}_A$ is an IVF set in U and λ_A is a fuzzy set in U.

A cubic set $\mathscr{A} = \{\langle x, \bar{\mu}_A(x), \lambda_A(x)\rangle \mid x \in U\}$ is simply denoted by $\mathscr{A} = \langle \bar{\mu}_A, \lambda_A\rangle$, and denote by $\mathcal{C}^U$ the collection of all cubic sets in U.

Definition 3.2 ([2])**.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

(a) (Equality) $\mathscr{A} = \mathscr{B} \Leftrightarrow A = B$ and $\lambda = \mu$.
(b) (P-order) $\mathscr{A} \subseteq_P \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu$.
(c) (R-order) $\mathscr{A} \subseteq_R \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu$.

Definition 3.3 ([2])**.** For any $\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\}$ where $i \in \Lambda$, we define

(a) $\bigcup_{i\in\Lambda}{}_P \mathscr{A}_i = \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} A_i\right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\}$ (P-union)

(b) $\bigcap_{i\in\Lambda}{}_P \mathscr{A}_i = \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} A_i\right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\}$ (P-intersection)

(c) $\bigcup_{i\in\Lambda}{}_R \mathscr{A}_i = \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} A_i\right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\}$ (R-union)

(d) $\bigcap_{i\in\Lambda}{}_R \mathscr{A}_i = \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} A_i\right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\}$ (R-intersection)

The complement of $\mathscr{A} = \langle A, \lambda\rangle$ is defined to be the cubic soft set

$$\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}.$$

Obviously, $(\mathscr{A}^c)^c = \mathscr{A}$, $\hat{0}^c = \hat{1}$, $\hat{1}^c = \hat{0}$, $\ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$. For any

$$\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\},\ i \in \Lambda,$$

we have $\left(\bigcup_{i\in\Lambda}{}_P \mathscr{A}_i\right)^c = \bigcap_{i\in\Lambda}{}_P (\mathscr{A}_i)^c$ and $\left(\bigcap_{i\in\Lambda}{}_P \mathscr{A}_i\right)^c = \bigcup_{i\in\Lambda}{}_P (\mathscr{A}_i)^c$. Also we have

$$\left(\bigcup_{i\in\Lambda}{}_R \mathscr{A}_i\right)^c = \bigcap_{i\in\Lambda}{}_R (\mathscr{A}_i)^c \quad \text{and} \quad \left(\bigcap_{i\in\Lambda}{}_R \mathscr{A}_i\right)^c = \bigcup_{i\in\Lambda}{}_R (\mathscr{A}_i)^c.$$

We now define cubic soft sets over an initial universe set.

Definition 3.4. Let U be an initial universe set and let E be a set of parameters. A *cubic soft set* over U is defined to be a pair $(\mathscr{F}, A)$ where $\mathscr{F}$ is a mapping from A to $\mathcal{C}^U$ and $A \subset E$. Note that the pair $(\mathscr{F}, A)$ can be represented as the following set:

$$(\mathscr{F}, A) := \{\mathscr{F}(e) \mid e \in A\} \text{ where } \mathscr{F}(e) = \langle \bar{\mu}_{\mathscr{F}(e)}, \lambda_{\mathscr{F}(e)} \rangle.$$

We provide an example of a cubic soft set.

Example 3.5. Suppose that there are six houses in the universe U given by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$, where

e_1 stands for the parameter 'expensive',
e_2 stands for the parameter 'beautiful',
e_3 stands for the parameter 'wooden',
e_4 stands for the parameter 'cheap',
e_5 stands for the parameter 'in the green surroundings'.

For $A = \{e_1, e_3, e_4\} \subseteq E$, the set $(\mathscr{F}, A) := \{\mathscr{F}(e_1), \mathscr{F}(e_3), \mathscr{F}(e_4)\}$ is a cubic soft set over U where

$$\begin{aligned} \mathscr{F}(e_1) = \{&\langle h_1, [0.5, 0.8], 0.6\rangle, \langle h_2, [1, 1], 0.7\rangle, \langle h_3, [0.1, 0.7], 0.5\rangle, \\ &\langle h_4, [0.2, 0.6], 0.9\rangle, \langle h_5, [0.3, 0.9], 0.4\rangle, \langle h_6, [0.2, 0.3], 0.3\rangle\} \\ \mathscr{F}(e_3) = \{&\langle h_1, [0.2, 0.5], 0.3\rangle, \langle h_2, [0.3, 0.6], 0.7\rangle, \langle h_3, [0.1, 0.2], 0.4\rangle, \\ &\langle h_4, [0.2, 0.7], 0.2\rangle, \langle h_5, [0.7, 0.9], 0.5\rangle, \langle h_6, [0.3, 0.5], 0.3\rangle\} \\ \mathscr{F}(e_4) = \{&\langle h_1, [0.4, 0.6], 0.7\rangle, \langle h_2, [0.1, 0.2], 0.7\rangle, \langle h_3, [0.1, 0.7], 0.3\rangle, \\ &\langle h_4, [0.3, 0.6], 0.2\rangle, \langle h_5, [0.4, 0.8], 0.7\rangle, \langle h_6, [0.6, 0.7], 0.8\rangle\} \end{aligned}$$

The cubic soft set $(\mathscr{F}, A)$ can be represented in tabular form as follows:

TABLE 1. Tabular representation of the cubic soft set $(\mathscr{F}, A)$

	e_1	e_3	e_4
h_1	$\langle[0.5, 0.8], 0.6\rangle$	$\langle[0.2, 0.5], 0.3\rangle$	$\langle[0.4, 0.6], 0.7\rangle$
h_2	$\langle[1.0, 1.0], 0.7\rangle$	$\langle[0.3, 0.6], 0.7\rangle$	$\langle[0.1, 0.2], 0.7\rangle$
h_3	$\langle[0.1, 0.7], 0.5\rangle$	$\langle[0.1, 0.2], 0.4\rangle$	$\langle[0.1, 0.7], 0.3\rangle$
h_4	$\langle[0.2, 0.6], 0.9\rangle$	$\langle[0.2, 0.7], 0.2\rangle$	$\langle[0.3, 0.6], 0.2\rangle$
h_5	$\langle[0.3, 0.9], 0.4\rangle$	$\langle[0.7, 0.9], 0.5\rangle$	$\langle[0.4, 0.8], 0.7\rangle$
h_6	$\langle[0.2, 0.3], 0.3\rangle$	$\langle[0.3, 0.5], 0.3\rangle$	$\langle[0.6, 0.7], 0.8\rangle$

Definition 3.6. Let U be an initial universe set and let E be a set of parameters. For any subsets A and B of E, let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U. We say that $(\mathscr{F}, A)$ is an *R-cubic soft subset* of $(\mathscr{G}, B)$ if

(i) $A \subseteq B$,
(ii) $(\forall e \in A)\,(\mathscr{F}(e) \subseteq_R \mathscr{G}(e))$, that is, $\bar{\mu}_{\mathscr{F}(e)}(x) \preceq \bar{\mu}_{\mathscr{G}(e)}(x)$ and $\lambda_{\mathscr{F}(e)}(x) \geq \lambda_{\mathscr{G}(e)}(x)$ for all $e \in A$ and $x \in U$.

Definition 3.7. Let U be an initial universe set and let E be a set of parameters. For any subsets A and B of E, let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U. We say that $(\mathscr{F}, A)$ is a *P-cubic soft subset* of $(\mathscr{G}, B)$ if

(i) $A \subseteq B$,
(ii) $(\forall e \in A)\,(\mathscr{F}(e) \subseteq_P \mathscr{G}(e))$, that is, $\bar{\mu}_{\mathscr{F}(e)}(x) \preceq \bar{\mu}_{\mathscr{G}(e)}(x)$ and $\lambda_{\mathscr{F}(e)}(x) \leq \lambda_{\mathscr{G}(e)}(x)$ for all $e \in A$ and $x \in U$.

Example 3.8. Consider the initial universe set $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and the set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$ which are provided in Example 3.5.

(1) For a subset $B = \{e_1, e_3, e_4, e_5\} \subseteq E$, consider a cubic soft set

$$(\mathscr{G}, B) := \{\mathscr{G}(e_1), \mathscr{G}(e_3), \mathscr{G}(e_4), \mathscr{G}(e_5)\}$$

over U which is given in the following tabular form:

TABLE 2. Tabular representation of the cubic soft set $(\mathscr{G}, B)$

	e_1	e_3	e_4	e_5
h_1	$\langle[0.55, 0.88], 0.56\rangle$	$\langle[0.22, 0.55], 0.23\rangle$	$\langle[0.44, 0.66], 0.67\rangle$	$\langle[0.44, 0.64], 0.67\rangle$
h_2	$\langle[1.0, 1.0], 0.67\rangle$	$\langle[0.33, 0.66], 0.67\rangle$	$\langle[0.11, 0.22], 0.67\rangle$	$\langle[0.44, 0.66], 0.67\rangle$
h_3	$\langle[0.11, 0.77], 0.45\rangle$	$\langle[0.11, 0.22], 0.34\rangle$	$\langle[0.11, 0.77], 0.23\rangle$	$\langle[0.44, 0.66], 0.67\rangle$
h_4	$\langle[0.22, 0.66], 0.89\rangle$	$\langle[0.22, 0.77], 0.12\rangle$	$\langle[0.33, 0.66], 0.12\rangle$	$\langle[0.44, 0.66], 0.67\rangle$
h_5	$\langle[0.33, 0.99], 0.34\rangle$	$\langle[0.77, 0.99], 0.45\rangle$	$\langle[0.44, 0.88], 0.67\rangle$	$\langle[0.44, 0.66], 0.67\rangle$
h_6	$\langle[0.22, 0.33], 0.23\rangle$	$\langle[0.33, 0.55], 0.23\rangle$	$\langle[0.66, 0.77], 0.78\rangle$	$\langle[0.44, 0.66], 0.67\rangle$

Then the cubic soft set $(\mathscr{F}, A)$ in Example 3.5 is an *R-cubic soft subset* of $(\mathscr{G}, B)$.

(2) For a subset $B = \{e_1, e_3, e_4, e_5\} \subseteq E$, consider a cubic soft set

$$(\mathscr{G}, B) := \{\mathscr{G}(e_1), \mathscr{G}(e_3), \mathscr{G}(e_4), \mathscr{G}(e_5)\}$$

over U which is given in the following tabular form:

TABLE 3. Tabular representation of the cubic soft set $(\mathscr{G}, B)$

	e_1	e_3	e_4	e_5
h_1	$\langle[0.55, 0.88], 0.62\rangle$	$\langle[0.22, 0.55], 0.31\rangle$	$\langle[0.44, 0.66], 0.71\rangle$	$\langle[0.44, 0.64], 0.71\rangle$
h_2	$\langle[1.0, 1.0], 0.71\rangle$	$\langle[0.33, 0.66], 0.71\rangle$	$\langle[0.11, 0.22], 0.71\rangle$	$\langle[0.44, 0.66], 0.71\rangle$
h_3	$\langle[0.11, 0.77], 0.51\rangle$	$\langle[0.11, 0.22], 0.41\rangle$	$\langle[0.11, 0.77], 0.31\rangle$	$\langle[0.44, 0.66], 0.71\rangle$
h_4	$\langle[0.22, 0.66], 0.91\rangle$	$\langle[0.22, 0.77], 0.21\rangle$	$\langle[0.33, 0.66], 0.21\rangle$	$\langle[0.44, 0.66], 0.71\rangle$
h_5	$\langle[0.33, 0.99], 0.41\rangle$	$\langle[0.77, 0.99], 0.51\rangle$	$\langle[0.44, 0.88], 0.71\rangle$	$\langle[0.44, 0.66], 0.71\rangle$
h_6	$\langle[0.22, 0.33], 0.31\rangle$	$\langle[0.33, 0.55], 0.31\rangle$	$\langle[0.66, 0.77], 0.81\rangle$	$\langle[0.44, 0.66], 0.71\rangle$

Then the cubic soft set $(\mathscr{F}, A)$ in Example 3.5 is a *P-cubic soft subset* of $(\mathscr{G}, B)$.

Definition 3.9. Let U be an initial universe set and let E be a set of parameters. For any subsets A and B of E, let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U.

(1) The *R-union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cup_R \mathscr{G}(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R (\mathscr{G}, B)$.

(2) The *P-union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cup_P \mathscr{G}(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_P (\mathscr{G}, B)$.

Definition 3.10. Let U be an initial universe set and let E be a set of parameters. For any subsets A and B of E, let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U.

(1) The *R-intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cap_R \mathscr{G}(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R (\mathscr{G}, B)$.

(2) The *P-intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cap_P \mathscr{G}(e) & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_P (\mathscr{G}, B)$.

Definition 3.11. Let U be an initial universe set and let E be a set of parameters. For any subsets A and B of E, let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U.

(1) The *restricted R-intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cap B$ and $\mathscr{H}(e) = \mathscr{F}(e) \cap_R \mathscr{G}(e)$ for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \tilde{\cap}_R (\mathscr{G}, B)$.

(2) The *restricted P-intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cap B$ and $\mathscr{H}(e) = \mathscr{F}(e) \cap_P \mathscr{G}(e)$ for all $e \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \tilde{\cap}_P (\mathscr{G}, B)$.

Definition 3.12. Let U be an initial universe set and let E be a set of parameters. The *complement* of a cubic soft set $(\mathscr{F}, A)$ over U is denoted by $(\mathscr{F}, A)^c$ and is defined by $(\mathscr{F}, A) = (\mathscr{F}^c, \rceil A)$ where $\mathscr{F}^c : \rceil A \to \mathcal{C}^U$ is a mapping given by $\mathscr{F}^c(\varepsilon) = \langle x, \bar{\mu}_{\mathscr{F}^c(\varepsilon)}(x), \lambda_{\mathscr{F}^c(\varepsilon)}(x) \rangle$ with $\bar{\mu}_{\mathscr{F}^c(\varepsilon)}(x) = \left[1 - \mu^+_{\mathscr{F}(\neg\varepsilon)}(x), 1 - \mu^-_{\mathscr{F}(\neg\varepsilon)}(x)\right]$ and $\lambda_{\mathscr{F}^c(\varepsilon)}(x) = 1 - \lambda_{\mathscr{F}(\neg\varepsilon)}(x)$ for all $x \in U$ and $\varepsilon \in \rceil A = \{\neg e \mid e \in A\}$.

Example 3.13. The complement $(\mathscr{F}, A)^c$ of the cubic soft set $(\mathscr{F}, A)$ in Example 3.5 is represented by the following tabular form.

Table 4. Tabular representation of the cubic soft set $(\mathscr{F}, A)^c$

	e_1	e_3	e_4
h_1	$\langle[0.2, 0.5], 0.4\rangle$	$\langle[0.5, 0.8], 0.7\rangle$	$\langle[0.4, 0.6], 0.3\rangle$
h_2	$\langle[0, 0], 0.3\rangle$	$\langle[0.4, 0.7], 0.3\rangle$	$\langle[0.8, 0.9], 0.3\rangle$
h_3	$\langle[0.3, 0.9], 0.5\rangle$	$\langle[0.8, 0.9], 0.6\rangle$	$\langle[0.3, 0.9], 0.7\rangle$
h_4	$\langle[0.4, 0.8], 0.1\rangle$	$\langle[0.3, 0.8], 0.8\rangle$	$\langle[0.4, 0.7], 0.8\rangle$
h_5	$\langle[0.1, 0.7], 0.6\rangle$	$\langle[0.1, 0.3], 0.5\rangle$	$\langle[0.2, 0.6], 0.3\rangle$
h_6	$\langle[0.7, 0.8], 0.7\rangle$	$\langle[0.5, 0.7], 0.7\rangle$	$\langle[0.3, 0.4], 0.2\rangle$

Proposition 3.14. *For any cubic soft sets $(\mathscr{F}, A)$, $(\mathscr{G}, B)$ and $(\mathscr{H}, C)$ over U, we have*

(1) $(\mathscr{F}, A) \Cup_R (\mathscr{G}, B) = (\mathscr{G}, B) \Cup_R (\mathscr{F}, A)$.
(2) $(\mathscr{F}, A) \Cup_P (\mathscr{G}, B) = (\mathscr{G}, B) \Cup_P (\mathscr{F}, A)$.

(3) $(\mathscr{F}, A) \Cap_R (\mathscr{G}, B) = (\mathscr{G}, B) \Cap_R (\mathscr{F}, A)$.
(4) $(\mathscr{F}, A) \Cap_P (\mathscr{G}, B) = (\mathscr{G}, B) \Cap_P (\mathscr{F}, A)$.
(5) $((\mathscr{F}, A) \Cup_R (\mathscr{G}, B)) \Cup_R (\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R ((\mathscr{G}, B) \Cup_R (\mathscr{H}, C))$.
(6) $((\mathscr{F}, A) \Cup_P (\mathscr{G}, B)) \Cup_P (\mathscr{H}, C) = (\mathscr{F}, A) \Cup_P ((\mathscr{G}, B) \Cup_P (\mathscr{H}, C))$.
(7) $((\mathscr{F}, A) \Cap_R (\mathscr{G}, B)) \Cap_R (\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R ((\mathscr{G}, B) \Cap_R (\mathscr{H}, C))$.
(8) $((\mathscr{F}, A) \Cap_P (\mathscr{G}, B)) \Cap_P (\mathscr{H}, C) = (\mathscr{F}, A) \Cap_P ((\mathscr{G}, B) \Cap_P (\mathscr{H}, C))$.

Proof. Straightforward. □

Proposition 3.15. *For any cubic soft sets $(\mathscr{F}, A)$, $(\mathscr{G}, B)$ and $(\mathscr{H}, C)$ over U, we have*

(1) $(\mathscr{F}, A)\tilde{\cup}_R(\mathscr{G}, B) = (\mathscr{G}, B)\tilde{\cup}_R(\mathscr{F}, A)$.
(2) $(\mathscr{F}, A)\tilde{\cup}_P(\mathscr{G}, B) = (\mathscr{G}, B)\tilde{\cup}_P(\mathscr{F}, A)$.
(3) $(\mathscr{F}, A)\tilde{\cap}_R(\mathscr{G}, B) = (\mathscr{G}, B)\tilde{\cap}_R(\mathscr{F}, A)$.
(4) $(\mathscr{F}, A)\tilde{\cap}_P(\mathscr{G}, B) = (\mathscr{G}, B)\tilde{\cap}_P(\mathscr{F}, A)$.
(5) $((\mathscr{F}, A)\tilde{\cup}_R(\mathscr{G}, B))\,\tilde{\cup}_R(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cup}_R\,((\mathscr{G}, B)\tilde{\cup}_R(\mathscr{H}, C))$.
(6) $((\mathscr{F}, A)\tilde{\cup}_P(\mathscr{G}, B))\,\tilde{\cup}_P(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cup}_P\,((\mathscr{G}, B)\tilde{\cup}_P(\mathscr{H}, C))$.
(7) $((\mathscr{F}, A)\tilde{\cap}_R(\mathscr{G}, B))\,\tilde{\cap}_R(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cap}_R\,((\mathscr{G}, B)\tilde{\cap}_R(\mathscr{H}, C))$.
(8) $((\mathscr{F}, A)\tilde{\cap}_P(\mathscr{G}, B))\,\tilde{\cap}_P(\mathscr{H}, C) = (\mathscr{F}, A)\tilde{\cap}_P\,((\mathscr{G}, B)\tilde{\cap}_P(\mathscr{H}, C))$.

Proof. Straightforward. □

The following proposition shows that the absorption law with respect to operations $\Cup_R$ (resp. $\Cup_P$) and $\tilde{\cap}_R$ (resp. $\tilde{\cap}_P$).

Proposition 3.16. *For any cubic soft sets $(\mathscr{F}, A)$, $(\mathscr{G}, B)$ and $(\mathscr{H}, C)$, over U, we have*

(1) $((\mathscr{F}, A) \Cup_R (\mathscr{G}, B))\,\tilde{\cap}_R(\mathscr{F}, A) = (\mathscr{F}, A)$.
(2) $((\mathscr{F}, A) \Cup_P (\mathscr{G}, B))\,\tilde{\cap}_P(\mathscr{F}, A) = (\mathscr{F}, A)$.
(3) $((\mathscr{F}, A)\tilde{\cap}_R(\mathscr{G}, B)) \Cup_R (\mathscr{F}, A) = (\mathscr{F}, A)$.
(4) $((\mathscr{F}, A)\tilde{\cap}_P(\mathscr{G}, B)) \Cup_P (\mathscr{F}, A) = (\mathscr{F}, A)$.

Jun et al. introduced the notions of internal (resp. external) cubic sets as follows.

Definition 3.17 ([2])**.** Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an

- *internal cubic set* (briefly, ICS) if $A^-(x) \le \lambda(x) \le A^+(x)$ for all $x \in X$.
- *external cubic set* (briefly, ECS) if $\lambda(x) \notin (A^-(x), A^+(x))$ for all $x \in X$.

Using these concepts, we introduce the notions of internal (resp. external) cubic soft sets.

Definition 3.18. Let U be an initial universe set and let E be a set of parameters.
(1) A cubic soft set $(\mathscr{F}, A)$ over U is said to be *internal* if it satisfies:

$$(\forall e \in A)\,(\forall x \in U)\left(\mu^-_{\mathscr{F}(e)}(x) \le \lambda_{\mathscr{F}(e)}(x) \le \mu^+_{\mathscr{F}(e)}(x)\right).$$

(2) A cubic soft set $(\mathscr{F}, A)$ over U is said to be *external* if it satisfies:

$$(\forall e \in A)\,(\forall x \in U)\left(\lambda_{\mathscr{F}(e)}(x) \notin \left(\mu^-_{\mathscr{F}(e)}(x), \mu^+_{\mathscr{F}(e)}(x)\right)\right).$$

Example 3.19. Consider the initial universe set $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and the set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$ which are provided in Example 3.5.

(1) For a subset $A = \{e_1, e_3, e_4\}$ of E, the cubic soft set $(\mathscr{F}, A)$ over U with the following tabular form is obviously an internal cubic soft set over U.

TABLE 5. Tabular representation of the cubic soft set $(\mathscr{F}, A)$

	e_1	e_3	e_4
h_1	$\langle[0.5, 0.8], 0.60\rangle$	$\langle[0.2, 0.5], 0.33\rangle$	$\langle[0.4, 0.6], 0.57\rangle$
h_2	$\langle[0.1, 0.2], 0.13\rangle$	$\langle[0.3, 0.6], 0.57\rangle$	$\langle[0.1, 0.2], 0.17\rangle$
h_3	$\langle[0.1, 0.7], 0.25\rangle$	$\langle[0.1, 0.2], 0.14\rangle$	$\langle[0.1, 0.7], 0.33\rangle$
h_4	$\langle[0.2, 0.6], 0.39\rangle$	$\langle[0.2, 0.7], 0.42\rangle$	$\langle[0.3, 0.6], 0.52\rangle$
h_5	$\langle[0.3, 0.9], 0.44\rangle$	$\langle[0.7, 0.9], 0.85\rangle$	$\langle[0.4, 0.8], 0.67\rangle$
h_6	$\langle[0.2, 0.3], 0.23\rangle$	$\langle[0.3, 0.5], 0.43\rangle$	$\langle[0.6, 0.7], 0.68\rangle$

(2) For a subset $B = \{e_1, e_2, e_5\}$ of E, the cubic soft set $(\mathscr{G}, B)$ over U with the following tabular form is obviously an external cubic soft set over U.

TABLE 6. Tabular representation of the cubic soft set $(\mathscr{G}, B)$

	e_1	e_2	e_5
h_1	$\langle[0.5, 0.8], 0.4\rangle$	$\langle[0.2, 0.5], 0.6\rangle$	$\langle[0.4, 0.6], 0.3\rangle$
h_2	$\langle[0.1, 0.2], 0.5\rangle$	$\langle[0.3, 0.6], 0.2\rangle$	$\langle[0.1, 0.2], 0.3\rangle$
h_3	$\langle[0.1, 0.7], 0.8\rangle$	$\langle[0.1, 0.2], 0.3\rangle$	$\langle[0.1, 0.7], 0.9\rangle$
h_4	$\langle[0.2, 0.6], 0.7\rangle$	$\langle[0.2, 0.7], 0.8\rangle$	$\langle[0.3, 0.6], 0.7\rangle$
h_5	$\langle[0.3, 0.9], 0.2\rangle$	$\langle[0.7, 0.9], 0.4\rangle$	$\langle[0.4, 0.8], 0.2\rangle$
h_6	$\langle[0.2, 0.3], 0.6\rangle$	$\langle[0.3, 0.5], 0.7\rangle$	$\langle[0.6, 0.7], 0.4\rangle$

Proposition 3.20. *The complement of an internal cubic soft set is an internal cubic soft set.*

Proof. Let $(\mathscr{F}, A)$ be an internal cubic soft set over U. Then

$$\mu^-_{\mathscr{F}(e)}(x) \leq \lambda_{\mathscr{F}(e)}(x) \leq \mu^+_{\mathscr{F}(e)}(x)$$

for all $e \in A$ and $x \in U$. It follows that

$$1 - \mu^+_{\mathscr{F}(e)}(x) \leq 1 - \lambda_{\mathscr{F}(e)}(x) \leq 1 - \mu^-_{\mathscr{F}(e)}(x)$$

for all $e \in A$ and $x \in U$, that is, $\mu^-_{\mathscr{F}^c(\neg e)}(x) \leq \lambda_{\mathscr{F}^c(\neg e)}(x) \leq \mu^+_{\mathscr{F}^c(\neg e)}(x)$ for all $\neg e \in \rceil A$ and $x \in U$. Therefore $(\mathscr{F}, A)^c$ is an internal cubic soft set over U. $\square$

Similarly, we have the following proposition.

Proposition 3.21. *The complement of an external cubic soft set is an external cubic soft set.*

Theorem 3.22. *The P-union of two internal cubic soft sets is also an internal cubic soft set.*

Proof. Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be internal cubic soft sets over U and $(\mathscr{H}, C)$ the P-union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$, that is, $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_P (\mathscr{G}, B)$. Then

$$\mu^-_{\mathscr{F}(e)}(x) \leq \lambda_{\mathscr{F}(e)}(x) \leq \mu^+_{\mathscr{F}(e)}(x)$$

for all $e \in A$ and $x \in U$, and

$$\mu^-_{\mathscr{G}(e)}(x) \leq \lambda_{\mathscr{G}(e)}(x) \leq \mu^+_{\mathscr{G}(e)}(x)$$

for all $e \in B$ and $x \in U$. If $e \in A \setminus B$ or $e \in B \setminus A$, then it is clear that $(\mathscr{H}, C)$ is an internal cubic soft set over U. Assume that $e \in A \cap B$. Then

$$\max\{\mu^-_{\mathscr{F}(e)}(x), \mu^-_{\mathscr{G}(e)}(x)\} \leq \max\{\lambda_{\mathscr{F}(e)}(x), \lambda_{\mathscr{G}(e)}(x)\} \leq \max\{\mu^+_{\mathscr{F}(e)}(x), \mu^+_{\mathscr{G}(e)}(x)\}$$

that is, $\left(\lambda_{\mathscr{F}(e)} \vee \lambda_{\mathscr{G}(e)}\right)(x) \in \text{rmax}\{\bar{\mu}_{\mathscr{F}(e)}(x), \bar{\mu}_{\mathscr{G}(e)}(x)\}$ for all $x \in U$. Therefore $(\mathscr{H}, C)$ is an internal cubic soft set over U. □

Similarly, we have the following theorem.

Theorem 3.23. *The P-intersection of two internal cubic soft sets is also an internal cubic soft set.*

Corollary 3.24. *The restricted P-intersection of two internal cubic soft sets is also an internal cubic soft set.*

We now pose questions.

Question 3.25. (1) Is the R-union (resp. R-intersetion) of two internal cubic soft sets an internal cubic soft set?

(2) Is the P-union (resp. P-intersetion) of two external cubic soft sets an external cubic soft set?

(3) Is the R-union (resp. R-intersetion) of two external cubic soft sets an external cubic soft set?

4. Applications to BCK/BCI-algebras

In what follows, let U be an initial universe set which is a BCK/BCI-algebra.

Definition 4.1. A cubic soft set $(\mathscr{F}, A)$ over U is said to be a *cubic soft BCK/BCI-algebra* over U based on a parameter ε if there exists a parameter $\varepsilon \in A$ such that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{4.1}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{4.2}$$

If $(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U based on all parameters, we say that $(\mathscr{F}, A)$ is a *cubic soft BCK/BCI-algebra* over U.

Example 4.2. Consider a BCK-algebra $U = \{0, a, b, c\}$ with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Consider a set of parameters $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Let $(\mathscr{F}, E)$ be a cubic soft set over U which is represented as the following tabular form.

TABLE 7. Tabular representation of the cubic soft set $(\mathscr{F}, E)$

	ε_1	ε_2	ε_3
0	$\langle[0.3, 0.8], 0.2\rangle$	$\langle[0.2, 0.5], 0.2\rangle$	$\langle[0.1, 0.3], 0.7\rangle$
a	$\langle[0.1, 0.5], 0.5\rangle$	$\langle[0.3, 0.6], 0.4\rangle$	$\langle[0.4, 0.6], 0.6\rangle$
b	$\langle[0.3, 0.8], 0.3\rangle$	$\langle[0.4, 0.7], 0.7\rangle$	$\langle[0.1, 0.7], 0.3\rangle$
c	$\langle[0.1, 0.5], 0.7\rangle$	$\langle[0.5, 0.8], 0.9\rangle$	$\langle[0.3, 0.6], 0.2\rangle$

Then $(\mathscr{F}, E)$ is a cubic soft BCK-algebra over U based on parameters ε_1 and ε_2, but it is not a cubic soft BCK-algebra over U based on the parameter ε_3 since $\lambda_{\mathscr{F}(\varepsilon_3)}(a * b) = 0.7 > 0.6 = \max\{\lambda_{\mathscr{F}(\varepsilon_3)}(a), \lambda_{\mathscr{F}(\varepsilon_3)}(b)\}$ and/or

$$\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(x * y) = [0.1, 0.3] \not\succeq [0.1, 0.6] = \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon_3)}(y)\}.$$

Proposition 4.3. *If $(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U based on the parameter ε in A, then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$ and $\lambda_{\mathscr{F}(\varepsilon)}(0) \le \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$.*

Proof. For any $x \in U$, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x * x) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$$

and $\lambda_{\mathscr{F}(\varepsilon)}(0) = \lambda_{\mathscr{F}(\varepsilon)}(x*x) \le \min\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$. □

Corollary 4.4. *If $(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U, then*

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \text{ and } \lambda_{\mathscr{F}(\varepsilon)}(0) \le \lambda_{\mathscr{F}(\varepsilon)}(x)$$

for all $\varepsilon \in A$ and $x \in U$.

Let $(\mathscr{F}, A)$ be a cubic soft set over U. For a parameter $\varepsilon \in A$, $r \in [0, 1]$ and $[s, t] \in [I]$, we define a set

$$U_\varepsilon(\mathscr{F}, A)^r_{[s,t]} := \{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s, t],\ \lambda_{\mathscr{F}(\varepsilon)}(x) \le r\}.$$

If we put $\bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]} := \{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s, t]\}$ and

$$\lambda_\varepsilon(\mathscr{F}, A)^r := \{x \in U \mid \lambda_{\mathscr{F}(\varepsilon)}(x) \le r\},$$

then $U_\varepsilon(\mathscr{F}, A)^r_{[s,t]} = \bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]} \cap \lambda_\varepsilon(\mathscr{F}, A)^r$.

Theorem 4.5. *For a cubic soft set $(\mathscr{F}, A)$ over U, the following are equivalent:*

(1) *$(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U based on the parameter ε in A.*
(2) *The sets $\bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]} := \{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s, t]\}$ and*

$$\lambda_\varepsilon(\mathscr{F}, A)^r := \{x \in U \mid \lambda_{\mathscr{F}(\varepsilon)}(x) \le r\}$$

are subalgebras of U for all $r \in [0, 1]$ and $[s, t] \in [I]$ whenever they are nonempty.

Proof. Assume that $(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U based on the parameter ε in A. Let $x, y \in U$ and $[s,t] \in [I]$ be such that $x, y \in \bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]}$. Then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s,t]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \succeq [s,t]$. It follows from (4.1) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \succeq \mathrm{rmin}\{[s,t],[s,t]\} = [s,t].$$

Hence $x * y \in \bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]}$, and therefore $\bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]}$ is a subalgebra of U. Now, let $x, y \in U$ and $r \in [0,1]$ be such that $x, y \in \lambda_\varepsilon(\mathscr{F}, A)^r$. Then $\lambda_{\mathscr{F}(\varepsilon)}(x) \le r$ and $\lambda_{\mathscr{F}(\varepsilon)}(y) \le r$. Using (4.2), we have $\lambda_{\mathscr{F}(\varepsilon)}(x * y) \le \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \le r$, and so $x * y \in \lambda_\varepsilon(\mathscr{F}, A)^r$. Therefore $\lambda_\varepsilon(\mathscr{F}, A)^r$ is a subalgebra of U.

Conversely, suppose that the sets

$$\bar{\mu}_\varepsilon(\mathscr{F}, A)_{[s,t]} := \{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s,t]\}$$

and

$$\lambda_\varepsilon(\mathscr{F}, A)^r := \{x \in U \mid \lambda_{\mathscr{F}(\varepsilon)}(x) \le r\}$$

are subalgebras of U for all $r \in [0,1]$ and $[s,t] \in [I]$ whenever they are nonempty. Assume that there exist $a, b \in U$ such that $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b) \not\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(a), \bar{\mu}_{\mathscr{F}(\varepsilon)}(b)\}$. If we take $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [s_a, t_a]$, $\bar{\mu}_{\mathscr{F}(\varepsilon)}(b) = [s_b, t_b]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b) = [s_0, t_0]$, then

$$[s_0, t_0] \not\succeq \mathrm{rmin}\{[s_a, t_a], [s_b, t_b]\} = [\min\{s_a, s_b\}, \min\{t_a, t_b\}].$$

Hence we have the following three cases:

(i) $s_0 \ge \min\{s_a, s_b\}$ and $t_0 < \min\{t_a, t_b\}$.
(ii) $s_0 < \min\{s_a, s_b\}$ and $t_0 \ge \min\{t_a, t_b\}$.
(iii) $s_0 < \min\{s_a, s_b\}$ and $t_0 < \min\{t_a, t_b\}$.

For the first case, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [s_a, t_a] \succeq [\min\{s_a, s_b\}, \min\{t_a, t_b\}] \succeq [\min\{s_a, s_b\}, t_0],$$

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(b) = [s_b, t_b] \succeq [\min\{s_a, s_b\}, \min\{t_a, t_b\}] \succeq [\min\{s_a, s_b\}, t_0]$$

and so $a, b \in \bar{\mu}_\varepsilon(\mathscr{F}, A)_{[\min\{s_a, s_b\}, t_0]}$. But $a * b \notin \bar{\mu}_\varepsilon(\mathscr{F}, A)_{[\min\{s_a, s_b\}, t_0]}$, a contradiction. By the similar way, the second and third cases induce a contradiction. Thus

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Now suppose that (4.2) is false. Then

$$\lambda_{\mathscr{F}(\varepsilon)}(a * b) > r \ge \max\{\lambda_{\mathscr{F}(\varepsilon)}(a), \lambda_{\mathscr{F}(\varepsilon)}(b)\}$$

for some $a, b \in U$ and $r_0 \in [0,1)$ which implies that $a, b \in \lambda_\varepsilon(\mathscr{F}, A)^{r_0}$ but $a * b \notin \lambda_\varepsilon(\mathscr{F}, A)^{r_0}$. This is a contradiction, and hence (4.2) is valid. Therefore $(\mathscr{F}, A)$ is a cubic soft BCK/BCI-algebra over U based on the parameter ε in A. □

Corollary 4.6. *If a cubic soft set $(\mathscr{F}, A)$ over U is a cubic soft BCK/BCI-algebra over U based on the parameter ε in A, then*

$$U_\varepsilon(\mathscr{F}, A) := \{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [s,t],\ \lambda_{\mathscr{F}(\varepsilon)}(x) \le r\}$$

is a subalgebra of U for all $r \in [0,1]$ and $[s,t] \in [I]$ whenever it is nonempty.

Theorem 4.7. *The R-intersection of two cubic soft BCK/BCI-algebras over U is also a cubic soft BCK/BCI-algebra over U.*

Proof. Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft BCK/BCI-algebras over U and let $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R (\mathscr{G}, B)$ be the R-intersection of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$. Then $C = A \cup B$. For any $\varepsilon \in C$, if $\varepsilon \in A \setminus B$, then

$$\begin{aligned}\bar{\mu}_{\mathscr{H}(\varepsilon)}(x * y) &= \bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \\ &\succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \\ &= \text{rmin}\{\bar{\mu}_{\mathscr{H}(\varepsilon)}(x), \bar{\mu}_{\mathscr{H}(\varepsilon)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathscr{H}(\varepsilon)}(x * y) = \lambda_{\mathscr{F}(\varepsilon)}(x * y) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \\ &= \max\{\lambda_{\mathscr{H}(\varepsilon)}(x), \lambda_{\mathscr{H}(\varepsilon)}(y)\}.\end{aligned}$$

for all $x, y \in U$. Similarly, if $\varepsilon \in B \setminus A$ then $\bar{\mu}_{\mathscr{H}(\varepsilon)}(x*y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{H}(\varepsilon)}(x), \bar{\mu}_{\mathscr{H}(\varepsilon)}(y)\}$ and

$$\lambda_{\mathscr{H}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{H}(\varepsilon)}(x), \lambda_{\mathscr{H}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Suppose that $\varepsilon \in A \cap B$. Then

$$\begin{aligned}\bar{\mu}_{\mathscr{H}(\varepsilon)}(x * y) &= \bar{\mu}_{\mathscr{F}(\varepsilon)\cap_R\mathscr{G}(\varepsilon)}(x * y) \\ &= \text{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{G}(\varepsilon)}(x * y)\right\} \\ &\succeq \text{rmin}\left\{\text{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\right\}, \text{rmin}\left\{\bar{\mu}_{\mathscr{G}(\varepsilon)}(x), \bar{\mu}_{\mathscr{G}(\varepsilon)}(y)\right\}\right\} \\ &= \text{rmin}\left\{\text{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{G}(\varepsilon)}(x)\right\}, \text{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{G}(\varepsilon)}(y)\right\}\right\} \\ &= \text{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)\cap_R\mathscr{G}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)\cap_R\mathscr{G}(\varepsilon)}(y)\right\} \\ &= \text{rmin}\left\{\bar{\mu}_{\mathscr{H}(\varepsilon)}(x), \bar{\mu}_{\mathscr{H}(\varepsilon)}(y)\right\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathscr{H}(\varepsilon)}(x * y) &= \lambda_{\mathscr{F}(\varepsilon)\cap_R\mathscr{G}(\varepsilon)}(x * y) \\ &= \lambda_{\mathscr{F}(\varepsilon)}(x * y) \vee \lambda_{\mathscr{G}(\varepsilon)}(x * y) \\ &= \max\left\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{G}(\varepsilon)}(x * y)\right\} \\ &\leq \max\left\{\max\left\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\right\}, \max\left\{\lambda_{\mathscr{G}(\varepsilon)}(x), \lambda_{\mathscr{G}(\varepsilon)}(y)\right\}\right\} \\ &= \max\left\{\max\left\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{G}(\varepsilon)}(x)\right\}, \max\left\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{G}(\varepsilon)}(y)\right\}\right\} \\ &= \max\left\{\lambda_{\mathscr{H}(\varepsilon)}(x), \lambda_{\mathscr{H}(\varepsilon)}(y)\right\}\end{aligned}$$

for all $x, y \in U$. Therefore $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R (\mathscr{G}, B)$ is a cubic soft BCK/BCI-algebra over U. □

Before ending our arguments, we pose questions.

Question 4.8. (1) Is the R-union of two cubic soft BCK/BCI-algebras a cubic soft BCK/BCI-algebra?

(2) Is the P-union (resp. P-intersetion) of two cubic soft BCK/BCI-algebras a cubic soft BCK/BCI-algebra?

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G. Muhiuddin (chishtygm@gmail.com)
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia

Abdullah M. Al-roqi (alroqi10@yahoo.com)
Department of Mathematics, King Abdulaziz University, Jeddah, KSA