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Some operations on bipolar fuzzy graphs

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Abstract. In this paper, we investigate some properties of complete bipolar fuzzy graphs, describe various methods of their construction. We also define strong product and categorical product on bipolar fuzzy graphs and give some of their properties. Moreover, we show that the strong product, join and intersection of two complete bipolar fuzzy graphs C_1 and C_2 is also complete bipolar fuzzy graph. Finally we define isometry on bipolar fuzzy graphs and show that isometry on bipolar fuzzy graphs is an equivalence relation.

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1. Introduction

Presently, science and technology is featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on an extension of the ordinary set theory, namely, fuzzy sets. Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, and transportation. In 1965, Zadeh [29] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, computer networks and automata theory.

In 1994, Zhang [31, 32] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [9], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean one of them is the negation of the other. The semantic of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination.

In 1975, Rosenfeld [20] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs. Bhattacharya [7] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [13]. Bhutani and Rosenfeld introduced the concept of M-strong fuzzy graphs in [8] and studied some of their properties. Talebi and Rashmanlou [23] studied properties of isomorphism and complement on interval- valued fuzzy graphs. The first definition of bipolar fuzzy graphs was proposed by Akram [1] . Akram and Davvaz discussed the properties of strong intuitionistic fuzzy graphs and they introduced the concept of intuitionistic fuzzy line graphs in [2]. Shannon and Atanassov [21] introduced the concept of intuitionistic fuzzy graphs and investigated some of their properties in [22].We have used standard definitions and terminologies in this paper. For the notations not mentioned in the paper, the readers are referred to [1]-[6],[10],[14],[15]-[19],[22]-[28].

2. Preliminaries

In this section, we first review some definitions of undirected graphs that are necessary of this paper.

Definition 2.1 ([10])**.** Recall that a graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^* such that every edge is corresponded to a two- element subset of V. We use the notation xy instead $\{x, y\}$.

Definition 2.2 ([10])**.** The complement graph $\overline{G^*}$ of a simple graph G^* has the same vertices as G^* and two vertices are adjacent in $\overline{G^*}$ if and only if they are not adjacency in G^*.

Definition 2.3 ([10])**.** The cartesian product of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is denoted by

$G^* = G_1 \times G_2 = (V, E)$ that $V = V_1 \times V_2$ and
$E = \{(x, x_2), (x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2, x_1y_1 \in E_1\}$.

Definition 2.4 ([10]). The composition of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$. Where
$E^0 = E \cup \{(x_1, x_2), (y_1, y_2) \mid x_1y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

Definition 2.5 ([10]). The union of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1^* and G_2^* is denoted by $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

Definition 2.6 ([10]). The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 = \phi$. The join of G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E')$.

Definition 2.7 ([28, 29]). A fuzzy subset μ on a set X is a map $\mu : X \to [0, 1]$. A map $\nu : X \times X \to [0, 1]$ is called a fuzzy relation on X if
$\nu(x, y) \leq \min(\mu(u), \mu(y))$ for all $x, y \in X$.
A fuzzy relation ν is symmetric if $\nu(x, y) = \nu(y, x)$, for all $x, y \in X$.

Definition 2.8 ([11, 30]). Let X be a nonempty set. A bipolar fuzzy set B in X is an object having the form $B = \{(x, \mu^P(x), \mu^N(x)) \mid x \in X\}$, where $\mu^P : X \to [0, 1]$ and $\mu^N : X \to [-1, 0]$ are mappings. We use the positive membership degree $\mu^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B and the negative membership degree $\mu^N(x) \neq 0$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set. B If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x to be such that $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of x .We shall use the symbol $B = (\mu^P, \mu^N)$ for the bipolar fuzzy set $B = \{(x, \mu^P(x), \mu^N(x)) \mid x \in X\}$.

Definition 2.9 ([12]). For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define
$(A \cap B)(x) = (\min(\mu_A^P(x), \mu_B^P(x)), \max(\mu_A^N(x), \mu_B^N(x)))$, for all $x \in X$,
$(A \cup B)(x) = (\max(\mu_A^P(x), \mu_B^P(x)), \min(\mu_A^N(x), \mu_B^N(x)))$, for all $x \in X$.

Definition 2.10 ([30]). Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [-1, 1] \times [-1, 1]$ a bipolar fuzzy relation on X such that $\mu_A^P(x, y) \in [0, 1]$ and $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.11 ([30]). Let $B = (\mu_B^P, \mu_B^N)$ be a bipolar fuzzy set on a set X. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy relation on X, then $A = (\mu_A^P, \mu_A^N)$ is called a bipolar fuzzy relation on $B = (\mu_B^P, \mu_B^N)$ if $\mu_A^P(x, y) \leq \min(\mu_B^P(x), \mu_B^P(y))$,
$\mu_A^N(x, y) \geq \max(\mu_B^N(x), \mu_B^N(y))$, for all $x, y \in X$. A bipolar fuzzy relation A on X is called symmetric if $\mu_A^P(x, y) = \mu_A^P(y, x)$, $\mu_A^N(x, y) = \mu_A^N(y, x)$ for all $x, y \in X$.

Definition 2.12. A bipolar fuzzy graph with an underlying set V is defined to be a pair $G = (A, B)$ where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy set such that $\mu_B^P(\{x, y\}) \leq \min(\mu_A^P(x), \mu_A^P(y))$,
$\mu_B^N(\{x, y\}) \geq \max(\mu_A^N(x), \mu_A^N(y))$ for all $x, y \in V$.
We call A the bipolar fuzzy vertex set of V, B the bipolar fuzzy edge set of E, respectively. Note that B is a symmetric bipolar fuzzy relation on A. Throughout this paper, G^* will be a crisp graph, and G a bipolar fuzzy graph.

3. Complete bipolar fuzzy graphs

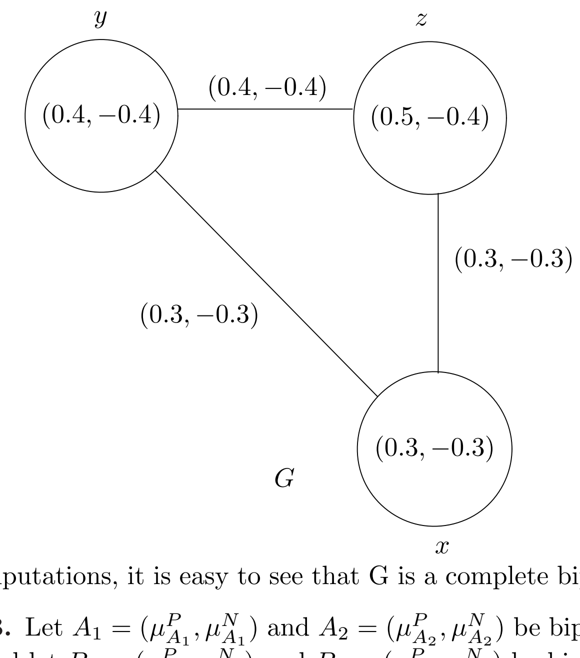
Throughout this paper, we suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two bipolar fuzzy graphs with underling fuzzy graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively.

Definition 3.1. A bipolar fuzzy graph $G = (A, B)$ is called complete if
$\mu_B^P(xy) = \min(\mu_A^P(x), \mu_A^P(y))$, $\mu_B^N(xy) = \max(\mu_A^N(x), \mu_A^N(y))$, for all $x, y \in V$.

Example 3.2. Consider the set $V = \{x, y, z\}$. Let A be a bipolar fuzzy subset of V and let B be a bipolar fuzzy subset of $V \times V$ defined by

	x	y	z
μ_{A^P}	0.3	0.4	0.5
μ_{A^N}	-0.3	-0.4	-0.4

	xy	yz	xz
μ_{B^P}	0.3	0.4	0.3
μ_{B^N}	-0.3	-0.4	-0.3



By routine computations, it is easy to see that G is a complete bipolar fuzzy graph.

Definition 3.3. Let $A_1 = (\mu_{A_1}^P, \mu_{A_1}^N)$ and $A_2 = (\mu_{A_2}^P, \mu_{A_2}^N)$ be bipolar fuzzy subsets of V_1 and V_2 and let $B_1 = (\mu_{B_1}^P, \mu_{B_1}^N)$ and $B_2 = (\mu_{B_2}^P, \mu_{B_2}^N)$ be bipolar fuzzy subsets of E_1 and E_2, respectively. Then, we denote the union of two bipolar fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* by $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ and define as follows

$$(A)\begin{cases} (\mu^P_{A_1} \cup \mu^P_{A_2})(x) = \mu^P_{A_1}(x) & \text{if } x \in V_1 - V_2, \\ (\mu^P_{A_1} \cup \mu^P_{A_2})(x) = \mu^P_{A_2}(x) & \text{if } x \in V_2 - V_1, \\ (\mu^P_{A_1} \cup \mu^P_{A_2})(x) = \max(\mu^P_{A_1}(x), \mu^P_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

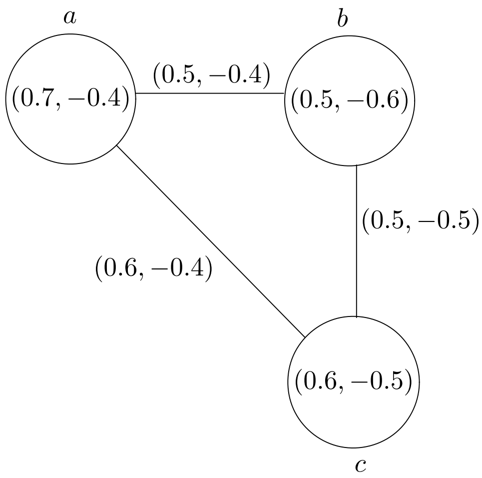
$$(B)\begin{cases} (\mu^N_{A_1} \cup \mu^N_{A_2})(x) = \mu^N_{A_1}(x) & \text{if } x \in V_1 - V_2, \\ (\mu^N_{A_1} \cup \mu^N_{A_2})(x) = \mu^N_{A_2}(x) & \text{if } x \in V_2 - V_1, \\ (\mu^N_{A_1} \cup \mu^N_{A_2})(x) = \min(\mu^N_{A_1}(x), \mu^N_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(C)\begin{cases} (\mu^P_{B_1} \cup \mu^P_{B_2})(xy) = \mu^P_{B_1}(xy) & \text{if } xy \in E_1 - E_2, \\ (\mu^P_{B_1} \cup \mu^P_{B_2})(xy) = \mu^P_{B_2}(xy) & \text{if } xy \in E_2 - E_1, \\ (\mu^P_{B_1} \cup \mu^P_{B_2})(xy) = \max(\mu^P_{B_1}(xy), \mu^P_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

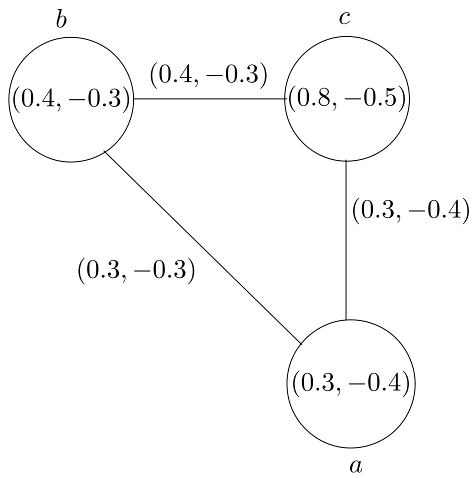
$$(D)\begin{cases} (\mu^N_{B_1} \cup \mu^N_{B_2})(xy) = \mu^N_{B_1}(xy) & \text{if } xy \in E_1 - E_2, \\ (\mu^N_{B_1} \cup \mu^N_{B_2})(xy) = \mu^N_{B_2}(xy) & \text{if } xy \in E_2 - E_1, \\ (\mu^N_{B_1} \cup \mu^N_{B_2})(xy) = \min(\mu^N_{B_1}(xy), \mu^N_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Remark 3.4. The union of two complete bipolar fuzzy graphs is not necessary a complete bipolar fuzzy graph.

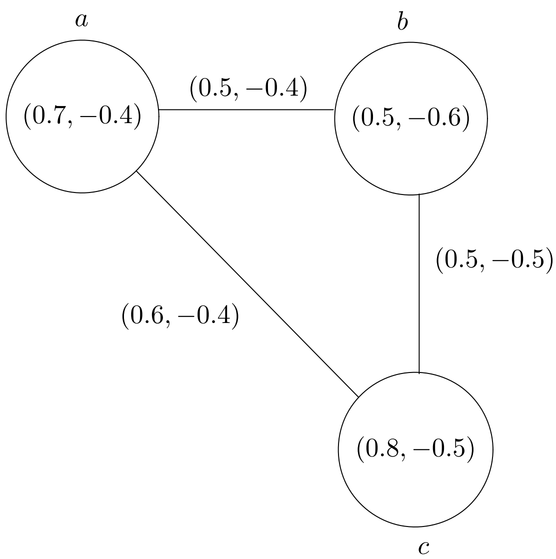
Example 3.5. Consider the complete bipolar fuzzy graphs.



G_1 is complete



G_2 is complete



$G_1 \cup G_2$ is not complete bipolar fuzzy graph

Definition 3.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs with underlying set V_1 and V_2, respectively. Then we denote the intersection. Then we denote the intersection G_1 and G_2 by $G_1 \cap G_2 = (A_1 \cap A_2, B_1 \cap B_2)$ and define as following

$(\mu^P_{A_1} \cap \mu^P_{A_2})(x) = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(x))$,
$(\mu^N_{A_1} \cap \mu^N_{A_2})(x) = \max(\mu^N_{A_1}(x), \mu^N_{A_2}(x))$, for all $x \in V_1 \cap V_2$.
$(\mu^P_{B_1} \cap \mu^P_{B_2})(xy) = \min(\mu^P_{B_1}(xy), \mu^P_{B_2}(xy))$,
$(\mu^N_{B_1} \cap \mu^N_{B_2})(xy) = \max(\mu^N_{B_1}(xy), \mu^N_{B_2}(xy))$, for all $xy \in V_1 \cap V_2$.

Proposition 3.7. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be complete bipolar fuzzy graphs. Then $G_1 \cap G_2$ is a complete bipolar fuzzy graph.*

Proof. Let $x, y \in V_1 \cap V_2$. we have
$(\mu^P_{B_1} \cap \mu^P_{B_2})(xy) = \min\{\mu^P_{B_1}(xy), \mu^P_{B_2}(xy)\}$
$= \min\{\min\{\mu^P_{A_1}(x), \mu^P_{A_1}(y)\}, \min\{\mu^P_{A_1}(x), \mu^P_{A_1}(y)\}\}$
$= \min\{\min\{\mu^P_{A_1}(x), \mu^P_{A_1}(x)\}, \min\{\mu^P_{A_1}(y), \mu^P_{A_1}(y)\}\}$
$= \min\{(\mu^P_{A_1} \cap \mu^P_{A_1}(x), (\mu^P_{A_1} \cap \mu^P_{A_1}(y)\}$,
$(\mu^N_{B_1} \cap \mu^N_{B_2})(xy) = \max\{\mu^N_{B_1}(xy), \mu^N_{B_2}(xy)\}$
$= \max\{\max\{\mu^N_{A_1}(x), \mu^N_{A_1}(y)\}, \max\{\mu^N_{A_2}(x), \mu^N_{A_2}(y)\}\}$
$= \max\{\max\{\mu^N_{A_1}(x), \mu^N_{A_2}(x)\}, \max\{\mu^N_{A_1}(y), \mu^N_{A_2}(y)\}\}$
$= \max\{(\mu^N_{A_1} \cap \mu^N_{A_1})(x), (\mu^N_{A_1} \cap \mu^N_{A_1})(y)\}$.
Hence $G_1 \cap G_2$ is a complete bipolar fuzzy graph. □

Definition 3.8. Let $A_1 = (\mu^P_{A_1}, \mu^N_{A_1})$ and $A_2 = (\mu^P_{A_2}, \mu^N_{A_2})$ be bipolar fuzzy subsets of V_1 and V_2 in which $V_1 \cap V_2 = \phi$, and let $B_1 = (\mu^P_{B_1}, \mu^N_{B_1})$ and $B_2 = (\mu^P_{B_2}, \mu^N_{B_2})$ be bipolar fuzzy subsets of $V_1 \times V_2$ and $V_2 \times V_1$, respectively. Then, we denote the join of two bipolar fuzzy graphs G_1 and G_2 by $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ and define as follows
$(\mu^P_{A_1} + \mu^P_{A_2})(x) = (\mu^P_{A_1} \cup \mu^P_{A_2})(x)$,
$(\mu^N_{A_1} + \mu^N_{A_2})(x) = (\mu^N_{A_1} \cap \mu^N_{A_2})(x)$, for all $x \in V_1 \cup V_2$.
$(\mu^P_{B_1} + \mu^P_{B_2})(xy) = (\mu^P_{B_1} \cup \mu^P_{B_2})(xy)$,
$(\mu^N_{B_1} + \mu^N_{B_2})(xy) = (\mu^N_{B_1} \cap \mu^N_{B_2})(xy)$, for all $xy \in E_1 \cup E_2$.
$(\mu^P_{B_1} + \mu^P_{B_2})(xy) = \max(\mu^P_{A_1}(x), \mu^P_{A_2}(y))$,
$(\mu^N_{B_1} + \mu^N_{B_2})(xy) = \min(\mu^N_{A_1}(x), \mu^N_{A_2}(y))$, for all $xy \in E'$,
where E' is the set of all edges joining the nodes of V_1 and V_2.

Proposition 3.9. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete bipolar fuzzy graphs, with underlying set V_1 and V_2, respectively, in which $V_1 \cap V_2 = \phi$, then $G_1 + G_2$ is a complete bipolar fuzzy graph.*

Proof. Let $xy \in E'$. Then

$$\begin{aligned}
(\mu^P_{B_1} + \mu^P_{B_2})(xy) &= \max(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = \max((\mu^P_{A_1} \cup \mu^P_{A_2})(x), (\mu^P_{A_1} \cup \mu^P_{A_2})(y)) \\
&= \max((\mu^P_{A_1} + \mu^P_{A_2})(x), (\mu^P_{A_1} + \mu^P_{A_2})(y)), \\
(\mu^N_{B_1} + \mu^N_{B_2})(xy) &= \min(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = \min((\mu^N_{A_1} \cup \mu^N_{A_2})(x), (\mu^N_{A_1} \cup \mu^N_{A_2})(y)) \\
&= \min((\mu^N_{A_1} + \mu^N_{A_2})(x), (\mu^N_{A_1} + \mu^N_{A_2})(y)).
\end{aligned}$$

For case $xy \in E_1 \cup E_2$, it is obvious. □

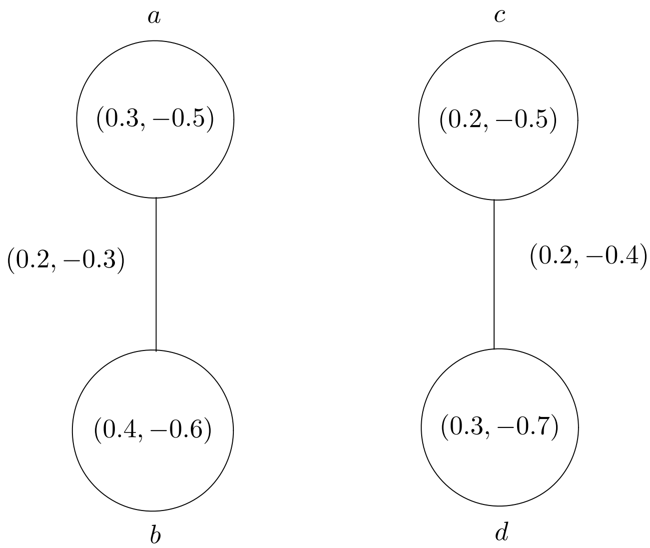
4. Some operations on bipolar fuzzy graphs

In this section, we introduce strong product and categorical product on bipolar fuzzy graphs and show that strong product and categorical product of two bipolar fuzzy graphs is a bipolar fuzzy graph.

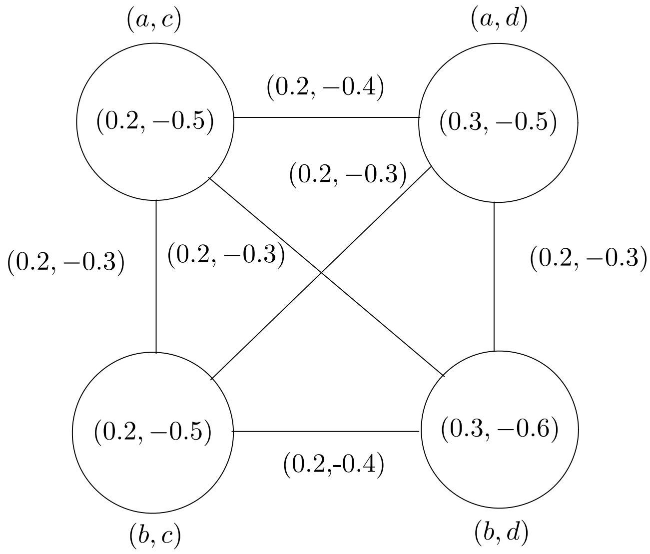
Definition 4.1. The strong product $G_1 \Box times G_2$ of two bipolar fuzzy graphs G_1 and G_2 is denoted by $(A_1 \Box times A_2, B_1 \Box times B_2)$ and is defined by
$(\mu^P_{A_1} \Box times\ \mu^P_{A_2})(x_1, x_2) = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$,
$(\mu^N_{A_1} \Box times\ \mu^N_{A_2})(x_1, x_2) = \max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$.
$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})(x_1, x_2)(y_1, y_2) = \min(\mu^P_{A_1}(x_1), \mu^P_{B_2}(x_2y_2))$,
$(\mu^N_{B_1} \Box times\ \mu^N_{B_2})(x_1, x_2)(y_1, y_2) = \max(\mu^N_{A_1}(x_1), \mu^N_{B_2}(x_2y_2))$, if $x_1 = y_1, x_2y_2 \in E_2$.
$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})(x_1, x_2)(y_1, y_2) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{A_2}(x_2))$,
$(\mu^N_{B_1} \Box times\ \mu^N_{B_2})(x_1, x_2)(y_1, y_2) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{A_2}(x_2))$, if $x_2 = y_2, x_1y_1 \in E_1$.
$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})(x_1, x_2)(y_1, y_2) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{B_2}(x_2y_2))$,
$(\mu^N_{B_1} \Box times\ \mu^N_{B_2})(x_1, x_2)(y_1, y_2) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{B_2}(x_2y_2))$, if $x_1 \neq y_1$,
$x_2 \neq y_2, x_1y_1 \in E_1, x_2y_2 \in E_2$.

Example 4.2. Consider two bipolar fuzzy graphs G_1 and G_2 defined as follows.

G_1 is bipolar fuzzy graph G_2 is bipolar fuzzy graph



G_1 G_2



$G_1\Box times G_2$

By a routine computation, it is easy to see that $G_1\Box times\ G_2$ is a bipolar fuzzy graph.

Proposition 4.3. *The strong product* $G_1\Box times\ G_2 = (A_1\Box times\ A_2, B_1\Box times\ B_2)$ *of two bipolar fuzzy graphs* G_1 *and* G_2 *is a bipolar fuzzy graph.*

Proof. We verify only conditions for $B_1\Box times B_2$, because conditions for $A_1\Box times A_2$ are obvious. If $x_1, y_1 \in V_1, x_2y_2 \in E_2$, then

$$\begin{aligned}
&(\mu^P_{B_1}\Box times\ \mu^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{A_1}(x_1), \mu^P_{B_2}(x_2y_2))\\
&\leq \min(\mu^P_{A_1}(x_1), \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2)))\\
&= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(y_2)))\\
&= \min((\mu^P_{A_1}\Box times\ \mu^P_{A_2})(x_1, x_2), (\mu^P_{A_1}\Box times\ \mu^P_{A_2})(y_1, y_2)),
\end{aligned}$$

$$\begin{aligned}
&(\mu^N_{B_1}\Box times\ \mu^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{A_1}(x_1), \mu^N_{B_2}(x_2y_2))\\
&\geq \max(\mu^N_{A_1}(x_1), \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2)))\\
&= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)), \max(\mu^N_{A_1}(y_1), \mu^N_{A_2}(y_2)))\\
&= \max((\mu^N_{A_1}\Box times\ \mu^N_{A_2})(x_1, x_2), (\mu^N_{A_1}\Box times\ \mu^N_{A_2})(y_1, y_2)).
\end{aligned}$$

If $x_2 = y_2 \in V_2$, $x_1y_1 \in E_1$, then

$$\begin{aligned}
&(\mu^P_{B_1}\Box times\ \mu^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{A_2}(x_2))\\
&\leq \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_1}(y_1)), \mu^P_{A_2}(x_2))\\
&= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(y_2)))\\
&= \min((\mu^P_{A_1}\Box times\ \mu^P_{A_2})(x_1, x_2), (\mu^P_{A_1}\Box times\ \mu^P_{A_2})(y_1, y_2)).
\end{aligned}$$

$$\begin{aligned}
&(\mu^N_{B_1}\Box times\ \mu^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{A_2}(x_2))\\
&\geq \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_1}(y_1)), \mu^N_{A_2}(x_2))\\
&= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)), \max(\mu^N_{A_1}(y_1), \mu^N_{A_2}(y_2)))\\
&= \max((\mu^N_{A_1}\Box times\ \mu^N_{A_2})(x_1, x_2), (\mu^N_{A_1}\Box times\ \mu^N_{A_2})(y_1, y_2)).
\end{aligned}$$

If $x_1 \neq y_1, x_2 \neq y_2, x_1y_1 \in E_1$, $x_2y_2 \in E_2$, then

$$\begin{aligned}
&(\mu^P_{B_1}\Box times\ \mu^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{B_2}(x_2y_2))\\
&\leq \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_1}(y_1)), \mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2)))\\
&= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(y_2)))\\
&= \min((\mu^P_{A_1}\Box times\ \mu^P_{A_2})(x_1, x_2), (\mu^P_{A_1}\Box times\ \mu^P_{A_2})(y_1, y_2)).
\end{aligned}$$

$$\begin{aligned}
&(\mu^N_{B_1}\Box times\ \mu^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{B_2}(x_2y_2))\\
&\geq \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_1}(y_1)), \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2)))\\
&= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)), \max(\mu^N_{A_1}(y_1), \mu^N_{A_2}(y_2)))\\
&= \max((\mu^N_{A_1}\Box times\ \mu^N_{A_2})(x_1, x_2), (\mu^N_{A_1}\Box times\ \mu^N_{A_2})(y_1, y_2)).
\end{aligned}$$

This complete the proof. □

Proposition 4.4. *If G_1 and G_2 are complete bipolar fuzzy graphs, then $G_1\Box times G_2$ is a complete bipolar fuzzy graph.*

Proof. If $x_1 \in V_1$, then for every $x_2, y_2 \in V_2$, we have

$$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})((x,x_2)(x,y_2)) = \min(\mu^P_{A_1}(x), \mu^P_{B_2}(x_2y_2))$$
$$= \min(\mu^P_{A_1}(x), \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2)))$$
$$= \min(\min(\mu^P_{A_1}(x), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y_2)))$$
$$= \min((\mu^P_{A_1} \Box times\ \mu^P_{A_2})(x,x_2), (\mu^P_{A_1} \Box times\ \mu^P_{A_2})(x,y_2)),$$

$$(\mu^N_{B_1} \Box times\ (\mu^N_{B_1} \times \mu^N_{B_2})(x,x_2)(x,y_2)) = \max(\mu^N_{A_1}(x), \mu^N_{B_2}(x_2y_2))$$
$$= \max(\mu^N_{A_1}(x), \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2)))$$
$$= \max(\max(\mu^N_{A_1}(x), \mu^N_{A_2}(x_2)), \max(\mu^N_{A_1}(x), \mu^N_{A_2}(y_2)))$$
$$= \max((\mu^N_{A_1} \Box times\ \mu^N_{A_2})(x,x_2), (\mu^N_{A_1} \Box times\ \mu^N_{A_2})(x,y_2)).$$

If $z \in V_2$, then for every $x_1, y_1 \in V_1$, we have

$$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})((x_1,z)(y_1,z)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{A_2}(z))$$
$$= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_1}(y_1)), \mu^P_{A_2}(z))$$
$$= \min(\min(\mu^P_{A_1}(x), \mu^P_{A_2}(z)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(z)))$$
$$= \min((\mu^P_{A_1} \Box times\ \mu^P_{A_2})(x,z), (\mu^P_{A_1} \Box times\ \mu^P_{A_2})(y_1,z)),$$

$$(\mu^N_{B_1} \Box times\ \mu^N_{B_2})((x_1,z)(y_1,z)) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{A_2}(z))$$
$$= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_1}(y_1)), \mu^N_{A_2}(z))$$
$$= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(z)), \max(\mu^N_{A_1}(y_1), \mu^N_{A_2}(z)))$$
$$= \max((\mu^N_{A_1} \Box times\ \mu^N_{A_2})(x_1,z), (\mu^N_{A_1} \Box times\ \mu^N_{A_2})(y_1,z)).$$

If $x_i, y_i \in V_i$ $(i = 12)$ are distinct, then we have

$$(\mu^P_{B_1} \Box times\ \mu^P_{B_2})(x_1,x_2)(y_1,y_2)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{B_2}(x_2y_2))$$
$$= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_1}(y_1)), \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2)))$$
$$= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(y_2)))$$
$$= \min(\mu^P_{A_1} \Box times\ \mu^P_{A_2})(x_1,x_2), \mu^P_{A_1} \Box times\ \mu^P_{A_2})(y_1,y_2)).$$

$$(\mu^N_{B_1} \Box times\ \mu^N_{B_2})((x_1,x_2)(y_1,y_2)) = \min(\mu^N_{B_1}(x_1y_1), \mu^N_{B_2}(x_2y_2))$$
$$= \min(\min(\mu^N_{A_1}(x_1), \mu^N_{A_2}(y_1)), \min(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2)))$$
$$= \min(\min(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)), \min(\mu^N_{A_1}(y_1), \mu^N_{A_2}(y_2)))$$
$$= \min(\mu^N_{A_1} \Box times\ \mu^N_{A_2})(x_1,x_2), \mu^N_{A_1} \Box times\ \mu^N_{A_2})(y_1,y_2)).$$

This complete the proof. □

Example 4.5. Let $V_1 = \{a, b\}$ and $V_2 = \{c, d\}$. Consider two complete bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

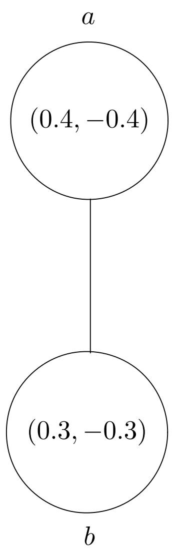
C_1 is bipolar fuzzy graph C_2 is bipolar fuzzy graph

	a	b
$\mu^P_{A_1}$	0.4	0.3
$\mu^N_{A_1}$	-0.4	-0.3

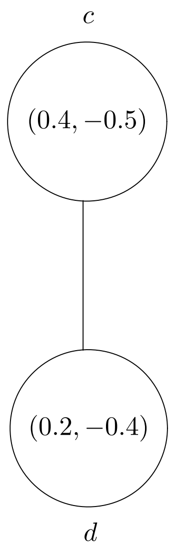
	ab
$\mu^P_{B_1}$	0.3
$\mu^N_{B_1}$	-0.3

	c	d
$\mu^P_{A_2}$	0.4	0.2
$\mu^N_{A_2}$	-0.5	-0.4

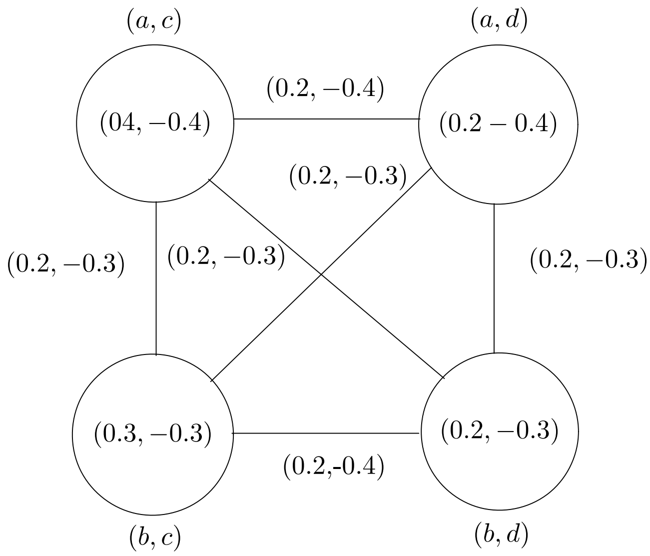
	cd
$\mu^P_{B_2}$	0.2
$\mu^N_{B_2}$	-0.4



G_1 is complete bipolar fuzzy graph



G_2 is complete bipolar fuzzy graph



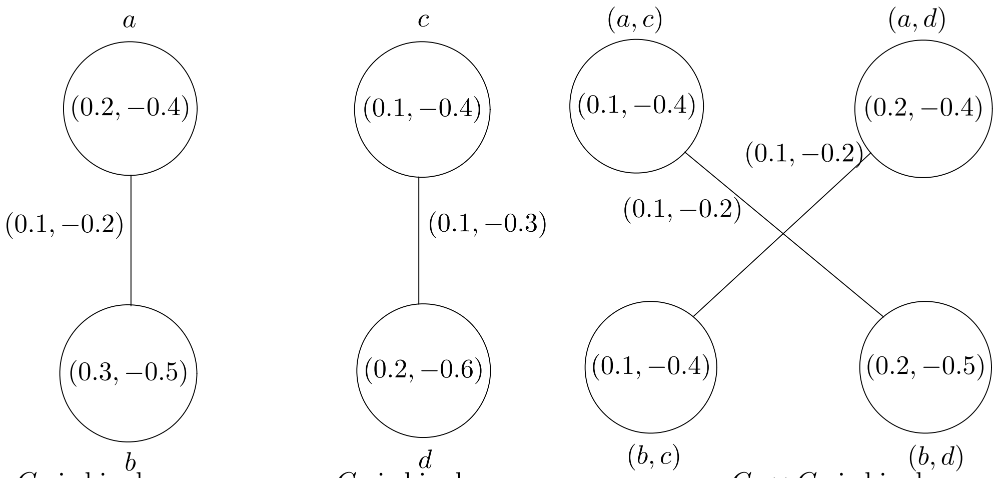
$G_1\Box times\ G_2$ is complete bipolar fuzzy graph

Definition 4.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two bipolar fuzzy graphs. The categorical product $G_1 \times G_2$ is defined by:
$(\mu^P_{A_1} \times \mu^P_{A_2})(x_1, x_2) = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$,
$(\mu^N_{A_1} \times \mu^N_{A_2})(x_1, x_2) = \max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$.

$$(\mu^P_{B_1} \times \mu^P_{B_2})((x_1,x_2)(y_1,y_2)) = \begin{cases} \min(\mu^P_{B_1}(x_1y_1), \mu^P_{B_2}(x_2y_2)) & \text{if} x_1 \neq y_1, x_2 \neq y_2 \\ 0 & \text{otherwise} \end{cases}$$

$$(\mu^N_{B_1} \times \mu^N_{B_2})((x_1,x_2)(y_1,y_2)) = \begin{cases} \max(\mu^N_{B_1}(x_1y_1), \mu^N_{B_2}(x_2y_2)) & \text{if} x_1 \neq y_1, x_2 \neq y_2 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.7. Let $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Consider two bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, and $G_1 \times G_2$ as follows.



G_1 is bipolar G_2 is bipolar $G_1 \times G_2$ is bipolar

By a routine computation, it is easy to see that $G_1 \times G_2$ is a bipolar fuzzy graph.

Proposition 4.8. *The categorical product $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$ of two bipolar fuzzy graphs G_1 and G_2 is a bipolar fuzzy graph.*

Proof. Similarly as in the previous proof we verify only conditions for $B_1 \times B_2$. Let $x_1 \neq y_1$, $x_1 \neq y_2$, then we have

$$\begin{aligned}
(\mu^P_{B_1} \times \mu^P_{B_2})(x_1,x_2)(y_1,y_2)) &= \min(\mu^P_{B_1}(x_1y_1), \mu^P_{B_2}(x_2y_2)) \\
&\leq \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_1}(y_1)), \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2))) \\
&= \min(\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)), \min(\mu^P_{A_1}(y_1), \mu^P_{A_2}(y_2))) \\
&= \min((\mu^P_{A_1} \times \mu^P_{A_2})(x_1,x_2), (\mu^P_{A_1} \times \mu^P_{A_2})(y_1,y_2)).
\end{aligned}$$

$$\begin{aligned}
(\mu^N_{B_1} \times \mu^N_{B_2})((x_1,x_2)(y_1,y_2)) &= \max(\mu^N_{B_1}(x_1y_1), \mu^N_{B_2}(x_2y_2)) \\
&\geq \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_1}(y_1)), \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2))) \\
&= \max(\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)), \max(\mu^N_{A_1}(y_1), \mu^N_{A_2}(y_2))) \\
&= \max((\mu^N_{A_1} \times \mu^N_{A_2})(x_1,x_2), \mu^N_{A_1} \times \mu^N_{A_2})(y_1,y_2)).
\end{aligned}$$

□

Definition 4.9. Let G_1 and G_2 be bipolar fuzzy graphs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:

(a) $\mu^P_{A_1}(x) \leq \mu^P_{A_2}(f(x)), \mu^N_{A_1}(x) \geq \mu^N_{A_2}(f(x))$, for all $x \in V_1$.

(b) $\mu^P_{B_1}(xy) \leq \mu^P_{B_2}(f(x)f(y)), \mu^N_{B_1}(xy) \geq \mu^N_{B_2}(f(x)f(y))$, for all $xy \in E_1$.

Definition 4.10. Let G_1 and G_2 be bipolar fuzzy graphs. An isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:

(c) $\mu^P_{A_1}(x) = \mu^P_{A_2}(f(x)), \mu^N_{A_1}(x) = \mu^N_{A_2}(f(x))$, for all $xy \in E_1$.
(d) $\mu^P_{B_1}(xy) = \mu^P_{B_2}(f(x)f(y)), \mu^N_{B_1}(xy) = \mu^N_{B_2}(f(x)f(y))$, for all $xy \in E_1$.

Definition 4.11. Let G_1 and G_2 be bipolar fuzzy graphs. Then a weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:

(e) f is a homomorphism,
(f) $\mu^P_{A_1}(x) = \mu^P_{A_2}(f(x)), \mu^N_{A_1}(x) = \mu^N_{A_2}(f(x))$, for all $x \in V_1$.

Definition 4.12. Let G_1 and G_2 be bipolar fuzzy graphs. Then a co-weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:

(g) f is a homomorphism,
(h) $\mu^P_{B_1}(xy) = \mu^P_{B_2}(f(x)f(y)), \mu^N_{B_1}(xy) = \mu^N_{B_2}(f(x)f(y))$, for all $x, y \in V_1$.

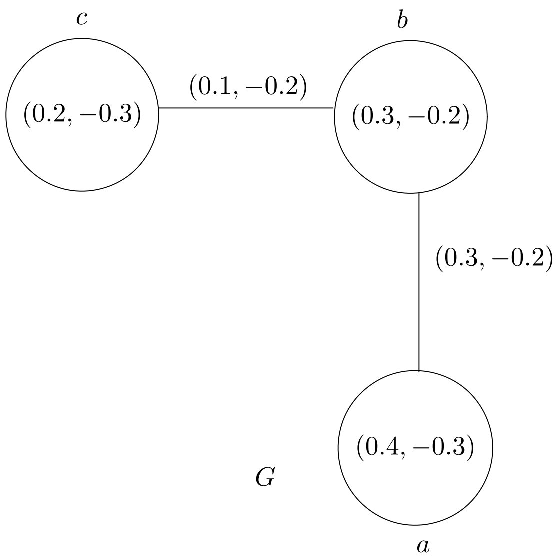
Thus a co-weak isomorphism preserves the weights of the arcs but not necessarily the weights of the nodes.

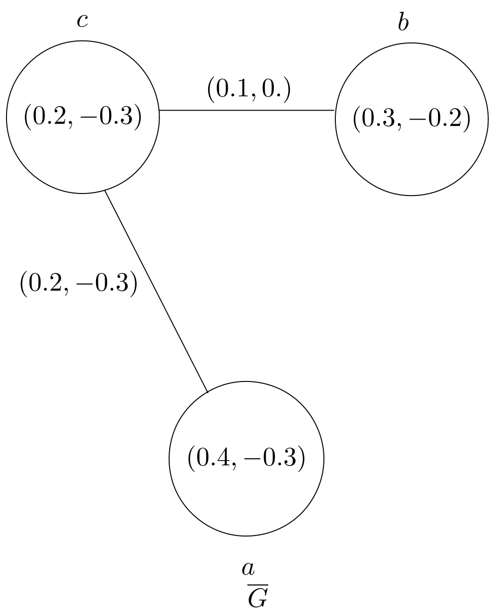
Definition 4.13. The complement of a bipolar fuzzy graph $G = (A, B)$ of $G^* = (V, E)$ is a bipolar fuzzy graph $\overline{G} = (\overline{A}, \overline{B})$ of $(V, V \times V)$, where $\overline{A} = (\overline{\mu^P_A}, \overline{\mu^N_A})$ and $\overline{B} = (\overline{\mu^P_B}, \overline{\mu^N_B})$ are defined by

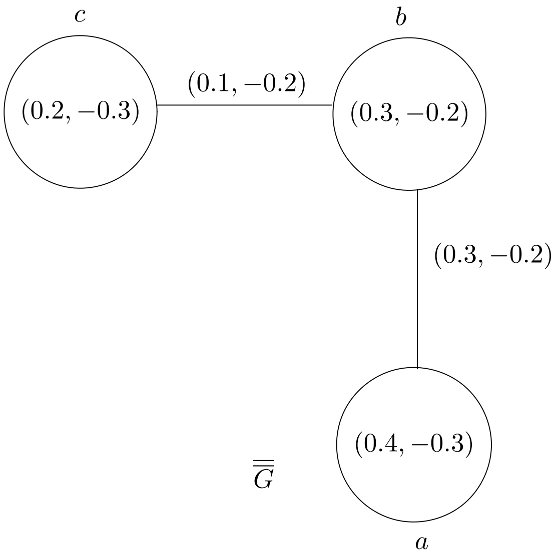
(i) $\overline{V} = V$,
(ii) $\overline{\mu^P_A}(x) = \mu^P_A(x), \overline{\mu^N_A}(x) = \mu^N_A(x)$, for all $x \in V$.
(iii) $\overline{\mu^P_B}(xy) = \min(\mu^P_A(x), \mu^P_A(y)) - \mu^P_B(xy)$, $\overline{\mu^N_B}(xy) = \max(\mu^N_A(x), \mu^N_A(y)) - \mu^N_B(xy)$.

Definition 4.14. A bipolar fuzzy graph G is called self- complementary if $G \cong \overline{\overline{G}}$.

Example 4.15. Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c\}$, $E = \{ab, bc\}$, and a bipolar fuzzy graph G.


c
b
(0.1, −0.2)
(0.2, −0.3)
(0.3, −0.2)
(0.3, −0.2)
(0.4, −0.3)
G
a


c
b
(0.1, 0.)
(0.2, −0.3)
(0.3, −0.2)
(0.2, −0.3)
(0.4, −0.3)
a
$\overline{G}$



$\overline{\overline{G}}$

Clearly, $\overline{\overline{G}} \cong G$. Hence G is self- complementary.

Proposition 4.16. *Let G_1 and G_2 be bipolar fuzzy graphs. If $G_1 \cong G_2$ then $\overline{G} \cong \overline{G_2}$.*

Proof. Assume that G_1 and G_2 are isomorphic, there exists a bijective map
$f : V_1 \to V_2$ satisfying
$\mu^P_{A_1}(x) = \mu^P_{A_2}(f(x)), \mu^N_{A_1}(x) = \mu^P_{A_2}(f(x))$ for all $x \in V_1$,
$\mu^P_{B_1}(xy) = \mu^P_{B_2}(f(x)f(y)), \mu^N_{B_1}(xy) = \mu^N_{B_2}(f(x)f(y))$ for all $xy \in E_1$.
By definition of complement, if $\mu^P_{B_1}(xy) > 0$, then $\mu^P_{B_2}(f(x)f(y)) > 0$ and we have
$\overline{\mu^P_{B_1}}(xy) = \overline{\mu^P_{B_2}}(f(x)f(y)) = 0$.
If $\mu^P_{B_1}(xy) = 0$ then $\mu^P_{B_2}(f(x)f(y)) = 0$, and we have
$\overline{\mu^P_{B_1}}(xy) = \min(\mu^P_{A_1}(x), \mu^P_{A_1}(y)) = \min(\mu^P_{A_2}(f(x)), \mu^P_{A_2}(f(y))) = \overline{\mu^P_{B_2}}(f(x)f(y))$.
Similarly, we can show that $\overline{\mu^N_{B_1}}(xy) = \overline{\mu^N_{B_2}}(f(x)f(y))$.
Hence $\overline{G_1} \cong \overline{G_2}$. □

Definition 4.17. A bipolar fuzzy graph $G = (A, B)$ is called strong if
$\mu^P_B(xy) = \min(\mu^P_A(x), \mu^P_A(y))$, $\mu^N_B(xy) = \max(\mu^N_A(x), \mu^N_A(y))$, for all $xy \in E$.

Remark 4.18. If $G = (A, B)$ is a strong bipolar fuzzy graph of $G^* = (V, E)$, then from Definition 4.14, it follows that $\overline{\overline{G}}$ is given by the bipolar fuzzy graph $\overline{\overline{G}} = (\overline{\overline{A}}, \overline{\overline{B}})$ on $G^* = (V, E)$ where $\overline{\overline{A}} = A$ and
$\overline{\overline{\mu^P_B}}(xy) = \min(\mu^P_A(x), \mu^P_A(y))$, $\overline{\overline{\mu^N_B}}(xy) = \min(\mu^N_A(x), \mu^N_A(y))$ for all $xy \in E$.
Thus $\overline{\overline{\mu^P_B}} = \mu^P_B$ and $\overline{\overline{\mu^N_B}} = \mu^N_B$ on V where $B = (\mu^P_B, \mu^N_B)$ is the strongest bipolar fuzzy relation on A. For any bipolar fuzzy graph G, $\overline{G}$ is a strong bipolar fuzzy graph and $G \subset \overline{\overline{G}}$.

Theorem 4.19. $G = \overline{\overline{G}}$ *if and only if G is strong bipolar fuzzy graph.*

Proof. If G is strong bipolar fuzzy graph then, by above Remark it is obvious that $G = \overline{\overline{G}}$. Conversely, suppose that $G = \overline{\overline{G}}$. Since for every bipolar fuzzy graph G, $\overline{G}$ is a strong bipolar fuzzy graph, $G = \overline{\overline{G}}$ implies G is strong bipolar fuzzy graph. □

Corollary 4.20. *By the Theorem 4.19, if G is strong bipolar fuzzy graph, then G is self-complementary, while the Example 4.15 shows that the converse of this statement is not true.*

5. Isometric bipolar fuzzy graphs

Definition 5.1. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs . Then G_2 is said to be isometric from G_1 if for each $v \in V_1$ there is a bijective $g_v : V_1 \to V_2$ such that $\delta_1^P(u, v) = \delta_2^P(g_v(u), g_v(v))$, $\delta_1^N(u, v) = \delta_2^N(g_v(u), g_v(v))$ for every $u \in V_1$, in which $\delta_i^P(i = 1, 2)$ and $\delta_i^N(i = 1, 2)$ are $\mu_{B_1}^P$ distance and $\mu_{B_1}^N$ distance, respectively. In addition we define $\delta^P(u, v) = \wedge \sum_{i=1}^{n} \frac{1}{\mu_B^P(u_{i-1}u_i)}$ and $\delta^N(u, v) = \vee \sum_{i=1}^{n} \frac{1}{\mu_B^N(u_{i-1}u_i)}$, where $u = u_0, u_1, \cdots, u_i, \cdots, u_n = v$ is a path from u to v.

Proposition 5.2. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two bipolar fuzzy graphs. Then G_1 is isomorphic to G_2 implies G_1 is isometric to G_2.*

Proof. As G_1 is isomorphic to G_2, there is a bijection $g : V_1 \to V_2$ such that
$\mu_{A_1}^P(x) = \mu_{A_2}^P(g(x)), \mu_{A_1}^N(x) = \mu_{A_2}^N(g(x))$ for all $x \in V_1$.
$\mu_{B_1}^P(xy) = \mu_{B_2}^P(g(x)g(y)), \mu_{B_1}^N(xy) = \mu_{B_2}^N(g(x)g(y))$ for all $x, y \in V_1$.
For each $u \in V_1$, we have

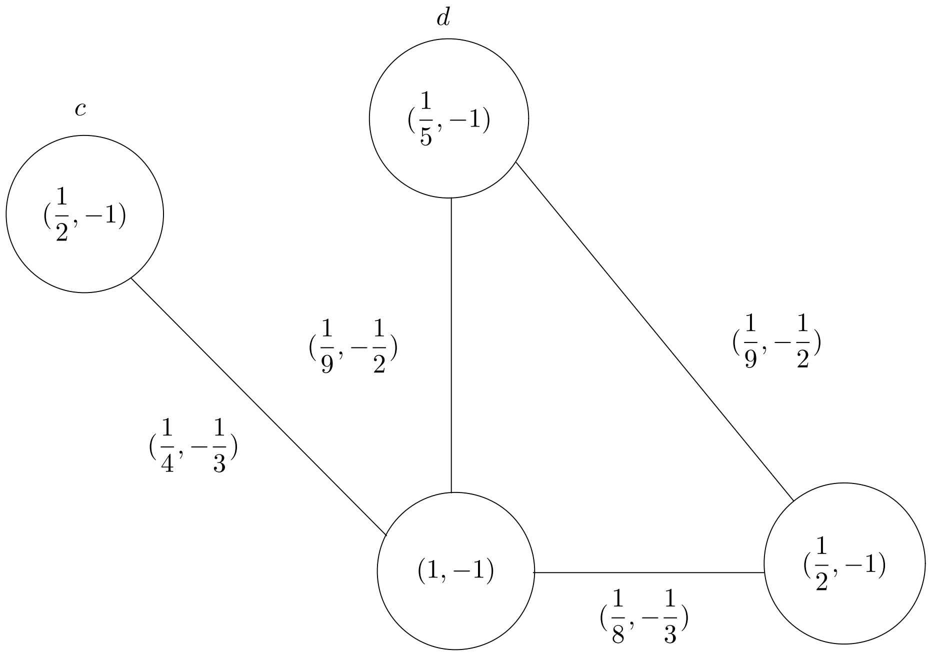
$$\delta_1^P(u, v) = \wedge\Big\{\sum_{i=1}^{n} \frac{1}{\mu_{B_1}^P(u_{i-1}u_i)}\Big\} = \wedge\Big\{\sum_{i=1}^{n} \frac{1}{\mu_{B_2}^P(g(u_{i-1})g(u_i))}\Big\} = \delta_2^P(g(u), g(v)),$$

$$\delta_1^N(u, v) = \vee\Big\{\sum_{i=1}^{n} \frac{1}{\mu_{B_1}^N(u_{i-1}u_i)}\Big\} = \vee\Big\{\sum_{i=1}^{n} \frac{1}{\mu_{B_2}^N(g(u_{i-1})g(u_i))}\Big\} = \delta_2^N(g(u), g(v)).$$

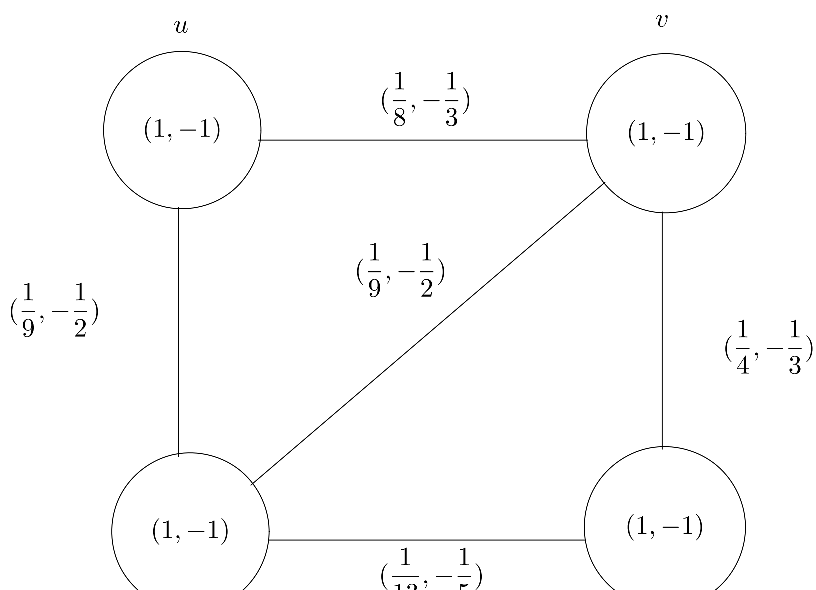
So, G_2 is isometric from G_1. □

Note 5.3. (i) The above result is true even G_1 is co-weak isomorphic to G_2 also. (ii) we know that, G_1 is isomorphic to G_1 implies $\overline{G_1}$ is isomorphic to $\overline{G_2}$. But this is not so in the case of isometry. In the following example, we show that G_2 is isometric from G_1 but $\overline{G_2}$ is not isometric from $\overline{G_1}$.

Example 5.4. Consider two bipolar fuzzy graphs G_1 and G_2 of G_1^* and G_2^*, respectively.

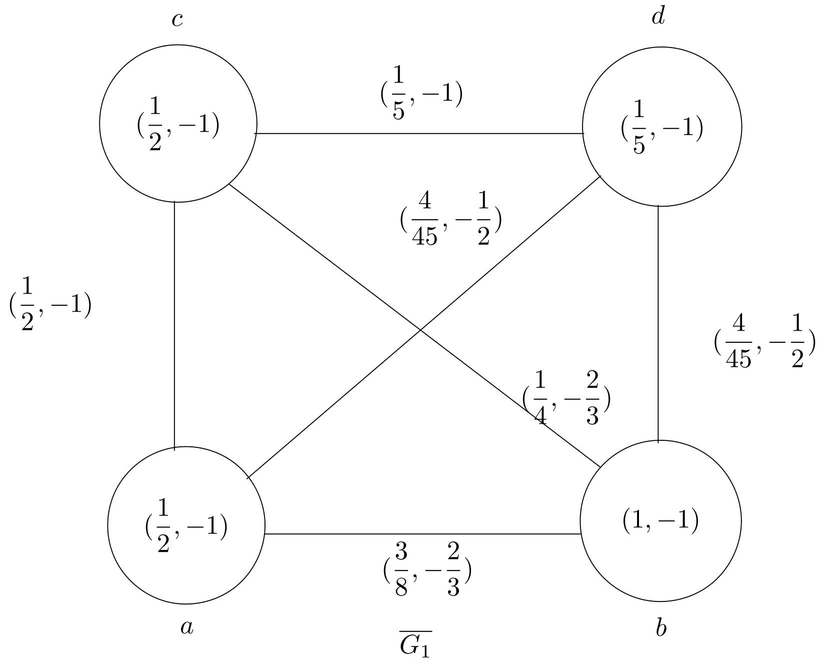

d
$(\frac{1}{5}, -1)$
c
$(\frac{1}{2}, -1)$
$(\frac{1}{9}, -\frac{1}{2})$
$(\frac{1}{9}, -\frac{1}{2})$
$(\frac{1}{4}, -\frac{1}{3})$
$(1, -1)$
$(\frac{1}{2}, -1)$
$(\frac{1}{8}, -\frac{1}{3})$
b
a

G_1

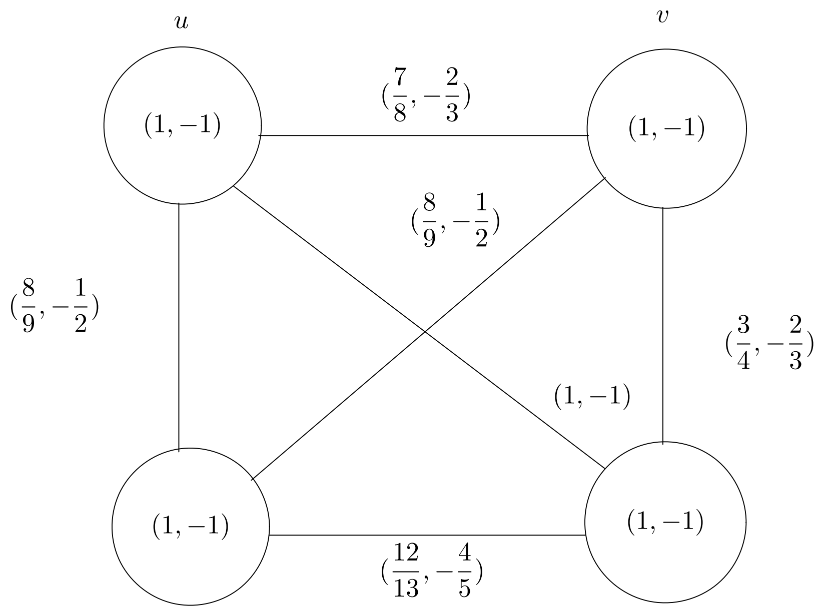

u
v
$(1, -1)$
$(\frac{1}{8}, -\frac{1}{3})$
$(1, -1)$
$(\frac{1}{9}, -\frac{1}{2})$
$(\frac{1}{9}, -\frac{1}{2})$
$(\frac{1}{4}, -\frac{1}{3})$
$(1, -1)$
$(1, -1)$
$(\frac{1}{13}, -\frac{1}{5})$
x
w

G_2

It is not difficult to verify that the flowing holds.
$\delta_1^P(a,b) = 8$, $\delta_1^P(a,c) = 12$, $\delta_1^P(a,d) = 9$, $\delta_1^P(b,c) = 4$, $\delta_1^P(b,d) = 9$
$\delta_1^N(a,b) = -3$, $\delta_1^N(a,c) = -6$, $\delta_1^N(a,d) = -2$, $\delta_1^N(b,c) = -3$, $\delta_1^N(b,d) = -2$
$\delta_1^P(c,d) = 13$, $\delta_1^N(c,d) = -5$. Likewise in bipolar fuzzy graph G_2 we have
$\delta_2^P(u,v) = 8$, $\delta_2^P(u,w) = 12$, $\delta_2^P(u,x) = 9$, $\delta_2^P(v,w) = 4$, $\delta_2^P(v,x) = 9$
$\delta_2^N(u,v) = -3$, $\delta_2^N(u,w) = -6$, $\delta_2^N(u,x) = -2$, $\delta_2^N(u,w) = -3$, $\delta_2^N(v,x) = -2$
$\delta_1^P(w,x) = 13$, $\delta_1^N(w,x) = -5$.
Defining $Q : V_1 \to V_2$ such that $Q(a) = u$, $Q = (b) = v$, $Q = (c) = w$, $Q = (d) = x$.
Clearly Q is a bijective that preserve the distance between every pair of vertices in G_1 and G_2. Hence G_2 is isometric from G_1.
Now we consider $\overline{G_1}$ and $\overline{G_2}$ of two bipolar fuzzy graph G_1 and G_2.



$\overline{G_1}$



$\overline{G_2}$

In $\overline{G_1}$, $\delta_1^P(a,b) = 2.6$, $\delta_1^P(a,c) = 2$, $\delta_1^P(a,d) = 7$, $\delta_1^P(b,c) = 4$,
$\delta_1^N(a,b) = -1.5$, $\delta_1^N(a,c) = -1$, $\delta_1^N(a,d) = -2$, $\delta_1^N(b,c) = -1.5$,
$\delta_1^P(b,d) = 9$, $\delta_1^N(b,d) = -2.5$, $\delta_1^P(c,d) = 5$, $\delta_1^N(c,d) = -1$.
Similarly, in $\overline{G_2}$ we have
$\delta_2^P(u,v) = 1.14$, $\delta_2^P(u,w) = 1$, $\delta_2^P(u,x) = 1.12$, $\delta_2^P(v,w) = 1.33$,
$\delta_2^N(u,v) = -1.5$, $\delta_2^N(u,w) = -1$, $\delta_2^N(u,x) = -2$, $\delta_2^N(u,w) = -1.5$,
$\delta_2^P(v,x) = 1.12$, $\delta_2^N(v,x) = -2$, $\delta_2^P(w,x) = 1.08$, $\delta_2^N(w,x) = -1.25$.
So there is not a bijective between $\overline{G_1}$ and $\overline{G_2}$ which preserving distance. Hence $\overline{G_2}$ is not isometric from $\overline{G_1}$.

Proposition 5.5. *Isometry on bipolar fuzzy graphs is an equivalence relation.*

Proof. Let $G_i = (A_i, B_i)$, $i = 1, 2, 3$ be the bipolar fuzzy graphs with underlying sets V_i. Considering the identity map $i : V_1 \to V_1$, G_1 is isometric to G_1. Therefore isometry is reflexive.
To prove the symmetric, assume that G_1 is isometric to G_2. Hence G_2 is isometric from G_1 and G_2 is isometric from G_2. By rearranging, G_2 is isometric to G_1. To prove the transitivity, let G_1 be isometric to G_2 and G_2 be isometric to G_3, i.e. G_2 is isometric from G_1 and G_3 is isometric from G_2. Then, for each $v \in V_1$, there exists a bijective map $g_v : V_1 \to V_2$ such that $\delta_1^P(v,u) = \delta_2^P(g_v(v), g_v(u))$,
$\delta_1^N(v,u) = \delta_2^N(g_v(v), g_v(u))$ for all $u \in V_1$.
Suppose that $g_v(v) = v'$. Similarly, for each $v' \in V_2$, there exists a bijective map $h_{v'} : V_2 \to V_3$ such that $\delta_2^P(v', u') = \delta_3^P(h_{v'}(v'), h_{v'}(u'))$,
$\delta_2^N(v', u') = \delta_3^N(h_{v'}(v'), h_{v'}(u'))$ for all $u' \in V_2$. Now if $v \in V_1$,

$\delta_1^P(v,u) = \delta_2^P(g_v(v), g_v(u)) = \delta_2^P(v', u') = \delta_3^P(h_{v'}(v'), h_{v'}(u'))$
$= \delta_3^P(h_{v'}(g_v), h_{v'}(g_v(u))) = \delta_3^P(h_{v'} \circ g_v(v), h_{v'} \circ g_v(u))$, for all $u \in V_1$.
$\delta_1^N(v,u) = \delta_2^N(g_v(v), g_v(u)) = \delta_3^N(h_{v'} \circ g_v(v), h_{v'} \circ g_v(u))$, for all $u \in V_1$.
Hence G_3 is isometric from G_1, using the composite map $h_{v'} \circ g_v : V_1 \to V_3$.
Thus isometry on bipolar fuzzy graphs is an equivalence relation. □

6. Conclusions

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including geometry, algebra, number theory, topology, operations research, optimization and computer science. The bipolar fuzzy sets constitute a generalization of Zadeh's fuzzy set theory. The bipolar fuzzy models give more precision, flexibility and compatibility to the systems as compared to the classical and fuzzy models. In this paper, we investigated some properties of complete bipolar fuzzy graphs, describe various methods of their construction. We also defined strong product and categorical product on bipolar fuzzy graphs and gave some of their properties. Finally we defined isometry on bipolar fuzzy graphs and show that isometry on bipolar fuzzy graphs is an equivalence relation.

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