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On acceleration convergence of double sequences of fuzzy numbers

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Abstract. In this article the notion of acceleration convergence of double sequences of fuzzy numbers in Pringsheim's sense has been introduced and some theorems related to that concept have been established using the four dimensional matrix transformations.

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1. Introduction

The faster convergence of sequences particularly the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations are widely used in finding solutions of mathematical as well as different scientific and engineering problems. The problem of acceleration convergence often occurs in numerical analysis. To accelerate the convergence, the standard interpolation and extrapolation methods of numerical mathematics are quite helpful. It is useful to study about the acceleration of convergence methods, which transform a slowly converging sequence into a new sequence, converging to the same limit faster than the original sequence. The speed of convergence of sequences is of the central importance in the theory of sequence transformation.

A sequence transformation T is a function $T : (x_k) \to (x_k^*)$ which maps a slowly convergent sequence to another sequence with better numerical properties. If $\lim_k x_k = x$ and $\lim_k x_k^* = x^*$ with r_k and r_k^* as the truncated errors. Then we have $x_k = x + r_k, x_k^* = x^* + r_k^*$. We say that the sequence (x_k) converge more rapidly than the sequence (x_k^*) if

$$\lim_k \frac{x_k^* - x^*}{x_k - x} = \lim_k \frac{r_k^*}{r_k} = 0.$$

The convergence rate of a sequence is defined as follows:

Let (x_k) be a real valued sequence with limit x. Then the convergence rate of (x_k) is characterized by

$$\alpha = \lim_k \frac{x_{k+1} - x_k}{x_k - x},$$

which closely resembles the ratio test in the theory of infinite series. If $0 < \alpha < 1$, then (x_k) is said to be linearly convergent. If $\alpha = 1$, then (x_k) is said to be logarithmically convergent, if $\alpha = 0$ then (x_k) is said to converge hyper-linearly and obviously $\alpha > 1$ stands for divergence of the sequence.

In 1968, D.F. Dawson [4] had characterized the summability field of matrix $A = (a_{kn})$ by showing A is convergence preserving over the set of all sequences which converge faster than some fixed sequence x or A only preserves the limit of a set of constant sequences. In 1979, Smith and Ford [23] introduced the concept of acceleration of linear and logarithmic convergence. Subsequently, Keagy and Ford [12] had established the results analogues to the results of Dawson [4] dealing with the acceleration field of subsequence transformation. Later, Brezinski, Delahaye and Gesmain-Bonne [3], Brezinski [2] and many other authors have worked in this areas (see [1, 18, 20, 24, 25]).

The sequence $x = (x_n)$ converges to σ faster than the sequence $y = (y_n)$ converges to λ, usually written as $x < y$, if

$$\lim_{n\to\infty} \frac{|x_n - \sigma|}{|y_n - \lambda|} = 0, \text{ provided } y_n - \lambda \neq 0 \text{ for all } n \in \mathbb{N}.$$

The matrix $A = (a_{kn})$ is said to accelerate the convergence of the sequence $x = (x_n)$ if $Ax < x$. The acceleration field of A is defined to be the class of sequences $\{x = (x_n) \in w : Ax < x\}$, where w is the space of all sequences.

The sequence $x = (x_n)$ converges to σ at the same rate as the sequence $y = (y_n)$ converges to λ, written as $x \approx y$, if

$$0 < \lim-\inf \left|\frac{x_n - \sigma}{y_n - \lambda}\right| \leq \lim-\sup \left|\frac{x_n - \sigma}{y_n - \lambda}\right| < \infty.$$

Let $A = (a_{kn})$ be an infinite matrix. For a sequence $x = (x_n)$, the A-transform of x is defined as

$$Ax = \sum_{k=1}^{\infty} a_{kn} x_n, \text{ for all } n \in \mathbb{N}.$$

The subsequence (x_{k_n}) of the sequence $x = (x_n)$ can be represented by a matrix transformation Ax, where the matrix $A = (a_{kn})$ is defined by

$$a_{i,k_i} = \begin{cases} 1, & \text{for all } i \in \mathbb{N}; \\ 0, & \text{otherwise} \end{cases}$$

For a matrix summabilty transformation A, we define the domain of A, usually denoted by $d(A)$, as

$$d(A) = \left\{ x = (x_n) \in w : \lim_k \sum_{n=1}^{\infty} a_{kn} x_n \text{ exists} \right\}.$$

We denote

$$l^1 = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

$S_\delta = \{x = (x_n) \in w : x_n \geq \delta > 0, \text{ for all } n\}$,
$S_0 =$ the set of all nonnegative sequences which have at most a finite number of zero entries.

The concept of fuzzy set was initially introduced by Zadeh [26]. It has a wide range of applications in almost all the branches of studied in particular in science, where mathematics is used. Now the notion of fuzziness are using by many researchers for Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition. Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting etc. The fuzziness of all the subjects of mathematical sciences has been investigated. In [13], Nanda studied on sequences of fuzzy numbers. Savas [21] introduced and discussed double convergent sequences of fuzzy numbers. For sequences of fuzzy numbers we refer to [5, 8, 10, 11, 19, 22].

Now, we recall some basic notions in the theory of fuzzy numbers.

A fuzzy number is a fuzzy set $X : \mathbb{R} \to J(= [0,1])$ which satisfies the following conditions:

(i) X is normal, i.e., there exists an $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$;
(ii) X is fuzzy convex i.e., $X(\lambda s + (1-\lambda)t) \geq \min\{X(s), X(t)\}$, for $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$;
(iii) X is said to be upper-semi continuous i.e., for each $\varepsilon > 0, X^{-1}([0, a+\varepsilon))$, for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$;
(iv) $[X]^0$ =closure of $\{t \in \mathbb{R} : X(t) > \alpha,\ if\ \alpha = 0\}$ is a compact set.

The properties (i) to (iv) implies that for each $\alpha \in (0,1]$, the α-level set,

$$[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\} = [x_1^\alpha, x_2^\alpha]$$

is a non-empty compact subset of $\mathbb{R}$. Let $\mathbb{R}(J)$ denote the set of all fuzzy numbers. The set $\mathbb{R}$ of real numbers can be embedded in $\mathbb{R}(J)$ if we define $\overline{r} \in \mathbb{R}(J)$ by

$$\overline{r}(t) = \begin{cases} 1 & , \quad \text{if } t = r; \\ 0 & , \quad \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of $\mathbb{R}(J)$ are defined by $\overline{0}$ and $\overline{1}$, respectively.

For $X, Y \in \mathbb{R}(J)$, the linear structure of $\mathbb{R}(J)$ induced by the operations addition $X+Y$ and scalar multiplication λX, for $\lambda \in \mathbb{R}$, in terms of α-level sets,

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha, \text{ for each } \alpha \in [0,1].$$

Define a mapping $\overline{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ by

$$\overline{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d\left([X^\alpha], [Y]^\alpha\right).$$

It is known that $(\mathbb{R}(J), \overline{d})$ is a complete metric space.

2. Definitions and Preliminaries

In 1900, A. Pringsheim [16] introduced the notion of double sequence and presented the definition of convergence of a double sequence. Robinson [17] studied the divergent of double sequences and series. Later on Hamilton ([6, 7]) introduced the transformation of multiple sequences. Subsequently Patterson [15] introduced the concept of rate of convergence of double sequences. Recently, Hazarika [9] introduced the notion of acceleration convergence of double sequences and proved some interesting results.

In this article various notions and definitions on fuzzy double sequence and fuzzy double sequence spaces have been presented. Some interesting results on convergent fuzzy double sequences and bounded fuzzy double sequences have been presented. Further, summability field of a four dimensional matrix $A = (a_{k,l,m,n})$ and the acceleration field of subsequence transformation of fuzzy double sequences have been characterized.

Definition 2.1 ([16])**.** A double sequence $x = (x_{m,n})$ is said to converge to a number L in Pringsheim's sense, symbolically we write $P - \lim_{m,n\to\infty} x_{m,n} = L$, if for every $\varepsilon > 0$, there exists a positive integer n_0 depending upon ε, such that $|x_{m,n} - L| < \varepsilon$, whenever $m, n \geq n_0$. The number L is called the Pringsheim's limit of the sequence x.

Definition 2.2 ([16])**.** A double sequence $x = (x_{m,n})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{m,n}| < M$, for all m and n.

Definition 2.3 ([14])**.** The double sequence y is a double subsequence of a sequence $x = (x_{m,n})$, if there exists two increasing index sequences $\{m_j\}$ and $\{n_j\}$ such that $y = (x_{m_j,n_j})$.

Definition 2.4 ([14])**.** A number β is said to be a Pringsheim limit point of the double sequence $x = (x_{m,n})$ if there exists a subsequence y of x such that $P - \lim y = \beta$.

Definition 2.5 ([14])**.** : Let $x = (x_{m,n})$ be a double sequence of real numbers and for each k, let $\alpha_k = \sup_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit superior of x is defined as follows:

(i) If $\alpha_k = +\infty$, for each k, then $P - \lim - \sup x = +\infty$;
(ii) If $\alpha_k < +\infty$, for each k, then $P - \lim - \sup x = \inf_k\{\alpha_k\}$.

Similarly, let $\beta_k = \inf_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit inferior of x is defined as follows:

(iii) If $\beta_k = -\infty$, for each k, then $P - \lim - \sup x = -\infty$;
(iv) If $\beta_k > -\infty$, for each k, then $P - \lim - \sup x = \sup_k\{\beta_k\}$.

Definition 2.6. A four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Definition 2.7. Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. The cardinality of A, usually denoted by $|A(m, n)|$, is defined to be the number of (i, j) in A such that $i \leq m$ and $j \leq n$.

Definition 2.8. A two-dimensional set of positive integers A is said to have a double natural density, if the sequence $\left(\frac{|A(m,n)|}{mn}\right)$ has a limit in Pringsheim's sense. If this exists, it is denoted by $\delta_2(A)$. Thus we have

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|A(m,n)|}{mn}.$$

Clearly we have $\delta_2(A^c) = \delta_2(\mathbb{N}\times\mathbb{N} - A) = 1 - \delta_2(A)$. Further, it is also clear that all finite subsets of $\mathbb{N}\times\mathbb{N}$ have zero double natural density. Moreover, some infinite subsets also have zero density. For example, the set

$$A = \big\{(i,j) : i \in [2^k, 2^k + k) \text{ and } j \in [2^l, 2^l + l), k, l = 1, 2, 3...\big\}$$

has double natural density zero.

Definition 2.9. A double sequence $x = (x_{m,n})$ is said to satisfy a property P for "almost all (m,n)" if it satisfies the property P for all (m,n) except a set of double natural density zero. We abbreviate this by "$a.a.(m,n)$".

Definition 2.10. A matrix transformation associated with the four-dimensional matrix A is said to be an ${}_2c_0 - {}_2c_0$ if Ax is in the set ${}_2c_0$, whenever x is in ${}_2c_0$ and is bounded (for details see [14]).

The following is an important result on the characterization of ${}_2c_0 - {}_2c_0$ matrices:

Lemma 2.11 ([6])**.** *A four-dimensional matrix $A = (a_{k,l,m,n})$ is an ${}_2c_0 - {}_2c_0$, if and only if,*

(a) $\sum_{p,q=1,1}^{\infty,\infty} |a_{k,l,p,q}| < \infty$ *for all* k, l;
(b) *for* $q = q_0$, *there exists* $C_q(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $q > C_q(k,l)$ *for all* k, l, p;
(c) *for* $p = p_0$, *there exists* $C_p(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $p > C_p(k,l)$ *for all* k, l, q;
(d) $P - \lim_{k,l} a_{k,l,p,q} = 0$, *for all* p *and* q.

Now, we recall some basic notions of double sequences of fuzzy numbers.

A fuzzy double sequence is a double infinite array of fuzzy numbers. We denote a fuzzy real-valued double sequence by $(X_{m,n})$, where $X_{m,n}$ are fuzzy numbers for each $m, n \in \mathbb{N}$.

Definition 2.12 ([21])**.** A double sequence $X = (X_{m,n})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P- convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\bar{d}\,(X_{m,n}, X_0) < \varepsilon \text{ for } m, n > N,$$

and we denote by $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of $(X_{m,n})$.

Definition 2.13. A double sequence $X = (X_{m,n})$ of fuzzy numbers is bounded if there exists a positive number M such that $d\,(X_{m,n}, X_0) < M$ for all m and n,

$$||X||_{(\infty,2)} = \sup_{m,n} \bar{d}\,(X_{m,n}, X_0) < \infty.$$

Now, we define the Pringsheim limit superior and limit inferior of double sequences of fuzzy numbers as follows:

Definition 2.14. Let $X = (X_{m,n})$ be a double sequence of fuzzy numbers and for each k, let $\alpha_k^F = \sup_k\{X_{m,n} : m, n \geq k\}$. Then the Pringsheim limit superior of X is defined as follows:

(i) If $\alpha_k^F = +\overline{\infty}$, for each k, then $P - \lim - \sup X = +\overline{\infty}$;
(ii) If $\alpha_k^F < +\overline{\infty}$, for each k, then $P - \lim - \sup X = \inf_k\{\alpha_k^F\}$.

Similarly, let $\beta_k^F = \inf_k\{X_{m,n} : m, n \geq k\}$. Then the Pringsheim limit inferior of X is defined as follows:

(iii) If $\beta_k^F = -\overline{\infty}$, for each k, then $P - \lim - \sup X = -\overline{\infty}$;
(iv) If $\beta_k^F > -\overline{\infty}$, for each k, then $P - \lim - \sup X = \sup_k\{\beta_k^F\}$.

Throughout this paper we use the following notations:
${}_2w^F$ = the space of all double sequences of fuzzy numbers.
${}_2l_\infty^F$ = the space of all bounded double sequences of fuzzy numbers.
${}_2c^F$ = the space of all convergent in Pringsheim's sense double sequences of fuzzy numbers.
${}_2c_0^F$ = the space of all null in Pringsheim's sense double sequences of fuzzy numbers.
${}_2c_0^{BF} = {}_2c_0^F \cap {}_2l_\infty^F$.
${}_2S_0^{BF}$ = the subset of the space ${}_2c_0^{BF}$.
${}_2S_\delta^F$ = the set of all fuzzy valued double sequences $X = (X_{m,n})$ such that $X_{m,n} \geq \delta > 0$, for all m and n.
${}_2S_0^F$ = the set of all non-negative sequences which have at most finite number of columns and /or rows with zero entries.

$$ {}_2l^F = \left\{ X = (X_{m,n}) : \sum_{m,n=1,1}^{\infty,\infty} \overline{d}(X_{m,n}, \overline{0}) < \infty \right\}, $$

$$ {}_2d^F(A) = \left\{ X = (X_{m,n}) : P - \lim_{k,l} \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} X_{m,n} \text{ exists} \right\}. $$

$$ \mu_{k,l}(X) = \sup_{m,n \geq k,l} \overline{d}(X_{m,n}, \overline{0}), \text{ for } X \in {}_2l^F. $$

Definition 2.15. Let $A = (a_{k,l,m,n})$ be a four-dimensional matrix. For any $X = (X_{m,n}) \in {}_2w^F$, the A-transform of X is defined as

$$ AX = \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} X_{m,n}, \text{ for all } m, n \in \mathbb{N}. \tag{2.1} $$

The subsequence (X_{m_k,n_l}) of the sequence $X = (X_{m,n})$ can be represented by a matrix transformation represented by (2.1), where

$$ a_{k,l,m,n} = \begin{cases} 1, & \text{if } (m,n) = (m_k, n_l); \\ 0, & \text{if } (m,n) \neq (m_k, n_l) \end{cases} \tag{2.2} $$

It can be easily verified that the matrix as defined in (2.2) is a RH-regular matrix.

3. Acceleration convergence of multiple sequences of fuzzy numbers

In this section some more definitions related to double sequences of fuzzy numbers have been defined and some interesting theorems regarding acceleration convergence of double sequences of fuzzy numbers have been discussed.

Recently, Hazarika [9] introduced the notion of acceleration convergence of double sequences of real nuumbers as follows:

Definition 3.1 ([9])**.** Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two double sequences of real numbers. Then the sequence x is said to converge P-faster than the sequence y, written as $x <^P y$, if

$$P - \lim_{m,n} \left| \frac{x_{m,n}}{y_{m,n}} \right| = 0.$$

Definition 3.2 ([9])**.** The sequence $x = (x_{m,n})$ is said to converge at the same rate in Pringsheim's sense as the sequence $y = (y_{m,n})$, written as $x \approx^P y$, if

$$0 < P - \lim - \inf \left| \frac{x_{m,n}}{y_{m,n}} \right| \leq P - \lim - \sup \left| \frac{x_{m,n}}{y_{m,n}} \right| < \infty.$$

Definition 3.3 ([9])**.** The four-dimensional matrix $A = (a_{k,l,m,n})$ is said to P-accelerate the convergence of the sequence $x = (x_{m,n})$ if $Ax <^P x$.

We define the P-acceleration field of A as the set

$$\{x = (x_{m,n}) \in {}_2w : Ax <^P x\}.$$

Now we define the acceleration convergence of double sequences of fuzzy numbers as follows:

Definition 3.4. Let $X = (X_{m,n})$ and $Y = (Y_{m,n})$ be two double sequences of fuzzy numbers with $X_{m,n} \xrightarrow{P} X_0$ and $Y_{m,n} \xrightarrow{P} Y_0$. Then the sequence X converges to X_0, P-faster than the sequence Y converges to Y_0, written as $X <^P Y$, if

$$P - \lim_{m,n} \frac{\overline{d}(X_{m,n}, X_0)}{\overline{d}(Y_{m,n}, Y_0)} = 0 \text{ provided } \overline{d}(Y_{m,n}, Y_0) \neq 0 \text{ for all } m, n \in \mathbb{N}.$$

Definition 3.5. The double sequence $X = (X_{m,n})$ converges to X_0 at the same rate in Pringsheim's sense as the sequence $Y = (Y_{m,n})$ converges to Y_0, written as $X \approx^P Y$, if

$$0 < P - \lim - \inf \frac{\overline{d}(X_{m,n}, X_0)}{\overline{d}(Y_{m,n}, Y_0)} \leq P - \lim - \sup \frac{\overline{d}(X_{m,n}, X_0)}{\overline{d}(Y_{m,n}, Y_0)} < \infty.$$

Definition 3.6. The four-dimensional matrix $A = (a_{k,l,m,n})$ is said to P-accelerate the convergence of the sequence $X = (X_{m,n})$ if $AX <^P X$.

We define the P-acceleration field of A as the set

$$\{X = (X_{m,n}) \in {}_2w^F : AX <^P X\}.$$

Definition 3.7. A matrix transformation associated with the four-dimensional matrix A is said to be an ${}_2c_0^F - {}_2c_0^F$ if AX is in the set ${}_2c_0^F$, whenever X is in ${}_2c_0^F$ and is bounded.

Theorem 3.8. *Let $X = (X_{m,n})$ and $Y = (Y_{m,n})$ be two elements of ${}_2S_0^{BF}$ such that $X <^P Y$, then there exists an element $Z = (Z_{m,n})$ in ${}_2S_0^{BF}$ such that $X <^P Z <^P Y$.*

Proof. Let $X, Y \in {}_2S_0^{BF}$ be such that $X <^P Y$. Define the sequence $Z = (Z_{m,n})$ as follows:

$$Z = X_{m,n}^{\frac{4}{5}} \otimes Y_{m,n}^{\frac{1}{5}}.$$

This implies that $X <^P Z <^P Y$. □

Theorem 3.9. *Let $X <^P Y$ and $Y \approx^P Z$, then $X <^P Z$.*

Proof. The proof is omitted as it is straight forward. □

Theorem 3.10. *Let A be a nonnegative ${}_2c_0^F - {}_2c_0^F$ summability matrix and let X and Y be two elements in ${}_2l^F$ such that $X <^P Y$ with $X \in {}_2S_0^{BF}$ and $Y \in {}_2S_\delta^F$ for some $\delta > 0$, then $\mu(AX) <^P \mu(AY)$.*

Proof. Since $X <^P Y$, then there exists a bounded double sequence $Z = (Z_{m,n})$ with Pringsheim's limit zero such that $X_{m,n} = Y_{m,n} \otimes Z_{m,n}$. For each k and l, we have

$$\begin{aligned}
\frac{\mu_{k,l}(AX)}{\mu_{k,l}(AY)} &= \frac{\sup_{r,s\geq k,l}(AX)_{r,s}}{\sup_{r,s\geq k,l}(AY)_{r,s}} \\
&= \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ X_{m,n}}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}} \\
&= \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}\ \otimes Z_{m,n}}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\overline{d}\left(\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}\otimes Z_{m,n}, \overline{0}\right)}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}\ \overline{d}\left(Z_{m,n}, \overline{0}\right)}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ Y_{m,n}\ \overline{d}\left(Z_{m,n}, \overline{0}\right)}{\delta\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}}.
\end{aligned}$$

Since Y and Z are bounded double sequences with Z is in ${}_2c_0^F$ and A is a nonnegative ${}_2c_0^F - {}_2c_0^F$ matrix , then

$$P-\lim_{k,l}\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\ y_{m,n}\ \overline{d}\left(Z_{m,n}, \overline{0}\right) = 0.$$

Hence

$$P-\lim_{k,l}\frac{\mu_{k,l}(AX)}{\mu_{k,l}(AY)} \leq 0. \tag{3.1}$$

In a similar manner we can establish

$$P-\lim_{k,l}\frac{\mu_{k,l}(AX)}{\mu_{k,l}(AY)} \geq 0. \tag{3.2}$$

Hence from (3.1) and (3.2), we have

$$P - \lim_{k,l} \frac{\mu_{k,l}(AX)}{\mu_{k,l}(AY)} = 0$$

which implies $\mu(AX) <^P \mu(AY)$. This establishes the result. □

Theorem 3.11. *Let $X = (X_{m,n}) \in {}_2S_0^{BF}$ and A be a subsequence transformation such that $AX <^P X$. Then there exists $Y = (Y_{m,n}) \in {}_2S_0^{BF}$ such that $X_{m,n} = Y_{m,n}$ a.a.(m,n) and $AY <^P Y$.*

Proof. Let $X = (X_{m,n}) \in {}_2S_0^{BF}$. Then there exists a subset $B_1 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_1) = 1$ such that

$$P - \lim\ X_{m,n} = 0, \text{ over } B_1.$$

Let $(X_{m_k,n_l}) \in {}_2S_0^{BF}$. Then there exists a subset $B_2 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_2) = 1$ such that

$$P - \lim\ X_{m_k,n_l} = 0, \text{ over } B_2.$$

Since $AX <^P X$, we have

$$P - \lim\ \frac{\overline{d}(X_{m_k,n_l}, \overline{0})}{\overline{d}(X_{m,n}, \overline{0})} = 0.$$

Then there exists a subset $B_3 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_3) = 1$ such that

$$P - \lim\ \frac{\overline{d}(X_{m_k,n_l}, \overline{0})}{\overline{d}(X_{m,n}, \overline{0})} = 0, \text{ over } B_3.$$

Let $D = B_1 \cap B_2 \cap B_3$. Then clearly $\delta_2(D) = 1$.
For $r \neq m_k, s \neq n_l, (k,l) \in \mathbb{N} \times \mathbb{N}$, let us define the sequence $Y = (Y_{m,n})$ as follows:

$$Y_{r,s} = \begin{cases} X_{r,s} & \text{if } (r,s) \in D; \\ (rs)^{-3} & \text{otherwise} \end{cases}$$

and

$$Y_{m_k,n_l} = \begin{cases} X_{m_k,n_l} & \text{if } (k,l) \in D; \\ Y_{m,n}\ (mn)^{-3} & \text{otherwise} \end{cases}$$

Then we have $Y = (Y_{m,n}) \in {}_2S_0^{BF}$ such that $X_{m,n} = Y_{m,n}$ *a.a.(m,n)* and this implies $AY <^P Y$. □

Theorem 3.12. *Let $X = (X_{m,n}) \in {}_2S_0^{BF}$ and A be a subsequence transformation such that $AX <^P X$. Then there exists $Y = (Y_{m,n}) \in {}_2S_0^{BF}$ such that $X <^P Y$ and $AY <^P Y$.*

Proof. Consider the sequence

$$Y_{m,n} = [\overline{d}(X_{m,n}, \overline{0})]^{\frac{1}{2}} \text{ for all } m, n \in \mathbb{N}.$$

Then clearly $Y = (Y_{m,n}) \in {}_2S_0^{BF}$ such that $X <^P Y$ and $AY <^P Y$. This establishes the theorem. □

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