Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 2, (August 2014), pp. 245–257
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Finite dimensional intuitionistic fuzzy normed linear spaces

T. Bag, S. K. Samanta

Received 30 September 2013; Revised 13 January 2014; Accepted 21 January 2014

ABSTRACT. In this paper we consider general t-norm in the definition of fuzzy normed linear space which is introduced by the authors in an earlier paper. It is proved that if t-norm is chosen other than "min" then decomposition theorem of a fuzzy norm into a family of crisp norms may not hold. We study some basic results on finite dimensional fuzzy normed linear spaces in general t-norm setting.

2010 AMS Classification: 54A40, 03E72

Keywords: Fuzzy norm, t-norm, Fuzzy normed linear space.

Corresponding Author: Tarapada Bag (tarapadavb@gmail.com)

1. Introduction

Theory of fuzzy sets was introduced by Zadeh [19] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments are made in the field of fuzzy metric spaces and fuzzy normed linear spaces [3, 4, 6, 7, 8, 9, 11]. The notion of intuitionistic fuzzy set has been introduced by Atanassov [1] as a generalized fuzzy set. J.H.Park [14], who first introduced the idea of intuitionistic fuzzy metric space and studied some basic properties. On the other hand, Saadati & Park [15] have an important contribution on the intuitionistic fuzzy topological spaces. They have also introduced the notion of intuitionistic fuzzy normed linear space and studied some basic properties in such spaces. There have been a good amount of work done in intuitionistic fuzzy set such as T.K. Mandal & S.K.Samanta [12, 13], N. Thillaigovindan et al.[17]. Recently Vijayabalaji et al.[18] introduced a concept of intuitionistic fuzzy n-normed linear space and developed some results. T.K.Samanta et al. [16] considered a fuzzy normed linear space which was introduced by Bag & Samanta [2, 5] and defined an intuitionistic fuzzy normed linear space in general setting (taking $*$ and $\diamondsuit$ as t-norm and t-co-norm respectively). They mainly studied different results on finite dimensional intuitionistic fuzzy normed linear space. But their results depend on the decomposition theorem of the intuitionistic fuzzy norm into a family of pairs of crisp norms for which they have taken the additional conditions on t-norm and t-conorm as $a * a = a$ and $a \diamondsuit a = a \quad \forall a \in [0\ ,\ 1]$ which resulted $*$=min and $a\diamondsuit a$=max. So effectively the generality of the t-norm and t-conorm are lost. On the other hand, because of the relation $M(x,t) + N(x,t) \leq 1$, some of the conditions involving the functions $M(x,t)$ and $N(x,t)$ in the definition considered by T.K.Samanta et al.[16] led to such a situation that in some definitions and results related to convergence and Cauchyness of a sequence statements involving one of the functions M and N follows from the other.

To avoid these triviality, in this paper, we have modified the definition of intuitionistic fuzzy normed linear space introduced by R. Saadati et al. [15] and study finite dimensional intuitionistic fuzzy normed linear space. In our definition both the conditions viz. (1) $a * a = a$, $a \diamondsuit a = a$ and (2) $M(x,t) + N(x,t) \leq 1$ are waived. In our present approach we have avoided the decomposition technique which is very much dependent on the restricted t-norm viz. min and t-conorm viz. max.

The organization of the paper is as in the following:
Section 1 comprises some preliminary results. In Section 2, we introduce a definition of intuitionistic fuzzy normed linear space. Some basic results on completeness and compactness are established in finite dimensional intuitionistic fuzzy normed linear spaces in Section 3.

2. Preliminaries

Definition 2.1 ([10]). A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $a * 1 = a \quad \forall a \in [0\ ,\ 1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.

If $*$ is continuous then it is called continuous t-norm. Following are examples of some t-norms that are frequently used as fuzzy intersections defined for all $a,\ b \in\ [0,\ 1]$.
(i) Standard intersection: $a * b = min(a,\ b)$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = max(0,\ a + b - 1)$.
(iv) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

The relations among these t-norms are

$$a * b(\text{Drastic}) \leq max(0,\ a + b - 1) \leq ab \leq min(a,\ b).$$

Definition 2.2 ([10]). A binary operation $\diamondsuit : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-co-norm if it satisfies the following conditions:

(1) $\diamondsuit$ is associative and commutative;
(2) $a \diamondsuit 0 = a \quad \forall a \in [0\ ,\ 1]$;
(3) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.

If $\diamondsuit$ is continuous then it is called continuous t-co-norm. Following are some examples of t-co-norms.
(i) Standard union: $a\diamondsuit b = max(a,\ b)$.
(ii) Algebraic sum: $a\diamondsuit b = a + b - ab$.
(iii) Bounded sum : $a\diamondsuit b = min(1\ ,\ a + b)$.
(iv) Drastic union:

$$a\diamondsuit b = \begin{cases} a & \text{for } b = 0 \\ b & \text{for } a = 0 \\ 1 & \text{for otherwise.} \end{cases}$$

Relations among these t-co-norms are $a\diamondsuit b$ (Drastic)$\geq\ min(1\ ,\ a + b) \geq\ a + b - ab \geq\ max(a,\ b)$

Definition 2.3 ([5]). Let U be a linear space over the field $\mathbf{F}$ ($\mathbf{C}$ or $\mathbf{R}$). A fuzzy subset N of $U \times R$ (R- set of real numbers) is called a fuzzy norm on U if
(N1) $\forall t \in \mathbf{R}$ with $t \leq 0,\ N(x\ ,\ t) = 0$;
(N2) $(\forall t \in \mathbf{R}, t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$;
(N3) $\forall t \in \mathbf{R},\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s, t \in \mathbf{R};\ x, u \in U$;
$N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$;
(N5) $N(x\ ,\ .)$ is a non-decreasing function of $\mathbf{R}$ and $\lim\limits_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space. In [2], particular t-norm "min" is taken for $*$.

Definition 2.4 ([15]). The 5-tuple $(V, \mu, \nu, *, \diamondsuit)$ is said to be an intuitionistic fuzzy normed linear space if V is a vector space, $*$ is continuous t-norm, $\diamondsuit$ is a continuous t-conorm and μ, ν are fuzzy sets on $V \times (0\ ,\ \infty)$ satisfying the following conditions for every $x, y \in V$ and $s, t > 0$;
(a) $\mu(x,t)\ +\ \nu(x,t) \leq 1$;
(b) $\mu(x\ ,\ t)\ > 0$;
(c) $\mu(x\ ,\ t) = 1$ if and only if $x = \underline{0}$;
(d) $\mu(cx\ ,\ t) = \mu(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(e) $\mu(x + u\ ,\ s + t) \geq\ \mu(x\ ,\ s) * \mu(u\ ,\ t)$;
(f) $\mu :\ (0, \infty) \to [0, 1]$ is continuous;
(g) $\lim\limits_{t\to\infty} \mu(x\ ,\ t) = 1$ and $\lim\limits_{t\to 0} \mu(x\ ,\ t) = 0$;
(h) $\nu(x\ ,\ t)\ < 1$;
(i) $\nu(x\ ,\ t) = 0$ iff $x = \underline{0}$;
(j) $\nu(cx\ ,\ t) = \nu(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(k) $\nu(x + u\ ,\ s + t) \leq\ \nu(x\ ,\ s)\diamondsuit\nu(u\ ,\ t)$;
(l) $\nu :\ (0, \infty) \to [0, 1]$ is continuous;
(m) $\lim\limits_{t\to\infty} \nu(x\ ,\ t) = 0$ and $\lim\limits_{t\to 0} \nu(x\ ,\ t) = 1$.
In this case $(\mu\ ,\ \nu)$ is called an intuitionistic fuzzy norm.

T.K.Samanta et al. [16] consider the above definition by omitting the conditions (f) and (l) as in the following.

Definition 2.5 ([16]). Let $*$ be a continuous t-norm, $\diamondsuit$ be a continuous t-conorm and V be a linear space over the field $\mathbf{F}(\mathbf{R}/\mathbf{C})$. An intuitionistic fuzzy norm (IFN) on V is an object of the form $A = \{((x\ ,\ t), N(x\ ,\ t),\ M(x\ ,\ t)) : (x\ ,\ t) \in V \times \mathbf{R}^+\}$ where N,M are fuzzy sets on $V \times \mathbf{R}^+$, N denotes the degree of membership and M denotes the degree of non-membership

(i) $N(x,t) + M(x,t) \leq 1 \ \ \forall (x\ ,\ t) \in V \times \mathbf{R}^+$;
(ii) $N(x\ ,\ t) > 0$;
(iii) $N(x\ ,\ t) = 1$ if and only if $x = \underline{0}$;
(iv) $N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0,\ c \in \mathbf{F}$;
(v) $\forall s, t \in \mathbf{R}^+;\ x, u \in V$;
$N(x + u\ ,\ s + t) \geq N(x\ ,\ s) * N(u\ ,\ t)$;
(vi) $N(x\ ,\ .)$ is a non-decreasing function of $\mathbf{R}^+$ and $\lim\limits_{t \to \infty} N(x\ ,\ t) = 1$;
(vii) $M(x\ ,\ t) > 0$;
(viii) $(\forall t \in \mathbf{R}, t > 0, M(x\ ,\ t) = 0)$ iff $x = \underline{0}$;
(ix) $M(cx\ ,\ t) = M(x\ ,\ \frac{t}{|c|})$ if $c \neq 0,\ c \in \mathbf{F}$;
(x) $\forall s, t \in \mathbf{R}^+;\ x, u \in V$;
$M(x + u\ ,\ s + t) \leq M(x\ ,\ s) \diamondsuit M(u\ ,\ t)$;
(xi) $M(x\ ,\ .)$ is a non-increasing function of $\mathbf{R}^+$ and $\lim\limits_{t \to \infty} M(x\ ,\ t) = 0$.
Then we say $(V\ ,\ A)$ is an intuitionistic fuzzy normed linear space.

Definition 2.6 ([16]). A sequence $\{x_n\}$ in an IFNLS $(V\ ,\ A)$ is said to converge to $x \in V$ if given $r > 0,\ t > 0,\ 0 < r < 1$ there exists a positive integer n_0 such that $N(x_n - x\ ,\ t) > 1 - r$ and $M(x_n - x\ ,\ t) < r \quad \forall n \geq n_0$.

Theorem 2.7 ([16]). *In an IFNLS $(V\ ,\ A)$, a sequence $\{x_n\}$ converges to x iff* $\lim\limits_{n \to \infty} N(x_n - x\ ,\ t) = 1$ *and* $\lim\limits_{n \to \infty} M(x_n - x\ ,\ t) = 0$.

Theorem 2.8 ([16]). *If a sequence $\{x_n\}$ in an IFNLS $(V\ ,\ A)$, is convergent, its limit is unique.*

Definition 2.9 ([16]). A sequence $\{x_n\}$ in an IFNLS $(V\ ,\ A)$ is said to be a Cauchy sequence if $\lim\limits_{n \to \infty} N(x_{n+p} - x_n\ ,\ t) = 1$ and $\lim\limits_{n \to \infty} M(x_{n+p} - x_n\ ,\ t) = 0$ uniformly on $p = 1, 2, ...,\ t > 0$.

Definition 2.10 ([16]). Let $(V\ ,\ A)$ be an IFNLS. A subset P of V is said to be closed if for any sequence $\{x_n\}$ in P converges to $x \in P$.

Definition 2.11 ([16]). Let $(V\ ,\ A)$ be an IFNLS and $P \subset V$. Then the closure of P denoted by $\bar{P}$, is defined by $\bar{P} = \{x \in V :\ \exists$ a sequence $\{x_n\}$ in P converging to $x\}$.

Definition 2.12 ([16]). Let $(V\ ,\ A)$ be an IFNLS. A subset P of V is said to be compact if any sequence $\{x_n\}$ in P has a subsequence which converges to some element in P.

3. Intuitionistic fuzzy normed linear spaces

Following is our modified definition of intuitionistic fuzzy normed linear space.

Definition 3.1. Let $*$ be a t-norm, $\diamondsuit$ be a t-conorm and V be a linear space over the field $\mathbf{F}$($\mathbf{R}$ or $\mathbf{C}$). An intuitionistic fuzzy norm (IFN) on V is an object of the form
$A = \{((x\ ,\ t), N(x\ ,\ t),\ M(x\ ,\ t)) :\ (x\ ,\ t) \in V \times \mathbf{R}\}$ where N,M are fuzzy sets on $V \times \mathbf{R}$, N denotes the degree of membership and M denotes the degree of non-membership $(x\ ,\ t) \in V \times \mathbf{R}$ satisfying the following conditions:

(IFN1) $\forall t \in \mathbf{R}$ with $t \leq 0,\ N(x\ ,\ t) = 0$;
(IFN2) $(\forall t \in \mathbf{R}, t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$;
(IFN3) $\forall t \in \mathbf{R},\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(IFN4) $\forall s, t \in \mathbf{R};\ x, u \in U$;
$N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$;
(IFN5) $\lim_{t\to\infty} N(x\ ,\ t) = 1$.
(IFN6) $\forall t \in \mathbf{R}$ with $t \leq 0,\ M(x\ ,\ t) = 1$;
(IFN7) $(\forall t \in \mathbf{R}, t > 0, M(x\ ,\ t) = 0)$ iff $x = \underline{0}$;
(IFN8) $\forall t \in \mathbf{R},\ t > 0,\ M(cx\ ,\ t) = M(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(IFN9) $\forall s, t \in \mathbf{R};\ x, u \in V$;
$M(x + u\ ,\ s + t) \leq\ M(x\ ,\ s) \diamondsuit M(u\ ,\ t)$;
(IFN10) $\lim_{t\to\infty} M(x\ ,\ t) = 0$.
Then we say $(V\ ,\ A)$ is an intuitionistic fuzzy normed linear space.

Remark 3.2. From (IFN2) and (IFN4), it follows that $N(x\ ,\ .)$ is a non-decreasing function of $\mathbf{R}$. From (IFN7) and (IFN9), it follows that $M(x\ ,\ .)$ is a non-increasing function of $\mathbf{R}$.

Example 3.3. Let $(V\ ,\ ||\ ||)$ be a normed linear space. Define two fuzzy subsets N and M $:V \times \mathbf{R} \to [0\ ,\ 1]$ by

$$N(x\ ,\ t) = \begin{cases} \frac{t}{t+||x||} & \text{for } t > ||x|| \\ 0 & t \leq ||x|| \end{cases}$$

$$M(x\ ,\ t) = \begin{cases} \frac{||x||}{||x||+t} & \text{for } t > ||x|| \\ 1 & \text{for } t \leq ||x|| \end{cases}$$

Take $a * b = ab$ and $a \diamondsuit b\ = min\{1,\ a + b\}$. Then $(V\ ,\ A)$ is an intuitionistic fuzzy normed linear space.

Proof. All the conditions except (IFN4) and (IFN9) are easily verified.
First we verify (IFN4); i.e., $N(x + y\ ,\ s + t) \geq\ N(x\ ,\ s) * N(y\ ,\ t)$.
Let $x, y \in V$ and $s, t \in \mathbf{R}$.
Suppose $s, t\ > 0$ (Since in other cases (IFN4) is obvious).
We have,

$$\frac{s+t}{s+t+||x+y||} - \frac{st}{(s+||x||)(t+||y||)}\ \geq \frac{s+t}{s+t+||x||+||y||} - \frac{st}{(s+||x||)(t+||y||)}$$
$$= \frac{s+t}{s+t+||x||+||y||} - \frac{st}{st+t||x||+s||y||+||x||||y||}\ = \frac{s^2||y||+(s+t)||x||||y||+t^2||x||}{A}\ \geq 0.$$

where $A = (s + t + ||x|| + ||y||)(st + t||x|| + s||y|| + ||x||||y||)$.
So $N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$.
Next we verify $M(x + u\ ,\ s + t) \leq\ M(x\ ,\ s) \diamondsuit M(u\ ,\ t)$.
We only consider the case when $s > ||x||$ and $t > ||y||$ (since in other cases (IFN9)

is obvious).
We have, $M(x\ ,\ s)\Diamond M(y\ ,\ t) - M(x+y\ ,\ s+t)\ = \frac{||x||}{||x||+s} + \frac{||y||}{||y||+s} - \frac{||x+y||}{||x+y||+s+t}$
$= [\{||x||(t+||y||) + ||y||(s+||x||)\}\{||x+y|| + s + t\} - ||x+y||(s+||x||)(t+||y||)]/A$
where $A = (||x|| + s)(t + ||y||)\{||x+y|| + s + t\}$.
$= [(s+t)||x||(t+||y||) + (s+t)||y||(s+||x||) + ||x||||x+y||(t+||y||) + ||x+y||||y||(s+||x||) - ||x+y||(s+||x||)(t+||y||)]/A$
$= [(s+t)||x||(t+||y||) + (s+t)||y||(s+||x||) + ||x||||x+y||(t+||y||) + ||x+y||(s+||x||)\{||y|| - t - ||y||\}]/A.$
$= [(s+t)||x||(t+||y||) + (s+t)||y||(s+||x||) + ||x+y||(t||x|| + ||x||||y|| - st - t||x||)]/A.$
$= [(s+t)||x||(t+||y||) + (s+t)||y||(s+||x||) + ||x||||y||||x+y|| - st||x+y||]/A.$
$\geq [(s+t)||x||(t+||y||) + (s+t)||y||(s+||x||) + ||x||||y||||x+y|| - st||x|| - st||y||]/A.$
i.e. $M(x\ ,\ s)\Diamond M(y\ ,\ t) - M(x+y\ ,\ s+t)\ \geq 0.$
i.e. $M(x\ ,\ s)\Diamond M(y\ ,\ t) \geq M(x+y\ ,\ s+t).$ □

Note 3.4. *In the context of modified Definition 3.1, we consider the same Definition 2.6, Definition 2.9, Definition 2.10, Definition 2.11, Definition 2.12 and it is easy to verify that the Theorem 2.7 and Theorem 2.8 are valid in respect of modified definition.*

4. Finite dimensional intuitionistic fuzzy normed linear spaces

In this section we study completeness and compactness properties of finite dimensional intuitionistic fuzzy normed linear spaces. Firstly consider the following Lemma which plays the key role in studying properties of finite dimensional intuitionistic fuzzy normed linear spaces.

Lemma 4.1. *Let $(V\ ,\ A)$ be an intuitionistic fuzzy normed linear space with the underlying t-norm $*$ continuous at (1 , 1) and the underlying t-conorm $\Diamond$ continuous at (0 , 0) and $\{x_1,\ x_2,\ ....,x_n\}$ be a linearly independent set of vectors in V. Then $\exists c_1,\ c_2 > 0$ and $\exists \delta_1,\ \delta_2 \in (0\ ,\ 1)$ such that for any set of scalars $\{\alpha_1,\ \alpha_2, ..., \alpha_n\}$;*

$$N(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,\ c_1 \sum_{j=1}^{n} |\alpha_j|)\ <\ 1 - \delta_1. \tag{4.1.1a}$$

$$M(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,\ c_2 \sum_{j=1}^{n} |\alpha_j|)\ >\ \delta_2. \tag{4.1.1b}$$

Proof. Let $s = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$.
If $s = 0$ then $\alpha_j = 0\ \forall j = 1, 2, ..., n$ and the relation (4.1.1a) holds for any $c > 0$ and $\delta \in (0\ ,\ 1)$.
Next we suppose that $s > 0$. Then (4.1.1a) is equivalent to

$$N(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n,\ c_1)\ <\ 1 - \delta_1 \tag{4.1.2a}$$

for some $c_1 > 0$ and $\delta_1 \in (0\ ,\ 1)$, and for all scalars β's with $\sum_{j=1}^{n} |\beta_j| = 1$.
If possible suppose that (4.1.2a) does not hold. Thus for each $c > 0$ and $\delta \in (0\ ,\ 1)$, $\exists$ a set of scalars $\{\beta_1,\ \beta_2, ..., \beta_n\}$ with $\sum_{j=1}^{n} |\beta_j| = 1$ for which
$N(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n,\ c)\ \geq 1 - \delta.$

Then for $c = \delta = \frac{1}{m},\ m = 1, 2, ...,\ \exists$ a set of scalars $\{\beta_1^{(m)},\ \beta_2^{(m)}, ..., \beta_n^{(m)}\}$ with $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$ such that $N(y_m\ ,\ \frac{1}{m})\ \geq 1 - \frac{1}{m}$ where $y_m =\ \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n$.

Since $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$, we have $0 \leq |\beta_j^{(m)}|\ \leq 1$ for $j = 1, 2, ..., n$.

So for each fixed j the sequence $\{\beta_j^{(m)}\}$ is bounded and hence $\{\beta_1^{(m)}\}$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ where $y_{n,m} =\ \sum_{j=1}^{n} \gamma_j^{(m)} x_j$ with $\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$ and $\gamma_j^{(m)} \to \beta_j$ as $m \to \infty$.

Let $y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n$.

Now we show that $\lim_{m\to\infty} N(y_{n,m} - y\ ,\ t) = 1\ \forall t > 0$. We have

$$N(y_{n,m} - y\ ,\ t) =\ N(\sum_{i=1}^{n} (\gamma_j^{(m)} - \beta_j\} x_j\ ,\ t)$$
$$\geq N(x_1, \tfrac{t}{n|\gamma_1^{(m)} - \beta_1|}) * .... * N(x_n, \tfrac{t}{n|\gamma_n^{(m)} - \beta_n|}).$$

So,

$\lim_{m\to\infty} N(y_{n,m} - y\ ,\ t)\ \geq \lim_{m\to\infty} N(x_1, \dfrac{t}{n|\gamma_1^{(m)} - \beta_1|}) * .... * \lim_{m\to\infty} N(x_n, \dfrac{t}{n|\gamma_n^{(m)} - \beta_n|})$.

$\Rightarrow \lim_{m\to\infty} N(y_{n,m} - y\ ,\ t)\ \geq 1 * .... * 1$ (by the continuity of t-norm $*$ at (1 , 1)).

$\Rightarrow \lim_{m\to\infty} N(y_{n,m} - y\ ,\ t) = 1\ \forall t > 0$ (4.1.3a).

Now for $k > 0$, choose m such that $\frac{1}{m}\ < k$.

We have $N(y_{n,m}\ ,\ k) = N(y_{n,m} + \underline{0}\ ,\ \frac{1}{m} + k - \frac{1}{m}) \geq N(y_{n,m}\ ,\ \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m})$ $\geq (1 - \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m})$.

i.e. $N(y_{n,m}\ ,\ k) \geq (1 - \frac{1}{m}) * N(\underline{0}\ ,\ k - \frac{1}{m}) = (1 - \frac{1}{m}) * 1 = 1 - \frac{1}{m})$.

i.e. $\lim_{m\to\infty} N(y_{n,m}\ ,\ k) \geq 1$.

i.e. $\lim_{m\to\infty} N(y_{n,m}\ ,\ k)\ = 1$. (4.1.4$a$)

Now $N(y\ ,\ 2k) = N(y - y_{n,m} + y_{n,m}\ ,\ k + k)\ \geq N(y - y_{n,m}\ ,\ k) * N(y_{n,m}\ ,\ k)$

$\Rightarrow N(y\ ,\ 2k) \geq \lim_{m\to\infty} N(y - y_{n,m}\ ,\ k) * \lim_{m\to\infty} N(y_{n,m}\ ,\ k)$ (by the continuity of t-norm $*$ at (1 , 1)).

$\Rightarrow N(y\ ,\ 2k) \geq 1 * 1$ by (4.1.3a)&(4.1.4a)

$\Rightarrow N(y\ ,\ 2k)\ = 1 * 1 = 1$.

Since $k > 0$ is arbitrary, by (IFN2) it follows that $y = \underline{0}$.

Again since $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$ and $\{x_1,\ x_2,\ ...., x_n\}$ are linearly independent set of vectors, so $y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n \neq \underline{0}$. Thus we arrive at a contradiction.

Now we prove the relation (4.1.1b).

If $s = 0$ then $\alpha_j = 0\ \forall j = 1, 2, ..., n$ and the relation (4.1.1b) holds for any $c > 0$

and $\delta \in (0\ ,\ 1)$.
Next we suppose that $s > 0$. Then (4.1.1b) is equivalent to
$M(\beta_1 x_1+\beta_2 x_2+.......+\beta_n x_n,\ c_2)\ >\ \delta_2$ $\quad (4.1.2b)$
for some $c_2 > 0$ and $\delta_2 \in (0\ ,\ 1)$, and for all scalars β_j's with $\sum_{j=1}^{n} |\beta_j| = 1$.
If possible suppose that (4.1.2b) does not hold. Thus for each $c > 0$ and $\delta \in (0\ ,\ 1)$, $\exists$ a set of scalars $\{\beta_1,\ \beta_2, ..., \beta_n\}$ with $\sum_{j=1}^{n} |\beta_j| = 1$ for which
$M(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n,\ c)\ \leq \delta$.
Then for $c = \delta = \frac{1}{m},\ m = 1, 2, ...,\ \exists$ a set of scalars $\{\gamma_1^{(m)},\ \gamma_2^{(m)}, ..., \gamma_n^{(m)}\}$ with $\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$ such that $M(z_m\ ,\ \frac{1}{m})\ \leq \frac{1}{m}$ where $z_m =\ \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + ... + \gamma_n^{(m)} x_n$.
Since $\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$, we have $0 \leq |\gamma_j^{(m)}|\ \leq 1$ for $j = 1, 2, ..., n$.
Then by same argument as above, we obtain a subsequence $\{z_{n,m}\}$ where
$z_{n,m} = \sum_{j=1}^{n} \eta_j^{(m)} x_j$ with $\sum_{j=1}^{n} |\eta_j^{(m)}| = 1$ and $\eta_j^{(m)} \to \eta_j$ as $m \to \infty$.
Thus $\sum_{j=1}^{n} |\eta_j| = 1$
Let $z = \eta_1 x_1 + \eta_2 x_2 + ... + \eta_n x_n$.
Then we have $\lim_{m\to\infty} M(z_{n,m} - z\ ,\ t) = 0\ \forall t > 0.$ $\quad (4.1.3b)$
Now for $k > 0$, choose m such that $\frac{1}{m}\ < k$.
We have $M(z_{n,m}\ ,\ k) = M(z_{n,m} + \underline{0}\ ,\ \frac{1}{m} + k - \frac{1}{m}) \leq M(z_{n,m}\ ,\ \frac{1}{m}) * M(\underline{0}\ ,\ k - \frac{1}{m})$
$\leq \frac{1}{m} * M(\underline{0}\ ,\ k - \frac{1}{m})$.
i.e. $M(z_{n,m}\ ,\ k) \leq \frac{1}{m} \diamondsuit M(\underline{0}\ ,\ k - \frac{1}{m}) = \frac{1}{m} \diamondsuit 0 = \frac{1}{m}$.
i.e. $\lim_{m\to\infty} M(z_{n,m}\ ,\ k) \leq 0$.
i.e. $\lim_{m\to\infty} M(z_{n,m}\ ,\ k)\ = 0.$ $\quad (4.1.4b)$
Now $M(z\ ,\ 2k) = M(z - z_{n,m} + z_{n,m}\ ,\ k + k)\ \leq M(z - z_{n,m}\ ,\ k) \diamondsuit M(z_{n,m}\ ,\ k)$
$\Rightarrow M(z\ ,\ 2k) \leq \lim_{m\to\infty} M(z - z_{n,m}\ ,\ k) \diamondsuit \lim_{m\to\infty} M(z_{n,m}\ ,\ k)$ (by the continuity of t-conorm $\diamondsuit$ at $(0\ ,\ 0)$).
$\Rightarrow M(z\ ,\ 2k) \leq 0 \diamondsuit 0$ by $(4.1.3b)$&$(4.1.4b)$
$\Rightarrow M(z\ ,\ 2k)\ = 0 \diamondsuit 0 = 0$.
Since $k > 0$ is arbitrary, by (IFN7) it follows that $z = \underline{0}$.
Again since $\sum_{j=1}^{n} |\eta_j^{(m)}| = 1$ and $\{x_1,\ x_2,\ ...., x_n\}$ are linearly independent set of vectors, so $z = \eta_1 x_1 + \eta_2 x_2 + ... + \eta_n x_n \neq \underline{0}$. Thus we arrive at a contradiction.
This completes the lemma. □

Theorem 4.2. *Every finite dimensional intuitionistic fuzzy normed linear space $(V\ ,\ A)$ with the continuity of the underlying t-norm $*$ at (1 , 1) and t-co-norm $\diamondsuit$ at (0 , 0) is complete.*

Proof. Let (V, A) be an intuitionistic fuzzy normed linear space and dimV =k (say). Let $\{e_1,\ e_2,\ ....,e_k\}$ be a basis for V and $\{x_n\}$ be a Cauchy sequence in V.
Let $x_n = \beta_1^{(n)}e_1 + \beta_2^{(n)}e_2 + ... + \beta_k^{(n)}e_k$ where $\beta_1^{(n)},\ \beta_2^{(n)}, ..., \beta_k^{(n)}$ are suitable scalars.
So $\lim\limits_{m,n\to\infty} N(x_m - x_n\ ,\ t) = 1 \forall t > 0$ $\quad(4.2.1a)$
and $\lim\limits_{m,n\to\infty} M(x_m - x_n\ ,\ t) = 0\ \forall t > 0$ $\quad(4.2.1b)$
Now from Lemma 4.1, it follows that $\exists c_1,\ c_2 > 0$ and $\delta_1, \delta_2 \in (0, 1)$ such that

$$N(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ c_1\sum_{i=1}^{k}|\beta_i^{(m)} - \beta_i^{(n)}|)\ < 1 - \delta_1. \quad (4.2.2a)$$

$$M(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ c_2\sum_{i=1}^{k}|\beta_i^{(m)} - \beta_i^{(n)}|)\ > \delta_2 \quad (4.2.2b)$$

Again for $1 > \delta_1 > 0$, from (4.2.1a), it follows that $\exists$ a positive integer $n_0(\delta_1, t)$ such that,

$$N(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ , t)\ > 1 - \delta_1\ \forall m, n \geq n_0(\delta_1, t). \quad (4.2.3a)$$

Now from (4.2.2a) and (4.2.3a), we have,

$$N(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ t)\ > 1 - \delta_1\ > N(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ c_1\sum_{i=1}^{k}|\beta_i^{(m)} - \beta_i^{(n)}|)\ \forall m, n \geq n_0(\delta_1, t)$$

$\Rightarrow c_1\sum\limits_{i=1}^{k}|\beta_i^{(m)} - \beta_i^{(n)}|\ <\ t\ \forall m, n \geq n_0(\delta_1, t)$ (since $N(x, .)$ is nondecreasing in t).

$\Rightarrow \sum\limits_{i=1}^{k}|\beta_i^{(m)} - \beta_i^{(n)}|\ <\ \dfrac{t}{c_1}\ \ \forall m, n \geq n_0(\delta_1, t)$

$\Rightarrow |\beta_i^{(m)} - \beta_i^{(n)}|\ <\ \frac{t}{c_1}\ \ \forall m, n \geq n_0(\delta_1, t)$ and $i = 1, 2, ..., k$.
Since $t > 0$ is arbitrary, from above we have,
$\lim\limits_{m,n\to\infty}|\beta_i^{(m)} - \beta_i^{(n)}| = 0$ for $i = 1, 2, ..., k$.
$\Rightarrow \{\beta_i^{(n)}\}$ is a Cauchy sequence of scalars for each $i = 1, 2, ..., k$.
So each sequence $\{\beta_i^{(n)}\}$ converges.

Let $\lim\limits_{n\to\infty}\beta_i^{(n)} = \beta_i$ for $i = 1, 2, ..., k$. and $x = \sum\limits_{i=1}^{k}\beta_i e_i$. Clearly $x \in V$
Now $\forall t > 0$,

$$N(x_n - x\ ,\ t) = N(\sum_{i=1}^{k}\beta_i^{(n)}e_i - \sum_{i=1}^{k}\beta_i e_i\ ,\ t) = N(\sum_{i=1}^{k}(\beta_i^{(n)} - \beta_i)e_i\ ,\ t).$$

i.e. $N(x_n - x\ ,\ t) \geq N(e_1\ ,\ \frac{t}{k|\beta_1^{(n)} - \beta_1|}) * N(e_2\ ,\ \frac{t}{k|\beta_2^{(n)} - \beta_2|}) * ....$
$.... * N(e_k\ ,\ \frac{t}{k|\beta_k^{(n)} - \beta_k|}).$ $\quad(4.2.4a)$

When $n \to \infty$ then $\frac{t}{k|\beta_i^{(n)} - \beta_i|} \to \infty$ (since $\beta_i^{(n)} \to \beta_i$) for $i = 1, 2, ..., k$ and $\forall t > 0$.
From (4.2.4a) we get, using the continuity of t-norm $*$ at (1 , 1),
$\lim_{n\to\infty} N(x_n - x\ ,\ t) \geq 1 * 1 * .... * 1\ \forall t > 0$
$\Rightarrow \lim_{n\to\infty} N(x_n - x\ ,\ t)\ = 1\ \forall t > 0.$ $\quad (4.2.5a)$
Now from (4.2.2b) and (4.2.3b), we have,
$M(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ t)\ < \delta_2\ <\ M(\sum_{i=1}^{k}(\beta_i^{(m)} - \beta_i^{(n)})e_i\ ,\ c_2 \sum_{i=1}^{k} |\beta_i^{(m)} - \beta_i^{(n)}|)\ \forall m, n \geq m_0(\delta, t)$
$\Rightarrow c_2 \sum_{i=1}^{k} |\beta_i^{(m)} - \beta_i^{(n)}|\ <\ t\ \forall m, n \geq n_0(\delta_2, t)$ (since $M(x, .)$ is non-increasing in t)
$\Rightarrow \sum_{i=1}^{k} |\beta_i^{(m)} - \beta_i^{(n)}|\ <\ \frac{t}{c_2}\ \ \forall m, n \geq n_0(\delta_2, t)$
$\Rightarrow |\beta_i^{(m)} - \beta_i^{(n)}|\ <\ \frac{t}{c_2}\ \ \forall m, n \geq n_0(\delta_2, t)$ and $i = 1, 2, ..., k$.
Since $t > 0$ is arbitrary, from above we have,
$\lim_{m,n\to\infty} |\beta_i^{(m)} - \beta_i^{(n)}| = 0$ for $i = 1, 2, ..., k$.
$\Rightarrow \{\beta_i^{(n)}\}$ is a Cauchy sequence of scalars for each $i = 1, 2, ..., k$.
So each sequence $\{\beta_i^{(n)}\}$ converges.
Let $\lim_{n\to\infty} \beta_i^{(n)} = \beta_i$ for $i = 1, 2, ..., k$. and $x = \sum_{i=1}^{k} \beta_i e_i$. Clearly $x \in V$.
Now $\forall t > 0$,
$M(x_n - x\ ,\ t) = M(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i e_i\ ,\ t) = M(\sum_{i=1}^{k}(\beta_i^{(n)} - \beta_i)e_i\ ,\ t).$
i.e. $M(x_n - x\ ,\ t) \leq M(e_1\ ,\ \frac{t}{k|\beta_1^{(n)} - \beta_1|}) \diamondsuit M(e_2\ ,\ \frac{t}{k|\beta_2^{(n)} - \beta_2|}) \diamondsuit ....$
$.... \diamondsuit M(e_k\ ,\ \frac{t}{k|\beta_k^{(n)} - \beta_k|}).$ $\quad (4.2.4b)$
When $n \to \infty$ then $\frac{t}{k|\beta_i^{(n)} - \beta_i|} \to \infty$ (since $\beta_i^{(n)} \to \beta_i$) for $i = 1, 2, ..., k$ and $\forall t > 0$.
From (4.2.4b) we get, using the continuity of t-co norm $\diamondsuit$ at (0 , 0),
$\lim_{n\to\infty} M(x_n - x\ ,\ t) \leq 0 * 0 * .... * 0\ \forall t > 0$
$\Rightarrow \lim_{n\to\infty} M(x_n - x\ ,\ t)\ = 0\ \forall t > 0.$ $\quad (4.2.5b)$
From (4.2.5a) and (4.2.5b), we have $x_n \to x$ as $n \to \infty$. Hence $(V\ ,\ A)$ is complete. □

Definition 4.3. Let $(V\ ,\ A)$ be an intuitionistic fuzzy normed linear space and $F \subset V$. F is said to be bounded if for each r, $0 < r < 1$, $\exists t_1,\ t_2 > 0$ such that $N(x\ ,\ t_1)\ > 1 - r$ and $M(x\ ,\ t_2)\ < r\ \ \forall x \in\ F$.

Theorem 4.4. *In a finite dimensional intuitionistic fuzzy normed linear space $(V\ ,\ A)$ in which the underlying t-norm $*$ is continuous at (1 , 1) and t-co-norm $\diamondsuit$ is continuous at (0 , 0), a subset F is compact iff it is closed and bounded.*

Proof. *First we suppose that F is compact. We have to show that F is closed and bounded.*

Let $x \in \bar{F}$. Then $\exists$ a sequence $\{x_n\}$ in F such that $\lim_{n\to\infty} x_n = x$.
Since F is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a point in F.
Again $\{x_n\} \to x$ so $\{x_{n_k}\} \to x$ and hence $x \in F$. So F is closed.
If possible suppose that F is not bounded. Then $\exists r_0$ with $0 < r_0 < 1$ such that for each positive integer n, $\exists x_n \in F$ such that $N(x_n \ , \ n) \ \leq 1-r_0$ or $M(x_n \ , \ n) \ \geq r_0$. So there exists a subsequence of $\{x_n\}$ (without loss of generality we assume $\{x_n\}$ to be that subsequence) for which at least one of the relations

$$N(x_{n_k} \ , \ n_k) \leq 1-r_0 \ \ \forall n \in N \tag{4.4.1a}$$

$$M(x_{n_k} \ , \ n_k) \geq r_0 \ \ \forall n \in N \tag{4.4.1b}$$

holds.
First we assume that (4.4.1a) holds.
Now for $t > 0$,

$1 - r_0 \geq N(x_{n_k} \ , \ n_k) = N(x_{n_k} - x + x \ , \ n_k - t + t)$ *where* $t > 0$

$\Rightarrow 1 - r_0 \geq N(x_{n_k} - x \ , \ t) * N(x \ , \ n_k - t)$

$\Rightarrow 1 - r_0 \geq \lim_{k\to\infty} N(x_{n_k} - x \ , \ t) * \lim_{k\to\infty} N(x \ , \ n_k - t)$

$\Rightarrow 1 - r_0 \geq 1 * 1 = 1$ *(using the continuity of t-norm at (1 , 1))*

$\Rightarrow r_0 \leq 0$ *which is a contradiction.*

In case (4.4.1b) holds, by considering the function $M(x \ , \ t)$, proceeding as above, we arrive at a contradiction. Hence F is bounded.
Conversely suppose that F is closed and bounded and we have to show that F is compact.
Let dim V=n and $\{e_1, \ , e_2, ..., e_n\}$ be a basis for V.
Choose a sequence $\{x_k\}$ in F and suppose $x_k = \ \beta_1^{(k)} e_1 + \beta_2^{(k)} e_2 + ... + \beta_n^{(k)} e_n$ where $\beta_1^{(k)}, \ \beta_2^{(k)}, ..., \beta_n^{(k)}$ are scalars.
Now from Lemma 4.1, $\exists c_1, \ c_2 > 0$ and $\exists \delta_1, \ \delta_2 \in (0,1)$ such that

$$N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ c_1 \sum_{i=1}^{n} |\beta_i^{(k)}|) \ < \ 1-\delta_1 \tag{4.4.2a}$$

and

$$M(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ c_2 \sum_{i=1}^{n} |\beta_i^{(k)}|) \ > \ \delta_2 \tag{4.4.2b}$$

Again since F is bounded, for $\delta_1 \in (0 \ , \ 1)$, $\exists t_1 > 0$ such that $N(x \ , \ t_1) \ > \ 1 - \delta_1$ and $\exists t_2 > 0$ such that$M(x \ , \ t_2) \ < \delta_1 \ \ \forall x \in F$.

$$\text{So } N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ t_1) \ > \ 1-\delta_1 \tag{4.4.3a}$$

and

$$M(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ t_2) \ < \ \delta_1 \tag{4.4.3b}$$

From (4.4.2a) and (4.4.3a) we get,

$$N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ c_1 \sum_{i=1}^{n} |\beta_i^{(k)}|) \ < \ 1 - \delta_1 \ < N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ t_1)$$

$$\Rightarrow N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ c_1 \sum_{i=1}^{n} |\beta_i^{(k)}|) \ < \ N(\sum_{i=1}^{n} \beta_i^{(k)} e_i \ , \ t_1)$$

$\Rightarrow c_1 \sum_{i=1}^{n} |\beta_i^{(k)}| \ < \ t_1$ *(since* $N(x \ , \ .)$ *is non-decreasing)*
$\Rightarrow |\beta_i^{(k)}| \ \leq \frac{t_1}{c_1}$ *for* $k = 1, 2, ...$ *and* $i = 1, 2, ..., n.$
So each sequence $\{\beta_i^{(k)}\}$ $(i \ = \ 1, 2, ..., n)$ *is bounded. By repeated applications of Bolzano-Weierstrass theorem, it follows that each of the sequences* $\{\beta_i^{(k)}\}$ *has a convergent subsequence say* $\{\beta_i^{k_l}\}, \ \forall i = 1, 2, ..., n.$
Let $x_{k_l} \ = \ \beta_1^{(k_l)} e_1 + \beta_2^{(k_l)} e_2 + ... + \beta_n^{(k_l)} e_n$ *and* $\{\beta_1^{(k_l)}\}, \ \{\beta_2^{(k_l)}\}, ..., \{\beta_n^{(k_l)}\}$ *are all convergent.*
Let $\beta_i = \lim_{l \to \infty} \beta_i^{(k_l)}, \ i = 1, 2, ..., n.$ *and* $x \ = \ \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n.$
Now for $t > 0$ *we have,*

$$N(x_{k_l} - x \ , \ t) \ = N(\textstyle\sum_{i=1}^{n} (\beta_i^{k_l} - \beta_i) e_i \ , \ t) \\ \geq N(e_1 \ , \ \frac{t}{n|\beta_1^{k_l} - \beta_1|}) * .... * N(e_n \ , \ \frac{t}{n|\beta_n^{k_l} - \beta_n|})$$

$\Rightarrow \lim_{l \to \infty} N(x_{k_l} - x \ , \ t) \ \geq 1 * 1 * .... * 1$ $(\beta_i^{k_l} \to \beta_i$ *as* $l \to \infty)$ *(using the continuity of t-norm* $*$ *at (1 , 1))*

$$\Rightarrow \lim_{l \to \infty} N(x_{k_l} - x \ , \ t) \ = 1. \tag{4.4.4a}$$

Now for $t > 0$ *we get,*

$$M(x_{k_l} - x \ , \ t) \ = M(\textstyle\sum_{i=1}^{n} (\beta_i^{k_l} - \beta_i) e_i \ , \ t) \\ \leq M(e_1 \ , \ \frac{t}{n|\beta_1^{k_l} - \beta_1|}) \diamondsuit .... \diamondsuit M(e_n \ , \ \frac{t}{n|\beta_n^{k_l} - \beta_n|})$$

$\Rightarrow \lim_{l \to \infty} M(x_{k_l} - x \ , \ t) \ \leq 0 \diamondsuit 0 \diamondsuit .... \diamondsuit 0$ $(\beta_i^{k_l} \to \beta_i$ *as* $l \to \infty)$ *(since* $\beta_i^{k_l} \to \beta_i$ *as* $l \to \infty$ *and using the continuity of t-co-norm* $\diamondsuit$ *at (0 , 0))*

$$\Rightarrow \lim_{l \to \infty} M(x_{k_l} - x \ , \ t) \ = 0. \tag{4.4.4b}$$

From (4.4.4a) and (4.4.4b), it follows that $x_{k_l} \to x$*. Thus* $x \in F$ *(since* F *is closed). Hence* F *is compact. This completes the proof.*

5. Conclusion

In this paper, the definition of intuitionistic fuzzy normed linear space (IFNLS) introduced by T.K.Samanta et al. is generalized. In the new definition of IFNLS, the underlying t-norm and t-conorm are considered in general setting in the sense that only continuity of t-norm and t-conorm at (1 , 1) and (0 , 0) respectively are used. We are able to establish some basic theorems in finite dimensional IFNLS in this setting and our approach is fundamentally different because we have not used the decomposition theorem of intuitionistic fuzzy norm whose validity require a stringent restriction that t-norm is *min* and t-conorm is *max*. Also we have only use the implicit intuitionistic properties among $N(x,t)$ and $M(x,t)$ functions and skip the explicit restriction $N(x,t) + M(x,t) \leq 1$ for which in several definitions involving convergence, boundeness both the functions $N(x,t)$ and $M(x,t)$ have equal role. We think that there is a scope of further work in this setting.

Acknowledgements. The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also thankful to the Editor-in-Chief of the journal (AFMI) for valuable comments which help us to revise the paper. The present work is partially supported by Special Assistance

Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009 (SAP-I)].

References

[1] K. Atanasov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11(3) (2003) 687–705.
[3] T. Bag and S. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems 151 (2005) 513–547.
[4] T. Bag and S. K. Samanta, Fixed point theorems on fuzzy normed linear spaces, Inform. Sci. 176 (2006) 2910–2931.
[5] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, Ann. Fuzzy Math. Inform. 6(2) (2013) 271–283.
[6] S. C. Cheng and J. N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. 86(5) (1994) 429–436.
[7] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets and Systems 48 (1992) 239–248.
[8] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215–229.
[9] A. K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems 12 (1984) 143–154.
[10] George. J. Klir and Bo Yuan, Fuzzy Sets and Fuzzy Logic, Printice-Hall of India Private Limited, New Delhi-110001 (1997).
[11] A. K. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika (Prague) 11(5) (1975) 336–344.
[12] Tapas Kumar Mondal and S. K. Samanta, Topology of interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 119 (2001) 483–494.
[13] Tapas Kumar Mondal and S. K. Samanta, On intuitionistic gradation of openness, Fuzzy Sets and Systems 131 (2002) 323–336.
[14] J. H. Park, Intuitionistic fuzzy mdetric spaces, Chaos Solitons Fractals 22 (2004) 1039–1046.
[15] R. Saadati and J. H. Park, On the intuitionistic fuzzy topoligical spaces , Chaos Solitons Fractals 27 (2006) 331–344.
[16] T. K. Samanta and Iqbal H. Jebril, Finite dimensional intuitionistic fuzzy normed linear space, Int. J. Open Probl. Comput. Sci. Math. 2(4) (2009) 574–591.
[17] N. Thillaigovindan, S. Anita Shanti and Y. B. Jun, On lacunary statistical convergence in intuitionistic n-normed linear spaces, Ann. Fuzzy Math. Inform. 1(2) (2011) 119–131.
[18] S. Vijayabalaji, N. Thillaigovindan and Y. B. Jun, Intuitionistic fuzzy n-normed linear space, Bull. Korean Math. Soc. 44 (2007) 291–308.
[19] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

T. Bag (tarapadavb@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India

Syamal Kumar Samanta (syamal123@yahoo.co.in)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India