Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 2, (August 2014), pp. 231–244
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

New analytical method for solving n-th order fuzzy differential equations

Smita Tapaswini, S. Chakraverty

Received 21 July 2013; Revised 24 December 2013; Accepted 21 January 2014

Abstract. This paper proposes a new method based on fuzzy centre for solving n-th order fuzzy linear differential equations. First the fuzzy differential equation is solved in term of fuzzy centre and then this solution is used to find the final solution of the original differential equation. The method is illustrated by considering three cases with examples and one application problem viz. circuit problem. We have compared the obtained results with the exact solutions in order to demonstrate the validity and applicability of the proposed method.

2010 AMS Classification: 34A07

Keywords: Fuzzy number, Fuzzy centre, Triangular fuzzy number, n-th order fuzzy differential equations.

Corresponding Author: S. Chakraverty (sne_chak@yahoo.com)

1. Introduction

In recent years, the topic of Fuzzy Differential Equations (FDEs) play an important role for modelling physical and engineering problems because those represent a natural way to model the systems under uncertainty. Since, it is too difficult to obtain the exact solutions of fuzzy differential equations, so one may need a reliable and efficient numerical technique to obtain the solutions.

There exist a good number of papers dealing with fuzzy differential equations and their applications in the open literature. Some of are reviewed and cited here for better understanding of the present analysis. Chang and Zadeh [11] first introduced the concept of a fuzzy derivative, followed by Dubois and Prade [12] who defined and used the extension principle in their approach. The fuzzy differential equations and fuzzy initial value problems are studied by Kaleva [23, 24] and Seikkala [45]. Mondal and Roy [33] described the solution procedure for a first order linear non homogeneous ordinary differential equation in fuzzy environment. Existence and uniqueness of fuzzy boundary value has been proved by Esfahani et al. [5].

Various numerical methods for solving fuzzy differential equations are introduced in [1, 3, 7, 8, 10, 13, 14, 16, 18, 19, 21, 25, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 40, 42, 48, 50, 51, 52, 53]. Ma et al. [31] developed a scheme based on the classical Euler method to solve fuzzy ordinary differential equations. A two-dimensional differential transform method for Fuzzy Partial Differential Equations (FPDEs) has been studied by Mikaeilvand and Khakrangin [32]. Recently, Tapaswini and Chakraverty [48, 50] proposed an improved Euler and homotopy perturbation method for the solution of fuzzy differential equations. Jayakumar et al. [21] developed Runge-Kutta method of order five for solving fuzzy differential equations and Palligkinis et al. [38] applied the Runge-Kutta method for more general problems and proved the convergence for s-stage Runge-Kutta methods. Extended Runge-Kutta-like formulae of order 4 has been applied by Ghazanfari et al. [16] for the numerical solutions of fuzzy differential equations. Generalized differentiability concept is used by Bede et al. [7] to investigate first order linear fuzzy differential equations. Abbasbandy et al. [1] developed a numerical method for solving fuzzy differential inclusions and in their proposed method, fuzzy reachable set is used to approximate the solution. Khastan et al. [27] used Nystrom method to solve fuzzy differential equations. Fard and Ghal-Eh [47] proposed an iterative method to get the approximate solution for the linear system of first-order fuzzy differential equations with fuzzy constant coefficients. Variation of constant formula has been handled by Khastan et al. [28] to solve first order fuzzy differential equations. Akin et al. [3] developed an algorithm based on $\alpha-$cut of a fuzzy set for the solution of second order fuzzy initial value problems. A new approach has been developed by Gasilov et al. [14] to get the solution of fuzzy initial value problem. Khastan and Nieto [29] investigated numerical algorithms for the solution of first-order fuzzy differential equations and hybrid fuzzy differential equations. Nieto et al. [37] obtained some interesting properties of the diameter and midpoint of the solution of linear first-order fuzzy differential. Numerical solution of second-order fuzzy differential equation is investigated by Rabiei et al. [42] using improved Runge-Kutta nystrom method. Pederson and Sambandham [40] used characterization theorem to obtain the numerical solution of hybrid fuzzy differential equations.

The concept of generalized H-differentiability is studied by Chalco-Cano and Roman Flores [10] to solve fuzzy differential equations. Lupulescu [30] developed a new concept of inner product on the fuzzy space for the solution of fuzzy initial value problems. Prakash and Kalaiselvi [41] implemented hybrid methods to obtain the numerical solution of fuzzy differential equations. Very Recently, Mosleh [34], Mosleh et al. [35] and Effati et al. [13] applied fuzzy neural network for the solution of fuzzy differential equations. Recently, Behera and Chakraverty [8] obtained uncertain impulse response of imprecisely defined half order mechanical system. Also homotopy perturbation method has been used by Tapaswini and Chakraverty [51] to obtain the solution of arbitrary order predator-prey equations and fuzzy quadratic Riccati differential equation [53]. A new double parametric form of fuzzy number has been developed by Tapaswini and Chakraverty [52] and using homotopy perturbation method, numerical solution of uncertain beam equations has been obtained. Hashemi et al. [18, 19] studied homotopy analysis method for the solution of system of fuzzy differential equations and obtained analytical solution of fuzzy wave-like equations with variable coefficients. Kanagarajan and Muthukumar [25] implemented extended Runge-Kutta method of order four for solving hybrid fuzzy differential equations. Variation iteration method is implemented by Narayanamoorthy and Murugan [36] for the solution of Fuzzy heat-like equations.

As regards, methods to solve $n-$th order fuzzy differential equations are discussed in [4, 15, 20, 26, 39, 44, 46, 49, 54]. However the Variational Iteration Method (VIM) was successfully applied by Jafari et al. [20] for solving $n-$th order fuzzy differential equations. A new result on multiple solutions for $n-$th order fuzzy differential equations under generalized differentiability has been proposed by Khastan et al. [26]. The existence and uniqueness of $n-$ th order fuzzy differential equations is proved by Georgiou et al.[15]. Based on the idea of collocation method , Allahviranloo et al.[4] solved $n-$th order fuzzy linear differential equations. Yue and Guangyuan [54] utilized time domain methods for the solutions of $n-$th order fuzzy differential equations. Parandin [39] discussed Runge-Kutta method for the numerical solution of fuzzy differential equations of $n-$th order. Integral form of $n-$th order fuzzy differential equations has been developed by Salahshour [44] under generalized differentiability. Mansouri and Ahmady [46] implemented characterization theorem for solving $n-$th order fuzzy differential equations. Also, Tapaswini and Chakraverty [49] implemented homotopy perturbation method for the solution of $n-$th order fuzzy linear differential equations.

Bede [6] described the exact solutions of fuzzy differential equations in his note in an excellent way. Ahmad et al. [2] studied analytical and numerical solutions of fuzzy differential equations based on the extension principle. Buckley and Feuring [9] applied two analytical methods for solving $n-$th order linear differential equations with fuzzy initial conditions. In the first method, they simply fuzzify the crisp solution to obtain a fuzzy function and then checked whether it satisfies the differential equation or not, and the second method was just the reverse of the first method. In the present study, we have to developed a new analytical approach using fuzzy centre to solve $n-$th order fuzzy differential equations.

This paper is organized as follows. In Section 2, we give some basic preliminaries related to the present investigation. The proposed technique has been discussed in Section 3. In Section 4, numerical examples are solved. Finally, in the last section conclusions are drawn.

2. Preliminaries

In this section, we present some notations, definitions and preliminaries which are used further in this paper [17, 22, 43, 55].

Definition 2.1. Fuzzy number

A fuzzy number $\tilde{U}$ is convex normalised fuzzy set $\tilde{U}$ of the real line R such that $\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall x \in R\}$ where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

Definition 2.2. Fuzzy Centre

Fuzzy centre of an arbitrary fuzzy number $u = [\underline{u}(r), \bar{u}(r)]$ is defined as $u^c = \frac{\underline{u}(r)+\bar{u}(r)}{2}$ for all $0 \le r \le 1$.

Definition 2.3. Triangular fuzzy number

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such that

i: there exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the memebership function of the fuzzy set.

ii: $\mu_{\tilde{U}}(x)$ is piecewise continuous.

Let us consider an arbitrary triangular fuzzy number $\tilde{U} = (a, b, c)$ as depicted in Fig. 1(i). The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ is defined as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, x \leq a \\ \frac{x-a}{b-a}, a \leq x \leq b \\ \frac{c-x}{c-b}, b \leq x \leq c \\ 0, x \geq c \end{cases}$$

Definition 2.4. Single parametric form of fuzzy numbers

A triangular fuzzy number $\tilde{U} = (a, b, c)$ can be represented by an ordered pair of functions through $r-$ cut approach viz. $[\underline{u}(r), \bar{u}(r)] = [(b-a)r + a,\ -(c-b)r + c]$ where, $r \in [0, 1]$

For all the above type of fuzzy numbers the left and right bound of the fuzzy numbers satisfy the following requirements

i: $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.

ii: $\bar{u}(r)$ is a bounded right continuous non-increasing function over $[0, 1]$.

iii: $\underline{u}(r) \leq \bar{u}(r)$,$0 \leq r \leq 1$.

Definition 2.5. Fuzzy arithmetic

For any two arbitrary fuzzy numbers $\tilde{x} = [\underline{x}(r), \bar{x}(r)]$, $\tilde{y} = [\underline{y}(r), \bar{y}(r)]$ and scalar k , the fuzzy arithmetic is defined as follows,

i: $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\bar{x}(r) = \bar{y}(r)$

ii: $\tilde{x} + \tilde{y} = [\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r)]$

iii: $\tilde{x} - \tilde{y} = [\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r)]$

iv: $\tilde{x} \times \tilde{y} = \begin{bmatrix} \min\left(\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \bar{y}(r), \bar{x}(r) \times \underline{y}(r), \bar{x}(r) \times \bar{y}(r)\right), \\ \max\left(\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \bar{y}(r), \bar{x}(r) \times \underline{y}(r), \bar{x}(r) \times \bar{y}(r)\right) \end{bmatrix}$

v: $k\tilde{x} = \left\{ \begin{array}{l} [k\overline{x}(r), k\underline{x}(r)], k < 0 \\ [k\underline{x}(r), k\overline{x}(r)], k \geq 0 \end{array} \right]$

3. Proposed method

In this section, we propose a new method based on fuzzy centre to solve the $n-$th order fuzzy differential equation. To compare the results obtained by the proposed method we have also applied the method of Bede [6] to find the exact solution.

Accordingly we consider the $n-$th order fuzzy differential equation in general form as

$$\tilde{y}^{(n)}(t;r) + a_{n-1}(t)\tilde{y}^{(n-1)}(t;r) + \cdots + a_1(t)\tilde{y}'(t;r) + a_0(t)\tilde{y}(t;r) = \tilde{g}(t;r), \tag{3.1}$$

where $a_i(t), 0 \leq i \leq n-1$. continuous on some interval subject to fuzzy initial conditions

$$\tilde{y}(0) = \tilde{b}_0, \tilde{y}'(0) = \tilde{b}_1, \ldots, \tilde{y}^{(n-1)}(0) = \tilde{b}_{n-1}.$$

where $\tilde{b}_i, 0 \le i \le n-1$ are fuzzy numbers. Here, $\tilde{y}(t;r)$ is the solution to be determined. Now three cases as below may arise,

Case 1: Coefficients $a_{n-1}(t), a_{n-2}(t), \cdots, a_1(t), a_0(t)$ are all positive.
Case 2: Coefficients $a_{n-1}(t), a_{n-2}(t), \cdots, a_1(t), a_0(t)$ are all negative.
Case 3: Coefficients $a_{n-1}(t), \cdots, a_{n-m}(t)$ are positive and
$a_{n-m-1}(t), a_{n-m-2}(t), \cdots, a_1(t), a_0(t)$ are negative.
Now we will discuss the above three cases in detail as follows,

Case 1: Coefficients $a_{n-1}(t), a_{n-2}(t), \cdots, a_1(t), a_0(t)$ are all positive.
First we will write Eq. (3.1) in terms of fuzzy centre as

$$y^{c^{(n)}}(t;r) + a_{n-1}(t)y^{c^{(n-1)}}(t;r) + \cdots + a_1(t)y^{c^{'}}(t;r) + a_0(t)y^c(t;r) = g^c(t;r), \tag{3.2}$$

with initial conditions

$$y^c(0) = b_0^c, y^{c^{'}}(0) = b_1^c, \ldots, y^{c^{(n-1)}}(0) = b_{n-1}^c.$$

Eq. (3.2) may easily be solved to obtain y^c by any standard method.
As per the single parametric form we may write the above fuzzy differential equation (3.1) and fuzzy initial condition as

$$\begin{aligned} &\left[\underline{y}^{(n)}(t;r), \bar{y}^{(n)}(t;r)\right] + a_{n-1}(t)\left[\underline{y}^{(n-1)}(t;r), \bar{y}^{(n-1)}(t;r)\right] + \cdots \\ &+a_1(t)\left[\underline{y}'(t;r), \bar{y}'(t;r)\right] + a_0(t)\left[\underline{y}(t;r), \bar{y}(t;r)\right] = \left[\underline{g}(t;r), \bar{g}(t;r)\right], \end{aligned} \tag{3.3}$$

subject to fuzzy initial conditions

$$\begin{aligned} &\left[\underline{y}(0;r), \bar{y}(0;r)\right] = \left[\underline{b}_0(r), \bar{b}_0(r)\right], \left[\underline{y}'(0;r), \bar{y}'(0;r)\right] = \left[\underline{b}_1(r), \bar{b}_1(r)\right], \ldots, \\ &\left[\underline{y}^{(n-1)}(0;r), \bar{y}^{(n-1)}(0;r)\right] = \left[\underline{b}_{n-1}(r), \bar{b}_{n-1}(r)\right], \text{where } r \in [0,1]. \end{aligned}$$

By using the definition of Hukuhara derivative one may write Eq. (3.3) as

$$\underline{y}^{(n)}(t;r) + a_{n-1}(t)\underline{y}^{(n-1)}(t;r) + \cdots + a_1(t)\underline{y}'(t;r) + a_0(t)\underline{y}(t;r) = \underline{g}(t;r), \tag{3.4}$$

and

$$\bar{y}^{(n)}(t;r) + a_{n-1}(t)\bar{y}^{(n-1)}(t;r) + \cdots + a_1(t)\bar{y}'(t;r) + a_0(t)\bar{y}(t;r) = \bar{g}(t;r). \tag{3.5}$$

Now solving Eq. (3.4) and (3.5) separately one may have $\underline{y}(t;r)$ and $\bar{y}(t;r)$ respectively. On the other hand one may solve either Eq. (3.4) or (3.5) to have the lower and upper bound solution respectively. Next, we may substitute the above values of y^c and $\underline{y}(t;r)$ or $(\bar{y}(t;r))$ with the fuzzy centre solution in the expression $\bar{y} = 2y^c - \underline{y}$ or $\underline{y} = 2y^c - \bar{y}$ to get the solution bound.

Case 2: Coefficients $a_{n-1}(t), a_{n-2}(t), \cdots, a_1(t), a_0(t)$ are all negative.
Eq. (3.1) may be written in term of fuzzy centre as

$$y^{c^{(n)}}(t;r) - a_{n-1}(t)y^{c^{(n-1)}}(t;r) - \cdots - a_1(t)y^{c^{'}}(t;r) - a_0(t)y^c(t;r) = g^c(t;r), \tag{3.6}$$

with initial conditions

$$y^c(0) = b_0^c, y^{c^{'}}(0) = b_1^c, \ldots, y^{c(n-1)}(0) = b_{n-1}^c$$

Again $y^c(t;r)$ may be obtained by solving Eq. (3.6) and by using the definition of Hukuhara derivative we have,

$$\underline{y}^{(n)}(t;r) + a_{n-1}(t)\bar{y}^{(n-1)}(t;r) + \cdots + a_1(t)\bar{y}'(t;r) + a_0(t)\bar{y}(t;r) = \underline{g}(t), \tag{3.7}$$

$$\bar{y}^{(n)}(t;r) + a_{n-1}(t)\underline{y}^{(n-1)}(t;r) + \cdots + a_1(t)\underline{y}'(t;r) + a_0(t)\underline{y}(t;r) = \vec{g}(t), \tag{3.8}$$

Using definition of fuzzy centre one may write Eq. (3.7) and Eq. (3.8) as

(3.9)

$$\begin{aligned}&\underline{y}^{(n)}(t;r) + a_{n-1}(t)\left(2y^c(t;r) - \underline{y}(t;r)\right)^{(n-1)} + \cdots + a_1(t)\left(2y^c(t;r) - \underline{y}(t;r)\right)' \\ &+ a_0(t)\left(2y^c(t;r) - \underline{y}(t;r)\right) = \underline{g}(t),\end{aligned}$$

(3.10)

$$\begin{aligned}&\bar{y}^{(n)}(t;r) + a_{n-1}(t)\left(2y^c(t;r) - \bar{y}(t;r)\right)^{(n-1)} + \cdots + a_1(t)\left(2y^c(t;r) - \bar{y}(t;r)\right)' \\ &+ a_0(t)\left(2y^c(t;r) - \bar{y}(t;r)\right) = \vec{g}(t),\end{aligned}$$

It may be seen that the above differential equations are now crisp differential equations. Solving Eqs. (3.9) and (3.10) one may get bounds of the solution as $\underline{y}(t;r)$ and $\bar{y}(t;r)$ respectively. Otherwise one may solve only Eq. (3.9) to obtain the lower $\underline{y}(t;r)$ of the solution. Next, using the expression obtained from the fuzzy centre $\bar{y}(t;r) = \left(2y^c - \underline{y}\right)$, we may have the upper bounds $\bar{y}(t;r)$ of the solution. Similarly one may also solve Eq. (3.10) to have the solution bounds accordingly.

Case 3: Coefficients $a_{n-1}(t), \cdots, a_{n-m}(t)$ are positive and $a_{n-m-1}(t), a_{n-m-2}(t), \cdots, a_1(t), a_0(t)$ are negative.
In this case we may write Eq. (3.1) in term of fuzzy centre as

$$\begin{aligned}&y^{c^{(n)}}(t;r) + a_{n-1}(t)y^{c^{(n-1)}}(t;r) + \cdots + a_{n-m}(t)y^{c^{(n-m)}}(t;r) \\ &-a_{n-m-1}(t)y^{c^{(n-m-1)}}(t;r) + \cdots - a_0(t)y^c(t;r) = g^c(t;r),\end{aligned} \tag{3.11}$$

with initial conditions

$$y^c(0) = b_0^c, y^{c'}(0) = b_1^c, \ldots, y^{c^{(n-1)}}(0) = b_{n-1}^c.$$

As in previous cases, we may solve for $y^c(t;r)$.
From Eq. (3.1) we have,

$$\begin{aligned}&\underline{y}^{(n)}(t;r) + a_{n-1}(t)\underline{y}^{(n-1)}(t;r) + \cdots + a_{n-m}(t)\underline{y}^{(n-m)}(t;r) \\ &+a_{n-m-1}(t)\bar{y}^{(n-m-1)}(t;r) + \cdots + a_0(t)\bar{y}(t;r) = \underline{g}(t;r),\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\bar{y}^{(n)}(t;r) + a_{n-1}(t)\bar{y}^{(n-1)}(t;r) + \cdots + a_{n-m}(t)\bar{y}^{(n-m)}(t;r) \\ &+a_{n-m-1}(t)\underline{y}^{(n-m-1)}(t;r) + \cdots + a_0(t)\underline{y}(t;r) = \bar{g}(t;r).\end{aligned} \tag{3.13}$$

Eqs. (3.12) and (3.13) are written as

$$\begin{aligned}&\underline{y}^{(n)}(t;r) + a_{n-1}(t)\underline{y}^{(n-1)}(t;r) + \cdots + a_{n-m}(t)\underline{y}^{(n-m)}(t;r) \\ &+a_{n-m-1}(t)\left(2y^c(t;r) - \underline{y}(t;r)\right)^{(n-m-1)} + \cdots \\ &+a_0(t)\left(2y^c(t;r) - \underline{y}(t;r)\right) = \underline{g}(t;r),\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\bar{y}^{(n)}(t;r) + a_{n-1}(t)\bar{y}^{(n-1)}(t;r) + \cdots + a_{n-m}(t)\bar{y}^{(n-m)}(t;r) \\ &+a_{n-m-1}(t)\left(2y^c(t;r) - \bar{y}(t;r)\right)^{(n-m-1)} + \cdots \\ &+a_0(t)\left(2y^c(t;r) - \bar{y}(t;r)\right) = \bar{g}(t;r).\end{aligned} \tag{3.15}$$

The lower and upper bounds of the solutions are obtained by solving Eqs. (3.14) and (3.15) respectively using the value of fuzzy centre $(y^c(t;r))$ in similar fashion as before. Otherwise, only one equation viz. Eq. (3.14) or (3.15) may be solved and using the expression $\bar{y}(t;r) = \left(2y^c(t;r) - \underline{y}(t;r)\right)$ (or $\underline{y}(t;r) = (2y^c(t;r) - \bar{y}(t;r))$) one may have the solution bounds.

4. Numerical implementation of the proposed method

In the following paragraphs example problems are solved using the proposed method with different cases and are also compared with exact solutions. We also obtain the exact solution by following the method of Bede [6].

Example 4.1. Let us consider the following second order fuzzy linear differential equation (Case 1)

$$\tilde{y}'' + 6\tilde{y}' + 9\tilde{y} = 0 \tag{4.1}$$

subject to the fuzzy initial conditions

$$\tilde{y}(0;r) = [0.2r + 1.8, 2.2 - 0.2r],\ \tilde{y}'(0;r) = [0.2r - 3.2, -2.8 - 0.2r].$$

The exact fuzzy solution are obtained by the method of Bede [6] as

$$\underline{Y}(t;r) = \left(\tfrac{1}{5}r + \tfrac{9}{5}\right)e^{-3t} + \left(\tfrac{4}{5}r + \tfrac{11}{5}\right)te^{-3t},$$
$$\bar{Y}(t;r) = \left(\tfrac{11}{5} - \tfrac{1}{5}r\right)e^{-3t} + \left(\tfrac{19}{5} - \tfrac{4}{5}r\right)te^{-3t}.$$

According to Eq. (3.2), the differential equation (Eq. (4.1)) can be written as

$$y^{c''} + 6y^{c'} + 9y^c = 0 \tag{4.2}$$

Solving Eq. (4.2) one may obtain $y^c = (2+3t)\,e^{-3t}$
As disused in Case 1, with the above value of y^c it gives the value of $\underline{y}(t) = \left(\tfrac{1}{5}r + \tfrac{9}{5}\right)e^{-3t} + \left(\tfrac{4}{5}r + \tfrac{11}{5}\right)te^{-3t}$ and $\bar{y}(t) = \left(\tfrac{11}{5} - \tfrac{1}{5}r\right)e^{-3t} + \left(\tfrac{19}{5} - \tfrac{4}{5}r\right)te^{-3t}$. Hence one may have the final solution as $\tilde{y}(t;r) = [\underline{y}(t;r), \bar{y}(t;r)]$.
One may note that the results obtained by the proposed method are exactly same as that of the exact solution obtained by the method of Bede [6]. Corresponding fuzzy plot is given in Fig. 1.

Example 4.2. Next, we consider the following second order fuzzy linear differential equation (case 2)

$$\tilde{y}'' - 3\tilde{y}' - 4\tilde{y} = 0 \tag{4.3}$$

subject to the fuzzy initial conditions

$$\tilde{y}(0) = [0.2r + 0.8, 1.2 - 0.2r],\ \tilde{y}'(0) = [0.2r + 1.8, 2.2 - 0.2r].$$

Exact fuzzy solution are obtained again by following the method of Bede [6] as

$$\begin{aligned}\underline{Y}(t;r) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} + \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)(-1+r)+ \\ \left(-\frac{1}{5} + \frac{1}{5}r\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right),\end{aligned} \tag{4.4}$$

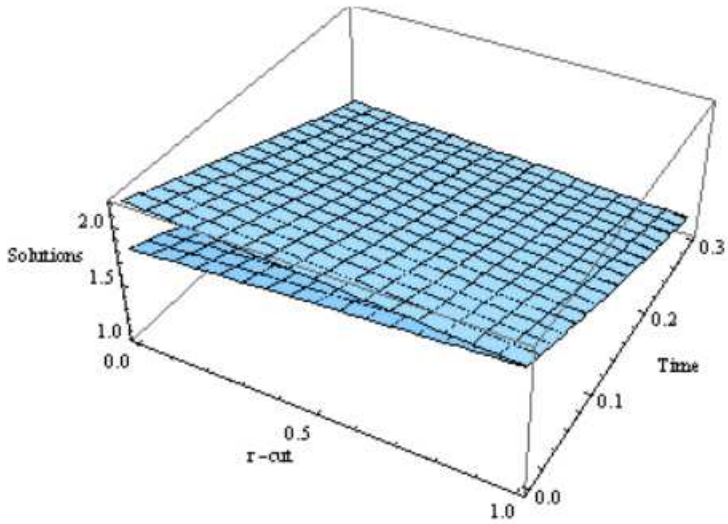

Fig.1 Fuzzy solution of Example 4.1 using the proposed method.

$$\overline{Y}(t;r) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} - \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)(-1+r) - \left(-\frac{1}{5}+\frac{1}{5}r\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right). \tag{4.5}$$

By using the proposed method we have,

$$y^c = \tfrac{2}{5}e^{-t} + \tfrac{3}{5}e^{4t}.$$

Subsequently, we get the solution

$$\underline{y}(t;r) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} + \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)(-1+r) + \left(-\frac{1}{5}+\frac{1}{5}r\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right), \tag{4.6}$$

$$\bar{y}(t;r) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} - \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)(-1+r) - \left(-\frac{1}{5}+\frac{1}{5}r\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right). \tag{4.7}$$

Again, it may be worth mentioning that the results obtained by proposed method exactly agree with exact solution. Plot for Example 2 is depicted in Fig. 2.

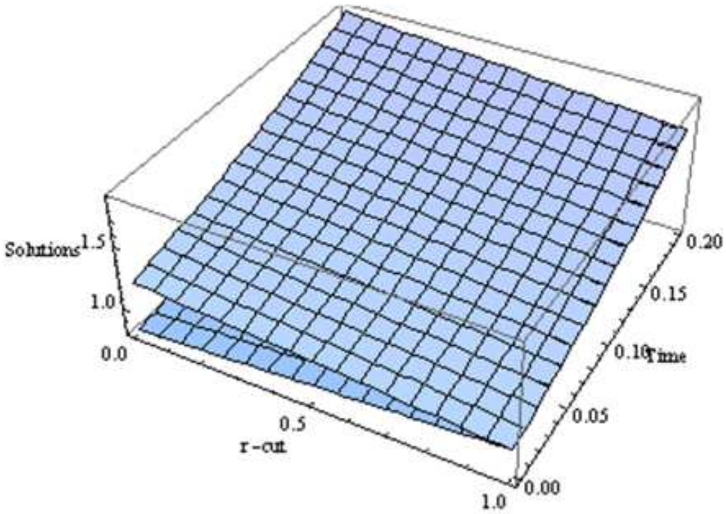

Fig.2 Fuzzy solution of Example 4.2 using the proposed method.

Example 4.3. Now, we take the following third order fuzzy differential equation (Case 3)

$$\tilde{y}''' - 6\tilde{y}'' + 11\tilde{y}' - 6\tilde{y} = 0 \tag{4.8}$$

subject to the fuzzy initial conditions

$$\tilde{y}(0;r) = [0.2r + 0.8, 1.2 - 0.2r],\ \tilde{y}'(0;r) = [0.2r + 0.8, 1.2 - 0.2r] \text{ and } \tilde{y}''(0;r) = [0.2r + 1.8, 2.2 - 0.2r].$$

The exact fuzzy solution may be obtained as [6],

$$\underline{Y}(t;r) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} + \left(\tfrac{3}{5}r - \tfrac{3}{5}\right)e^{-3t} + \left(-\tfrac{8}{5}r + \tfrac{8}{5}\right)e^{-2t} + \left(\tfrac{6}{5}r - \tfrac{6}{5}\right)e^{-t},$$
$$\bar{Y}(t;r) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} - \left(\tfrac{3}{5}r - \tfrac{3}{5}\right)e^{-3t} - \left(-\tfrac{8}{5}r + \tfrac{8}{5}\right)e^{-2t} - \left(\tfrac{6}{5}r - \tfrac{6}{5}\right)e^{-t}$$

Following the proposed method we have the solution as

$$\underline{y}(t;r) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} + \left(\tfrac{3}{5}r - \tfrac{3}{5}\right)e^{-3t} + \left(-\tfrac{8}{5}r + \tfrac{8}{5}\right)e^{-2t} + \left(\tfrac{6}{5}r - \tfrac{6}{5}\right)e^{-t},$$
$$\bar{y}(t;r) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} - \left(\tfrac{3}{5}r - \tfrac{3}{5}\right)e^{-3t} - \left(-\tfrac{8}{5}r + \tfrac{8}{5}\right)e^{-2t} - \left(\tfrac{6}{5}r - \tfrac{6}{5}\right)e^{-t}.$$

Again one may see that the solution obtained by proposed method exactly matches with the exact solution. Plot for this example is also shown in Fig. 3

Example 4.4. Finally, Let us consider the electrical circuit shown in Fig. 4 [9] where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 20\cos t$. If Q is the charge on the capacitor at time $t > 0$, then we have the second order fuzzy differential equation

$$\tilde{Q}''(t) + 2\tilde{Q}'(t) + 4\tilde{Q}(t) = 50\cos t \tag{4.9}$$

subject to the fuzzy initial conditions

$$\tilde{Q}(0;r) = [4 + r, 6 - r],\ \tilde{Q}'(0;r) = [r, 2 - r]$$

Exact fuzzy solution for this problem may be obtained as

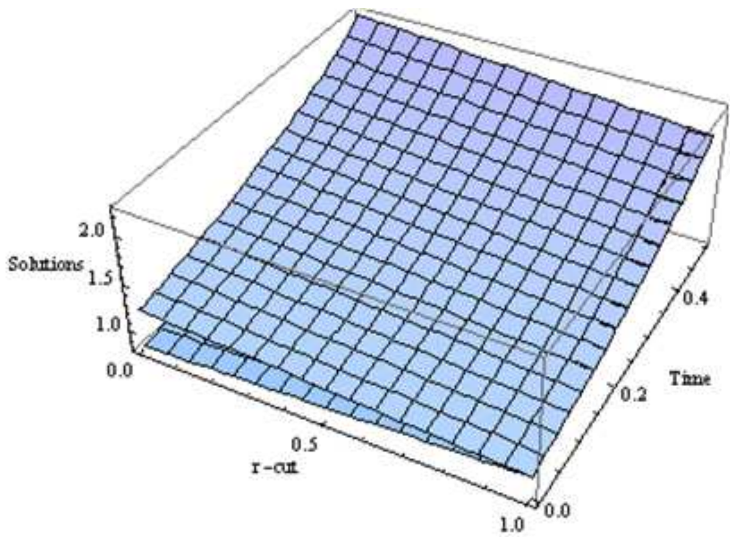

Fig.3 Fuzzy solution of Example 4.3 for Case 3 using the proposed method.

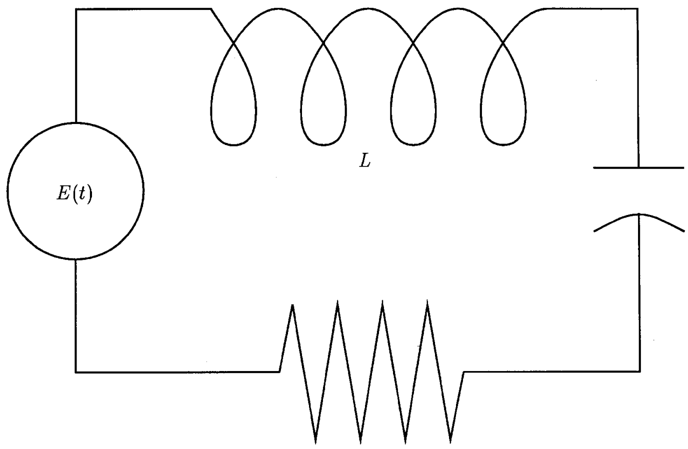

Fig.4 Electrical circuit in Example 4.4 [9]

$$\underline{Q}(t;r) = \frac{2}{39}e^{-t}\sin(\sqrt{3}t)(13r-99)\sqrt{3}+e^{-t}\cos(\sqrt{3}t)\left(r-\frac{98}{13}\right)+\left(\frac{150}{13}\right)\cos(t)+\left(\frac{100}{13}\right)\sin(t),$$

$$\bar{Q}(t;r) = -\frac{2}{39}e^{-t}\sin(\sqrt{3}t)(73+13r)\sqrt{3}+e^{-t}\cos(\sqrt{3}t)\left(-r-\frac{72}{13}\right)+\left(\frac{150}{13}\right)\cos(t)+\left(\frac{100}{13}\right)\sin(t).$$

By following the proposed method, we get the solution for Eq. (4.9) as

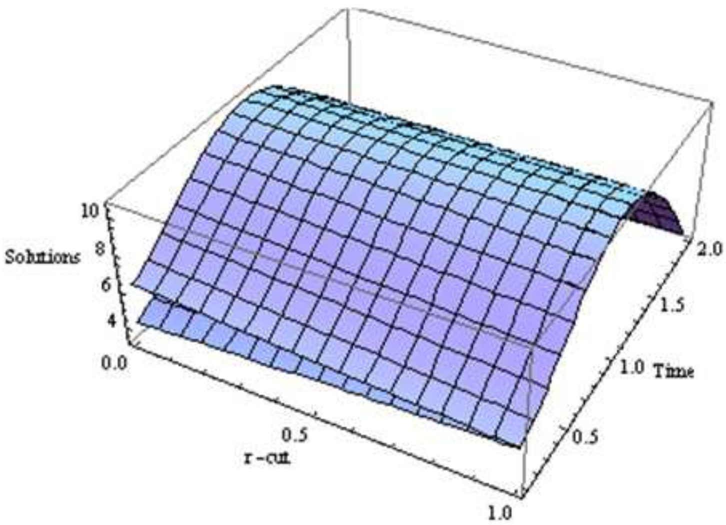

Fig.5 Fuzzy solution of Example 4.4 using the proposed method.

$$\underline{q}(t;r) = \frac{2}{39}e^{-t}\sin(\sqrt{3}t)(13r-99)\sqrt{3} + e^{-t}\cos(\sqrt{3}t)\left(r-\frac{98}{13}\right) + \left(\frac{150}{13}\right)\cos(t) + \left(\frac{100}{13}\right)\sin(t),$$

$$\bar{q}(t;r) = -\frac{2}{39}e^{-t}\sin(\sqrt{3}t)(73+13r)\sqrt{3} + e^{-t}\cos(\sqrt{3}t)\left(-r-\frac{72}{13}\right) + \left(\frac{150}{13}\right)\cos(t) + \left(\frac{100}{13}\right)\sin(t).$$

Here also, results from the proposed method are same as that of the exact solution and corresponding plot for this example is also cited in Fig. 5. The main value of the paper is not the example problems as discussed above. But here the main contribution is the new analytical method to handle $n-$th order fuzzy differential equation giving all the possible cases. As such the known differential equations are solved as test problems to have the confidence of the proposed method. The solutions by the proposed method in all the test problems exactly matches with the exact solution. The proposed method gives us a straightforward, alternate and computationally efficient way to handle $n-$th order fuzzy differential equations.

5. Conclusions

In this paper, a new method has been proposed to solve general $n-$th order fuzzy differential equations. First, the fuzzy differential equation is solved in term of fuzzy centre then this solution is used to get the final solution of the original $n-$th order fuzzy differential equation. The proposed method has been applied to three numerical example problems and an application problem viz. Circuit problem. Also, the obtained results are compared with the exact solutions to show the efficiency and powerfulness of the methodology. The solutions obtained are shown graphically too.

Acknowledgements. The first author would like to thank the UGC, Government of India, for financial support under Rajiv Gandhi National Fellowship (RGNF).

References

[1] S. Abbasbandy, T. Allahviranloo, O. Lopez-Pouso and J. J. Nieto, Numerical methods for fuzzy differential inclusions, Comput. Math. Appl. 48 (2004) 1633–1641.

[2] M. Z. Ahmad, M. K. Hasan and B. D Baets, Analytical and numerical solutions of fuzzy differential equations, Inform. Sci. 236 (2013) 156–167.

[3] O. Akin, T. Khaniyev, O. Oruc and I. B. Turksen, An algorithm for the solution of second order fuzzy initial value problems, Expert Syst. Appl. 40 (2013) 953-957.

[4] T. Allahviranloo, E. Ahmady and N. Ahmady, $N-$th order fuzzy linear differential equations, Inform. Sci. 178 (2008) 1309–1324.

[5] A. Esfahani, O. S. Fard, T. A. Bidgoli, On the existence and uniqueness of solutions to fuzzy boundary value problems, Ann. Fuzzy Math. Inform. 7 (2014) 15–29.

[6] B. Bede, Note on "Numerical solutions of fuzzy differential equations by predictor-corrector method", Inform. Sci. 178 (2008) 1917–1922.

[7] B. Bede, I. J. Rudas and A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Inform. Sci. 177 (2007) 1648–1662.

[8] D. Behera and S. Chakraverty, Uncertain impulse response of imprecisely defined half order mechanical system, Ann. Fuzzy Math. Inform. 7 (2014) 401–419.

[9] J. J. Buckley and T. Feuring, Fuzzy initial value problem for nth-order linear differential equations, Fuzzy Sets and Systems 121(2) (2001) 247–255.

[10] Y. Chalco-Cano and H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos Solitons Fractals 38 (2008) 112–119.

[11] S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, IEEE Trans. Systems, Man, and Cybernet. SMC-2 (1972) 30–34.

[12] D. Dubois and H. Prade, Towards fuzzy differential calculus. III. Differentiation, Fuzzy Sets and Systems 8(3) (1982) 225–233.

[13] S. Effati and M. Pakdaman, Artificial neural network approach for solving fuzzy differential equations, Inform. Sci. 180 (2010) 1434–1457.

[14] N. A. Gasilov, A. G. Fatullayev, S. E. Amrahov and A. Khastan, A new approach to fuzzy initial value problem, Soft Comput. 18 (2014) 217–225.

[15] D. N. Georgiou, J. J. Nieto and R. Rodriguez-Lopez, Initial value problems for higher-order fuzzy differential equations, Nonlinear Anal. 63(4) (2005) 587–600.

[16] B. Ghazanfari and A. Shakerami, Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order 4, Fuzzy Sets and Systems 189 (2012) 74–91.

[17] M. Hanss, Applied fuzzy arithmetic: An introduction with engineering applications, Berlin, 2005.

[18] M. S. Hashemi, J. Malekinagad and H. R. Marasi, Series solution of fuzzy differential equations, Advances Fuzzy Syst. 2012 (2012) 1–16.

[19] M. S. Hashemi and J. Malekinagad, Series solution of fuzzy wave-like equations with variable coefficients, J. Intell. Fuzzy Systems 25(2) (2013) 415–428.

[20] H. Jafari, M. Saeidy and D. Baleanu, The variational iteration method for solving $n-$th order fuzzy differential equations, Central Eur. J. Phys. 10 (2012) 76–85.

[21] T. Jayakumar, K. Kanagarajan and S. Indrakumar, Numerical solution of nth-order fuzzy differential equation by runge-kutta method of order five, Int. J. Math. Anal. 6 (2012) 2885–2896.

[22] L. Jaulin, M. Kieffer, O. T. Didri and E. Walter, Applied interval analysis, Springer 2001.

[23] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[24] O. Kaleva, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems 35 (1990) 389–396.

[25] K. Kanagarajan and S. Muthukumar, Extended Runge-Kutta method of order four for hybrid fuzzy differential equations, Int. J. Appl. Math. Comput. 4 (2012) 312–320.

[26] A. Khastan, F. Bahrami and K. Ivaz, New results on multiple solutions for nth-order fuzzy differential equations under generalized differentiability, Boundary Value Probl. 2009 (2009) 1–13. doi:10.1155/2009/395714.
[27] A. Khastan and K. Ivaz, Numerical solution of fuzzy differential equations by nystrom method, Chaos Solitions Fractals 41 (2009) 859–868.
[28] A. Khastan, J. J. Nieto and R. Rodriguez- Lopez, Variation of constant formula for first order fuzzy differential equations, Fuzzy Sets and Systems 177 (2011) 20–33.
[29] A. Khastan and J. J. Nieto, A numerical method for fuzzy differential equations and hybrid fuzzy differential equations, Abstr. Appl. Anal. 2013, Art. ID 735128, 10 pp.
[30] V. Lupulescu, Initial value problem for fuzzy differential equations under dissipative conditions, Inform. Sci. 178 (2008) 4523–4533.
[31] M. Ma, M. Friedman and A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133–138.
[32] N. Mikaeilvand and S. Khakrangin, Solving fuzzy partial differential equations by fuzzy two-dimensional differential transform method, Neural Comput. Appl. 21 (2012) S307–S312.
[33] S. P. Mondal and T. K. Roy, First order linear non homogeneous ordinary differential equation in fuzzy environment, Math. Theory Model. 3 (2013) 85–95.
[34] M. Mosleh, Fuzzy neural network for solving a system of fuzzy differential equations, Appl. Soft Comput. 13 (2013) 3597–3607.
[35] M. Mosleh and M. Otadi, Simulation and evaluation of fuzzy differential equations by fuzzy neural network, Appl. Soft Comput. 12 (2012) 2817–2827
[36] S. Narayanamoorthy and K. Murugan, Solutions of fuzzy heat-like equations by variational iteration method, Ann. Fuzzy Math. Inform. 7 (2014) 133–142
[37] J. J. Nieto, R. Rodriguez-Lopez and D. Franco, Linear first-order fuzzy differential equations, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 14(6) (2006) 687–709.
[38] S. Palligkinis, G. Papageorgiou and I. Famelis, Runge-Kutta methods for fuzzy differential equations, Appl. Math. Comput. 209 (2009) 97–105.
[39] N. Parandin, Numerical solution of fuzzy differential equations of n-th order by Runge-Kutta method, Neural Comput. Appl. 21 (2012) S347–S355.
[40] S. Pederson and M. Sambandham, Numerical solution of hybrid fuzzy differential equation IVP by characterization theorem, Inform. Sci. 179 (2009) 319–328.
[41] P. Prakash and V. Kalaiselvi, Numerical solutions of fuzzy differential equations by using hybrid methods, Fuzzy Inf. Eng. 4 (2012) 445–455
[42] F. Rabiei, Fudziah Ismail, Ali Ahmadian, and Soheil Salahshour, Numerical solution of second-order fuzzy differential equation using improved Runge-Kutta nystrom method, Math. Prob. Eng. 2013 (2013) 1–10.
[43] T. J. Ross, Fuzzy logic with engineering applications, Wiley Student Edition, 2007.
[44] S. Salahshour, Nth-order fuzzy differential equations under generalized differentiability, J. Fuzzy Set Valued Anal. 2011 (2011) 1–14.
[45] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.
[46] S. Siah Mansouri and N. Ahmady, A numerical method for solving Nth-order fuzzy differential equation by using characterization theorem, Commun. Numer. Analy. 2012 (2012) 1–12
[47] O. Solaymani Fard and N. Ghal-Eh, Numerical solutions for linear system of first-order fuzzy differential equations with fuzzy constant coeficients, Inform. Sci. 181 (2011) 4765–4779
[48] S. Tapaswini and S. Chakraverty, A new approach to fuzzy initial value problem by improved euler method, Fuzzy Inf. Eng. 4(3) (2012) 293–312.
[49] S. Tapaswini and S. Chakraverty, Numerical solution of n-th order fuzzy linear differential equations by homotopy perturbation method, Int. J. Computer Appl. 64 (2013) 5–10.
[50] S. Tapaswini and S. Chakraverty, Euler based new solution method for fuzzy initial value problems, Int. J. Artificial Intell. Soft Comput. 4 (2014) 58–79.
[51] S. Tapaswini and S. Chakraverty, Numerical solution of fuzzy arbitrary order predator-prey equations, Appl. Appl. Math. 8(2) (2013) 647–672.
[52] S. Tapaswini and S. Chakraverty, Numerical solution of uncertain beam equations using double parametric form of fuzzy numbers, Appl. Comput. Intell. Soft Comput. 2013 (2013) 1–8.

[53] S. Tapaswini and S. Chakraverty, Numerical solution of fuzzy quadratic Riccati diferential equation, Coupled Syst. Mech. 3 (2013) 255–269.
[54] Z. Yue and W. Guangyuan, Time domain methods for the solutions of N-order fuzzy differential equations, Fuzzy Sets and Systems 94 (1998) 77–92.
[55] H. J. Zimmermann, Fuzzy Set Theory and its Application, Kluwer Academic Publishers,2001.

Smita Tapaswini (smitatapaswini@gmail.com)
Department of Mathematics, National Institute of Technology Rourkela, Odisha 769 008, India

S. Chakraverty (sne_chak@yahoo.com)
Department of Mathematics, National Institute of Technology, Rourkela, Odisha 769 008, India