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Pseudo-addition and fuzzy ideals in BL-algebras

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Abstract. A pseudo-addition in BL-algebras is a binary operation that is associative and non commutative. The aim of this paper is to use the notion of pseudo-addition to study the notion of ideal and fuzzy ideal as a natural generalization of that of ideal and fuzzy ideal in MV-algebras. Using a pseudo-addition, we characterize fuzzy ideals and also obtain a concrete description of the fuzzy ideals generated by a fuzzy subset. Among other things, we establish the extension property of fuzzy prime ideal, prove the fuzzy prime ideal Theorem and show that the set of cosets of a fuzzy ideal is a linearly ordered MV-algebra. Finally, we use the operators N and D to make a link between fuzzy ideals and fuzzy filters.

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1. Introduction

The notion of fuzzy ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideals theory is a very effective tool for studying various algebraic and logical systems. BL-algebras were invented by Hájek [7] in order to study the basic logic framework of fuzzy set theory and MV-algebras were introduced by C.C. Chang in order to give an algebraic proof of the completeness theorem of Lukasiewicz system of many valued logic. MV-algebras and BL-algebras are closely related as MV-algebras are simply BL-algebras satisfying the double negation. The study of these algebras has been carried out from both a logical and algebraic standpoints. Most results from the theory of MV-algebras remain unchanged in BL-algebras, though their proofs are usually quite different in the BL-algebras settings. In the theory of MV-algebras, as in various algebraic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also called filters. The study of BL-algebras has experienced tremendous growth over the recent years and the main focus has been on filters and fuzzy filters [1][5],[9],[19],[21],[25]. In the meantime, several authors have claimed in recent works ([3],[14],[24]) that the notion of ideal is missing in BL-algebras. This has been partly associated to the fact that there was no suitable algebraic addition in BL-algebras.

Nowadays, filters and ideals are tools of extreme importance in many areas of classical mathematics. For example, in topology, they enhance the concept of convergence and in measure theory, prime filters can be interpreted as basic components of probability measures, in fuzzy mathematics, filters and ideals have been conceived in various manners [12], [18]. Fuzzy ideals and fuzzy filters are also particularly interesting because they are closely related to congruence relations [4]. In [12], Kroupa introduced the concept of filters of fuzzy class theory and investigated graded properties of filters, prime filters and related construction. From a logical point of view, various filters and ideals correspond to various sets of provable formulae. The sets of provable formulas in the corresponding inference systems from the point of view of uncertain information can be described by fuzzy ideals of those algebraic semantics. Many research papers have been publishing on BL-algebras and related structures [14],[15],[16],[17],[10],[11], [20],[26],[27],[29],[30],[31]. But so far, mostly on filters and fuzzy filters while the study of ideals and fuzzy ideals in BL-algebras have been for the most part ignored. Given the importance of ideals and congruences in classification problems, data organization, formal concept analysis, to name a few; it is meaningful to make and intensive study of ideals in BL-algebras. The main goal of this work is to study the notion of ideal and fuzzy ideals. We shall also introduce in the process a natural algebraic pseudo-addition in BL-algebras, which as suspected is closely tied to the notion of ideals. This notion must generalize in a natural sense the existing notion in MV-algebras and subsequently all the results about ideals and fuzzy ideals in MV-algebras [8]. We do this by using the concept of pseudo-addition to study fuzzy ideals in BL-algebras and show that fuzzy ideals are useful to obtaining results on classical ideals in BL-algebras. Moreover, we also study fuzzy ideals for their own sake. The paper is organized as follows. In Section 2, we discuss some important properties of ideals. In Section 3, we introduce the concept of fuzzy ideal, we construct some important examples which show that the notion of fuzzy ideal has a proper meaning in BL-algebras. Unlike in MV-algebras, we observe that fuzzy ideals and fuzzy filters behave quite differently in BL-algebras. Moreover, we give several characterizations of fuzzy ideals with a concrete description of the fuzzy ideal generated by a fuzzy subset. In Section 4, we study the concept of fuzzy prime ideals. Among other things, we establish the extension property of fuzzy prime ideal and prove the fuzzy prime ideal Theorem. In Section 5 , we prove that the set of cosets of a fuzzy ideal is a linearly ordered MV-algebra, not just BL-algebras as it is the case with fuzzy filters. In the last Section, we use the operators N and D to make a link between fuzzy ideals and fuzzy filters. It is our hope that this work will settle once and for all the existence of ideals and fuzzy ideals in BL-algebras settings.

2. Preliminaries and Notations

All the results included in this section and their proofs can be found in [13].

Definition 2.1. A *BL-algebra* is a nonempty set L with four binary operations $\wedge, \vee, \otimes, \rightarrow$, and two constants $0, 1$ satisfying:

BL-1 $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
BL-2 $(L, \otimes, 1)$ is a commutative monoid;
BL-3 $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. (Residuation);
BL-4 $x \wedge y = x \otimes (x \rightarrow y)$ (Divisibility);
BL-5 $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (Prelinearity).

The main examples of BL-algebras are from the unit interval $[0, 1]$ endowed with the structure induced by continuous t-norms. Every BL-algebra has the complementation operation defined by $\bar{x} = x \rightarrow 0$.
A BL-algebra satisfying the double negation is called an MV-algebra, that is $\bar{\bar{x}} = x$.
The following is the most comprehensive list of properties of BL-algebras.

Proposition 2.2 ([6], [7], [22], [23])**.** *For any BL-algebra $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, the following properties hold for every $x, y, z \in L$:*

1. $x \leq y$ *iff* $x \rightarrow y = 1$;
2. $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$;
3. $x \otimes y \leq x \wedge y$;
4. $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
5. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
6. *If* $x \leq y$*, then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$;
7. *If* $x \vee \bar{x} = 1$*, then* $x \wedge \bar{x} = 0$;
8. $x \leq y \rightarrow (x \otimes y)$;
9. $x \otimes (x \rightarrow y) \leq y$;
10. $1 \rightarrow x = x; x \rightarrow x = 1; x \rightarrow 1 = 1; x \leq y \rightarrow x, x \leq \bar{\bar{x}}, \bar{\bar{\bar{x}}} = \bar{x}$;
11. $x \otimes \bar{x} = 0$, $x \otimes y = 0$ *iff* $x \leq \bar{y}$;
12. $x \leq y$ *implies* $x \otimes z \leq y \otimes z$;
13. $\overline{(x \otimes y)} = x \rightarrow \bar{y}$;
14. $\overline{(x \wedge y)} = \bar{x} \vee \bar{y}, \overline{(x \vee y)} = \bar{x} \wedge \bar{y}$;
15. $\overline{\overline{(x \rightarrow y)}} = \bar{\bar{x}} \rightarrow \bar{\bar{y}}, \overline{\overline{(x \wedge y)}} = \bar{\bar{x}} \wedge \bar{\bar{y}}, \overline{\overline{(x \vee y)}} = \bar{\bar{x}} \vee \bar{\bar{y}}; \bar{\bar{x}} \otimes \bar{\bar{y}} = \overline{\overline{x \otimes y}}$;
16. $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z), x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$;
17. $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

We would like to point out that some of theses properties are redundant. For instance, 20 can be obtained from 2 by setting $z = 0$, but we prefer to list all these to make their uses obvious.

A subset F of a BL-algebra $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called a filter if it satisfies:

F1 : $1 \in F$;
F2 : For every $x, y \in F$, $x \otimes y \in F$; and
F3 : For every $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$.

It is clear from $F3$ and $x \leq \bar{\bar{x}}$, that $x \in F$ implies $\bar{\bar{x}} \in F$.
A deductive system of a BL-algebra L is a subset F containing 1 such that for all $x, y \in L$;

$$x \rightarrow y \in F \quad \text{and} \quad x \in F \quad \text{imply} \quad y \in F.$$

It is known that in a BL-algebra, filters and deductive systems coincide [23]. It is also known that the filters of a BL-algebra L form a lattice commonly denoted by

$\mathcal{F}(L)$.
In the literature, as for example in [2], MV-algebras are also defined as algebras $(M, \oplus, ^*, 0)$ satisfying:

MV-1 $(M, \oplus, 0)$ is an Abelian monoid;
MV-2 $(x^*)^* = x$;
MV-3 $0^* \oplus x = 0^*$;
MV-4 $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

The two definitions of MV-algebras are equivalent through the following transfer.
Given a BL-algebra $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ satisfying the double negation, define $\oplus$ and * by: $x^* = \bar{x}$ and $x \oplus y = \bar{x} \rightarrow y$.
Then $(L, \oplus, 0)$ satisfies MV-1 through MV-4.
Conversely, given an algebra $(M, \oplus, 0)$ satisfying MV-1 through MV-4, define the operations $\wedge, \vee, \otimes, \rightarrow$ by:
$x \otimes y = (x^* \oplus y^*)^*$; $x \rightarrow y = x^* \oplus y$; $x \wedge y = x \otimes (y \oplus x^*)$; $x \vee y = x \oplus (y \otimes x^*)$; $x^* = \bar{x}$ and $1 = \bar{0}$ where $\bar{x} = x \rightarrow 0$.
Then $(M, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a BL-algebra satisfying the double negation.
For any BL-algebra L, the subset $MV(L) = \{\bar{x}, x \in L\}$ is the largest MV-sub algebra of L and is called the MV-center of L [24].
The addition in the MV-center is defined by $\bar{x} \oplus \bar{y} = \overline{x \otimes y}$ for any $\bar{x}, \bar{y} \in MV(L)$.
A detailed treatment of the MV-center is found in [24].
We recall that for any subset X of a BL-algebra L, $\bar{X} = \{\bar{x}, x \in X\}$.
If L is a BL-algebra and $x, y \in L$, $x \oslash y := \bar{x} \rightarrow y$. The associative and non commutative operation $\oslash$ is called the (natural) pseudo-addition of the BL-algebra.

Definition 2.3. Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and I a non empty subset of L. We say that I is an *ideal* of L if it satisfies:

I1 : For every $x, y \in I$, $x \oslash y \in I$; and
I2 : For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

The following result is a characterization of ideals.

Theorem 2.4. *In any BL-algebra L and $I \subseteq L$, the following conditions are equivalent.*

I1. *I is an ideal of L.*
I2. *$0 \in I$ and for every $x, y \in L$, $\bar{x} \otimes y \in I$ and $x \in I$ imply $y \in I$.*

The following result is another characterization of ideals.

Theorem 2.5. *A subset I of a BL-algebra L is a ideal if and only if the following conditions hold:*

J1: *$0 \in I$; and*
J2: *For every $x, y \in L$, if $x \in I$ and $\overline{(\bar{x} \rightarrow \bar{y})} \in I$, then $y \in I$.*

Proof. Assume that I is a ideal. It is clear that $0 \in I$.
Let $x, y \in L$ such that $x \in I$ and $\overline{(\bar{x} \rightarrow \bar{y})} \in I$. We must prove that $y \in I$. First, we observe that $\bar{x} \otimes \bar{\bar{y}} = \overline{\overline{\bar{x} \otimes \bar{\bar{y}}}} = \overline{(\bar{x} \rightarrow \bar{y})} \in I$. We have $x, \bar{x} \otimes \bar{\bar{y}} \in I$ and we apply Theorem 2.4 and obtain $\bar{\bar{y}} \in I$, from which it follows that $y \in I$.
Conversely, assume that J1 and J2 hold.

Setting $y = \bar{\bar{x}}$ in J2, we obtain that $x \in I$ implies $\bar{\bar{x}} \in I$. To show that I is an ideal, by Theorem 2.4, let $x, y \in L$ such that $x, \bar{x} \otimes y \in I$. Since $\bar{x} \otimes y \in I$, we have $\overline{\overline{(\bar{x} \otimes y)}} \in I$. But since $\overline{(\bar{x} \rightarrow \bar{y})} = \overline{\overline{\bar{x} \otimes \bar{\bar{y}}}} = \overline{\overline{(\bar{x} \otimes y)}}$, then $\overline{(\bar{x} \rightarrow \bar{y})} \in I$. Now, we apply J2 and obtain that $y \in I$. $\square$

Remark 2.6. The above Theorem enables us to see that our definition of ideal coincide with the definition given in[32]. It is worth noting that the definition of ideal given in [32] is hard to use. In addition, our definition and characterization offers more insight in terms of comparing this notion to the well-studied notions of fuzzy ideals in MV-algebras, and of fuzzy filters in BL-algebras.

Example 2.7. Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and X a nonempty set. The set of functions from $X \rightarrow L$, L^X has a natural structure of BL-algebra with the operations defined pointwise. Fix any element $x_0 \in X$ and consider
$I = \{f \in L^X, f(x_0) = 0\}$. Routine computations prove that I is an ideal of L^X.

We recall that the smallest ideal containing A is called the ideal generated by the subset A and it is denoted by $\langle A \rangle$. It is also the intersection of all the ideals containing A.

Theorem 2.8. *For every subset A of a BL-algebra L*

(i) *If A is empty, then $\langle A \rangle = \{0\}$.*

(ii) *If A is not empty, then*
$\langle A \rangle = \{a \in L : a \leq (...((x_1 \oslash x_2) \oslash x_3) \oslash ...) \oslash x_n; x_1, x_2, x_3 ... x_n \in A\}$.

Theorem 2.9. *Let I be an ideal of a BL-algebra L. Define the relation $\sim_I$ on L by: for every $x, y \in L$, $x \sim_I y$ if and only if $\bar{x} \otimes y$ and $\bar{y} \otimes x \in I$. Then $\sim_I$ is a congruence on L and the quotient BL-algebra is an MV-algebra.*

Proposition 2.10. *Let X, Y be two BL-algebras, $\theta : X \rightarrow Y$ a BL-homomorphism and I an ideal of Y. Then $\theta^{-1}(I)$ is an ideal of X.*

Proposition 2.11. *(The Main Isomorphism Theorem)*
Let X, Y be two BL-algebras, and $\theta : X \rightarrow Y$ be a BL-homomorphism. Then $X/ker(\theta)$ is isomorphic to $MV(Im(\theta))$

Proposition 2.12. *A proper ideal P of a BL-algebra L is a prime ideal if and only if it satisfies one of the following equivalent conditions:*

P1 : *The BL-algebra quotient L/P is an MV-chain.*
P2 : *For any $x, y \in L$, $x \wedge y \in P$ implies that $x \in P$ or $y \in P$.*
P3 : *For any $x, y \in L$, $\overline{x \rightarrow y} \in P$ or $\overline{y \rightarrow x} \in P$.*

We recall the extension property of prime ideals and the prime ideal Theorem.

Proposition 2.13. *(The Extension Property for Prime Ideals)*
Let L be a BL-algebra and I a prime ideal of L. Then every proper ideal J of L containing I is also prime.

Proposition 2.14. *(Prime ideal Theorem)*
Let L be a BL-algebra, I an ideal of L with $x \in L$, but $x \notin I$. Then there exits a prime ideal P of L containing I with $x \notin P$.

Definition 2.15. Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra an X any subset of L. The set of complement elements denoted by $N(X)$ is defined by

$$N(X) = \{x \in L; \bar{x} \in X\}.$$

We recall some properties of the set of complement elements $N(X)$.

Theorem 2.16. *Let F be a BL-filter and I an ideal of L, we have the following results:*

N1. *The set of complement elements $N(I)$ is a filter and $\bar{I} \subseteq N(I)$.*
N2. *The set of complement elements $N(F)$ is an ideal generated by $\bar{F}$.*
N3. $I = N(N(I))$.
N4. $F \subseteq N(N(F))$.
N5. $N(F) = N(N(N(F)))$.
N6. *The MV-center of the quotient BL-algebra L/F is isomorphic to the MV-algebra $L/N(F)$.*
N7. *The MV-center of the quotient BL-algebra $L/N(I)$ is isomorphic to the MV-algebra L/I.*

Proposition 2.17. *Let X, Y be two BL-algebras, $f : X \rightarrow Y$ a BL-homomorphism and I a prime ideal of Y. Then $f^{-1}(I)$ is a prime ideal of X*

Proposition 2.18. *Let I be a prime ideal and F be a prime filter of L, then $N(I)$ is a prime filter and $N(F)$ is a prime ideal .*

3. Fuzzy Ideals

We recall [28] that a fuzzy set of a set X is a function $\mu : X \longrightarrow [0; 1]$. For a fuzzy set μ in X and $t \in [0; 1]$, define μ_t to be the set $\mu_t = \{x \in X/\mu(x) \geq t\}$. μ_t is called the $t-$cut set or $t-$level set. In this section, we use the pseudo-addition and introduce the notion of fuzzy ideal in general BL-algebras which coincides with the notion of fuzzy ideal in MV-algebras.

Definition 3.1. Let μ be a fuzzy set of a BL-algebra L. Then μ is a fuzzy ideal L if for every $t \in [0; 1]$, the $t-$cut set $\mu_t := \{x \in X/\mu(x) \geq t\}$ is either empty or an ideal. The set of fuzzy ideals in L will be denoted by $FI(L)$.

The following example justify that the newly introduced class of fuzzy ideals exists and has a proper meaning in a general BL-algebra.

Example 3.2. Let $X = \{0, a, b, c, d, e, f, 1\}$ be such that $0 < a < b < c < 1$, $0 < d < e < f < 1$, $a < e$ and $b < f$. Define $\otimes$ and $\rightarrow$ as follows:

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(X, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a BL-algebra which is not MV-algebra. Define a fuzzy set μ by $\mu(0) = t_1, \mu(a) = \mu(b) = \mu(c) = t_2, \mu(d) = \mu(e) = \mu(f) = \mu(1) = t_3$ with $0 \leq t_3 < t_2 < t_1 \leq 1$. Simple computations prove that μ is a fuzzy ideal of X.

Remark 3.3. A non-empty subset I of L is an ideal of L if and only if its characteristic function μ_I is a fuzzy ideal of L.

The following theorems characterize fuzzy ideals.

Theorem 3.4. *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and μ a fuzzy subset of L. μ is a fuzzy ideal of L if and only if the following conditions hold:*

FI1 *: For every $x, y \in L$, $\mu(x \oslash y) \geq min(\mu(x), \mu(y))$; and*
FI2 *: For every $x, y \in L$, if $x \leq y$, then $\mu(x) \geq \mu(y)$.*

Proof. Suppose that μ satisfies $FI1$ and $FI2$, we must show that for all $t \in [0; 1]$, $\mu_t := \{x \in X/\mu(x) \geq t\}$ is either empty or an ideal.
Let $t \in [0, 1]$ such that $\mu_t := \{x \in X/\mu(x) \geq t\}$ is non empty. We shall prove that μ_t satisfies $I1$ and $I2$.
Let $x, y \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$, by $FI1$, we obtain $\mu(x \oslash y) \geq min(\mu(x), \mu(y)) \geq t$ and $x \oslash y \in \mu_t$.
Let $x, y \in L$ with $x \leq y$ and $y \in \mu_t$, we have $\mu(y) \geq t$ and by $FI2$, we have $\mu(x) \geq \mu(y) \geq t$ and $y \in \mu_t$.
Let $x, y \in L$, by setting $t = min(\mu(x), \mu(y))$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$, that is $x, y \in \mu_t$. Since μ_t is an ideal, by $I1$, $x \oslash y \in \mu_t$ and $\mu(x \oslash y) \geq t = min(\mu(x), \mu(y))$, from which we conclude that $\mu(x \oslash y) \geq min(\mu(x), \mu(y))$.
Let $x, y \in L$ with $x \leq y$, by setting $t = \mu(y)$, we have $y \in \mu_t$. Since μ_t is an ideal, by $I2$, $x \in \mu_t$ and $\mu(x) \geq t = \mu(y)$, we conclude that $\mu(x) \geq \mu(y)$. □

It is easy to see that for any fuzzy ideal μ, $\mu(0) \geq \mu(x)$ and $\mu(\bar{\bar{x}}) = \mu(x)$ for every $x \in L$. It is also clear that the intersection of any family of fuzzy ideals of a BL-algebra L is again a fuzzy ideal of L.

Remark 3.5. If a BL-algebra is an MV-algebra, $x \oslash y = x \oplus y$ and the above characterization show that the concept of fuzzy ideal coincides with the well known notion of fuzzy ideal in MV-algebras [8].

Theorem 3.6. *A fuzzy subset μ of a BL-algebra L is a fuzzy ideal if and only if the following conditions hold:*

FJ1 *: For every x, $\mu(0) \geq \mu(x)$; and*
FJ2 *: For every $x, y \in L$, $\mu(y) \geq min(\mu(x), \mu(\bar{x} \otimes y)$.*

Proof. Assume that μ is a fuzzy ideal and let $x, y \in L$.
We observe that $y \leq x \oslash (\bar{x} \otimes y)$ and apply the fact that μ is a fuzzy ideal and obtain that $\mu(y) \geq \mu(x \oslash (\bar{x} \otimes y)) \geq min(\mu(x), \mu(\bar{x} \otimes y)$.
Conversely, assume that μ satisfies $FJ1$ and $FJ2$, we must show that μ satisfies $FI1$ and $FI2$. Let $x, y \in L$ such that $x \leq y$.
Since $x \leq y$, we have $\bar{y} \leq \bar{x}$ and $\bar{y} \otimes x \leq \bar{x} \otimes x = 0$. Using the hypothesis, we obtain $\mu(y) \geq min(\mu(x), \mu(\bar{x} \otimes y) = min(\mu(x), \mu(0)) \geq \mu(x)$.
In addition, let $x, y \in I$. We observe that $\bar{x} \otimes (x \oslash y) = \bar{x} \wedge y \leq y$. So, $\mu(\bar{x} \otimes (x \oslash y)) \geq \mu(y)$. On the other hand, $\mu(x \oslash y) \geq min(\mu(x), \mu(\bar{x} \otimes (x \oslash y))$ and we conclude that $\mu(x \oslash y) \geq min(\mu(x), \mu(y))$. □

Theorem 3.7. *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and μ a fuzzy subset of L. μ is a fuzzy ideal of L if and only if the following conditions hold:*

FJ1 : *For every $x \in L$, $\mu(0) \geq \mu(x)$; and*

FJ3 : *For every $x, y \in L$, $\mu(y) \geq min(\mu(x), \mu(\overline{(\bar{x} \rightarrow \bar{y})})$.*

Proof. Assume that μ is a fuzzy ideal, for every $x \in L$, $\mu(0) \geq \mu(x)$.
Let $x, y \in L$, we must prove that $\mu(y) \geq min(\mu(x), \mu(\overline{(\bar{x} \rightarrow \bar{y})})$. First, we observe that $\bar{x} \otimes \bar{\bar{y}} = \overline{\overline{\bar{x} \otimes \bar{\bar{y}}}} = \overline{(\bar{x} \rightarrow \bar{y})}$. By Theorem 3.6 , we have $\mu(\bar{\bar{y}}) \geq min(\mu(x), \mu(\bar{x} \otimes \bar{\bar{y}}) = min(\mu(x), \mu(\overline{(\bar{x} \rightarrow \bar{y})})$. Since $\mu(y) \geq \mu(\bar{\bar{y}})$, it follows that

$$\mu(y) \geq min(\mu(x), \mu(\overline{(\bar{x} \rightarrow \bar{y})}).$$

Conversely, assume that FJ1 and FJ3 hold. Setting $y = \bar{\bar{x}}$ in FJ3, we obtain that for every $x \in L$, $\mu(\bar{\bar{x}}) \geq \mu(x)$. To show that μ is an fuzzy ideal, let $x, y \in L$, we must show that $\mu(y) \geq min(\mu(x), \mu(\bar{x} \otimes y))$.
But $\overline{\overline{(\bar{x} \otimes y)}} = \bar{x} \otimes \bar{\bar{y}} = \overline{\overline{\bar{x} \otimes \bar{\bar{y}}}} = \overline{(\bar{x} \rightarrow \bar{y})}$ and we obtain

$$\mu(y) \geq min(\mu(x), \mu(\overline{(\bar{x} \rightarrow \bar{y})}) = min(\mu(x), \mu(\overline{\overline{(\bar{x} \otimes y)}}) \geq min(\mu(x), \mu(\bar{x} \otimes y)).$$

□

Remark 3.8. The above Theorem enables us to see that our definition of fuzzy ideal coincide with the definition given in [32]. It is worth noting that the definition of fuzzy ideal given in [32] is hard to use and the authors did not mention that it is an extension of the definition of fuzzy ideal in MV-algebras [8].

Remark 3.9. Let μ be a fuzzy ideal, then for every $x, y \in L$, we have

$$min(\mu(x), \mu(y)) = \mu(x \vee y);\ \mu(x \wedge y) \geq max(\mu(x), \mu(y));\ \mu(\bar{x} \otimes y) \geq \mu(y).$$

We conclude that a fuzzy ideal is a fuzzy lattice ideal. But the following example shows that the converse is not true in general.

Example 3.10. Let $(X, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra of Example 3.2. Define a fuzzy set μ by $\mu(0) = \mu(a) = t_1, \mu(b) = \mu(c) = \mu(d) = \mu(e) = \mu(f) = \mu(1) = t_2$ with $0 \leq t_2 < t_1 \leq 1$. It is easy to see that μ is a fuzzy lattice ideal which is not a fuzzy ideal.

Our next aim is to give a concrete description of the fuzzy ideal generated by a fuzzy subset μ of a BL-algebra L.

Definition 3.11. Let μ be a fuzzy subset of L. A fuzzy ideal ν of L is said to be **generated** (or **induced**) **by** μ, if $\mu \leq \nu$ and for any fuzzy ideal λ of L, $\mu \leq \lambda$ implies $\nu \leq \lambda$. The fuzzy ideal induced by μ will be denoted by $\langle \mu \rangle$.

Remark 3.12. The fuzzy ideal of L induced by μ is the **least fuzzy ideal** of L containing μ. It is also the intersection of all the fuzzy ideals of L containing μ.

Theorem 3.13. *Let μ and ν be fuzzy subsets of L. The following properties hold :*

(i) *If $\mu \in FI(L)$, then $\langle \mu \rangle = \mu$;*
(ii) *If $\mu \leq \nu$, then $\langle \mu \rangle \leq \langle \nu \rangle$;*
(iii) *$\langle 0_L \rangle = 0_L$;*

(iv) $\langle 1_L \rangle = 1_L$.

Lemma 3.14. *Let μ be a fuzzy subset of L.*
Then $\mu(x) = \sup\{\alpha \in [0;1]/x \in \mu_\alpha\}$ for all $x \in L$.

Proof. Let $x \in L$ and let $\beta = \sup\{\alpha \in [0;1]/x \in \mu_\alpha\}$. For any $\epsilon > 0$, there is $\alpha_0 \in [0;1]$, such that $\beta - \epsilon < \alpha_0$ and $x \in \mu_{\alpha_0}$. Thus for any $\epsilon > 0$, $\beta - \epsilon < \mu(x)$. i.e., $\beta \le \mu(x)$. Since $\mu(x) \in \{\alpha \in [0;1]/x \in \mu_\alpha\}$, we have $\mu(x) \le \beta$. Then, $\mu(x) = \beta$. □

Let Γ be a subset of $[0,1]$.

Theorem 3.15. *Let $\{I_\alpha/\alpha \in \Gamma\}$ be a collection of ideals of L such that :*

(i) $L = \bigcup_{\alpha\in\Gamma} I_\alpha$.

(ii) $\alpha > \beta$ *implies* $I_\alpha \subseteq I_\beta$ *for all* $\alpha, \beta \in \Gamma$.

Define a fuzzy subset ν of L by $\nu(x) = \sup\{\alpha \in \Gamma/x \in I_\alpha\}$ for all $x \in L$. Then ν is a fuzzy ideal of L.

Proof. It is sufficient to prove that ν_α is an ideal of $\mathcal{L}$, for every $\alpha \in [0;1]$ with $\nu_\alpha \neq \varnothing$.

Let $\alpha \in [0;1]$ and consider the following two cases:
(1) $\alpha = \sup\{\beta \in \Gamma/\beta < \alpha\}$, and (2) $\alpha \neq \sup\{\beta \in \Gamma/\beta < \alpha\}$.
In Case (1) one shows that $\nu_\alpha = \bigcap_{\substack{\beta < \alpha \\ \beta\in\Gamma}} I_\beta$, which is an ideal of L.

For the case (2), there exists $\epsilon > 0$ such that $[\alpha - \epsilon; \alpha] \cap \Gamma = \varnothing$. If $x \in \bigcup_{\substack{\beta \geq \alpha \\ \beta\in\Gamma}} I_\beta$, then $x \in I_\beta$ for some $\beta \ge \alpha$. It follows that $\nu(x) \ge \beta \ge \alpha$ so that $x \in \nu_\alpha$. That is $\bigcup_{\substack{\beta \geq \alpha \\ \beta\in\Gamma}} I_\beta \subseteq \nu_\alpha$. Conversely, if $x \notin \bigcup_{\substack{\beta \geq \alpha \\ \beta\in\Gamma}} I_\beta$, then $x \notin I_\beta$ for all $\beta \ge \alpha$. Which implies that $x \notin I_\beta$ for all $\beta > \alpha - \epsilon$, that is, if $x \in I_\beta$ then $\beta \le \alpha - \epsilon$. Thus $\nu(x) \le \alpha - \epsilon$ and so $x \notin \nu_\alpha$. Consequently $\nu_\alpha = \bigcup_{\substack{\beta \geq \alpha \\ \beta\in\Gamma}} I_\beta$, which is an ideal of $\mathcal{L}$. □

The following theorem shows how to construct the fuzzy ideal induced by a fuzzy subset.

Theorem 3.16. *Let μ be a fuzzy subset of L. Then the fuzzy subset μ^* of L defined by $\mu^*(x) = \sup\{\alpha \in [0;1]/x \in \langle\mu_\alpha\rangle\}$ for all $x \in L$ is the fuzzy ideal of L induced by μ.*

Proof. We first prove that μ^* is a fuzzy ideal of L.

Let $\Gamma = [0;1]$ and consider the family $(\langle\mu_\alpha\rangle)_{\alpha\in\Gamma}$ of ideals of L (unless $\mu_t = \varnothing$). We have:

(i) $L = \bigcup_{\alpha\in\Gamma} \langle\mu_\alpha\rangle$;

(ii) For all α, $\beta \in \Gamma$,

$$\begin{aligned} \alpha \ge \beta &\Rightarrow \mu_\alpha \subseteq \mu_\beta \\ &\Rightarrow \langle\mu_\alpha\rangle \subseteq \langle\mu_\beta\rangle \end{aligned}$$

Thus by Theorem 3.15, μ^* is a fuzzy ideal of L.

For any $x \in L$, let $\beta \in \{\alpha \in [0,1]/x \in \mu_\alpha\}$. Then $x \in \mu_\beta$, and so $x \in \langle\mu_\beta\rangle$. Thus $\beta \in \{\alpha \in [0,1]/x \in \langle\mu_\alpha\rangle\}$. Which implies that $\{\alpha \in [0,1]/x \in \mu_\alpha\} \subseteq \{\alpha \in [0,1]/x \in \langle\mu_\alpha\rangle\}$. Then $\sup\{\alpha \in [0,1]/x \in \mu_\alpha\} \leq \sup\{\alpha \in [0,1]/x \in \langle\mu_\alpha\rangle\}$, i.e., $\mu(x) \leq \mu^*(x)$. Therefore, $\mu \leq \mu^*$.

Finally let ν be a fuzzy ideal of L containing μ. Let $x \in L$, we have

$$\begin{aligned}\mu^*(x) &= \sup\{\alpha \in [0;1]/x \in \langle\mu_\alpha\rangle\}\\ &\leq \sup\{\alpha \in [0;1]/x \in \langle\nu_\alpha\rangle\} \text{ (since } \mu_\alpha \subseteq \nu_\alpha)\\ &= \sup\{\alpha \in [0;1]/x \in \nu_\alpha\} \text{ (since } \langle\nu_\alpha\rangle = \nu_\alpha)\\ &= \nu(x) \text{ (by lemma 3.14).}\end{aligned}$$

Thus $\mu^* \leq \nu$. Hence, μ^* is the least fuzzy ideal of L containing μ. □

So we have proved that $(FI(L), \wedge, \vee, 0_L, 1_L)$ is a bounded lattice, where for μ, $\nu \in FI(L)$, $\mu \wedge \nu$ is the usual intersection and $\mu \vee \nu = \langle\mu \vee \nu\rangle$.

4. Fuzzy prime ideals

In Section 4, we study the concept of fuzzy prime ideals. Among other things, we establish the extension property of fuzzy prime ideal and prove the fuzzy prime ideal Theorem.

Definition 4.1. Let μ be a fuzzy ideal of a BL-algebra L. Then μ is a fuzzy prime ideal of a BL-algebra L if for every $t \in [0;1]$, the $t-$cut set $\mu_t := \{x \in X/\mu(x) \geq t\}$ is either empty or a prime ideal if it is proper.

Now we give an example of prime ideal.

Example 4.2. Let $(X, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra of Example 3.2. Define a fuzzy set μ by $\mu(0) = t_1, \mu(d) = t_2, \mu(a) = \mu(b) = \mu(c) = \mu(e) = \mu(f) = \mu(1) = t_3$ with $0 \leq t_3 < t_2 < t_1 \leq 1$. It is easy to see that μ is a fuzzy prime ideal.

Theorem 4.3. *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and μ a non constant fuzzy ideal of L. Then, μ is a fuzzy prime ideal of L if and only if for every $x, y \in L$, $max(\mu(x), \mu(y)) \geq \mu(x \wedge y)$.*

Proof. Assume that μ is a fuzzy prime ideal. Let $x, y \in L$, and $t := \mu(x \wedge y)$, then $x \wedge y \in \mu_t$, and since μ_t is a prime ideal, we obtain that $x \in \mu_t$ or $y \in \mu_t$. Thus, $\mu(x) \geq t$ or $\mu(y) \geq t$ from which it follows that $max(\mu(x), \mu(y)) \geq \mu(x \wedge y)$.
Conversely, assume that μ is a non constant fuzzy ideal and for every $x, y \in L$, $max(\mu(x), \mu(y)) \geq \mu(x \wedge y)$.
Let $t \in [0,1]$ such that μ_t is non empty, and $\mu_t \neq L$. Let $x, y \in L$ such that $x \wedge y \in \mu_t$, then $\mu(x \wedge y) \geq t$. We apply the hypothesis and obtain $max(\mu(x), \mu(y)) \geq \mu(x \wedge y) \geq t$. So $\mu(x) \geq t$ or $\mu(y) \geq t$, which means $x \in \mu_t$ or $y \in \mu_t$. □

Remark 4.4. Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra and μ a non constant fuzzy ideal of L. μ is a fuzzy prime ideal of L if and only if for every $x, y \in L$,

$$max(\mu(x), \mu(y)) = \mu(x \wedge y).$$

Remark 4.5. Let L be a BL-algebra and I a proper ideal of L. Then I is a prime ideal if and only if it characteristic function μ_I is a fuzzy prime ideal.

Proposition 4.6. *Let* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *be a BL-algebra and* μ *a non constant fuzzy ideal of* L*. Then* μ *is a fuzzy prime ideal of* L *if and only if* $\mu_{\mu(0)}$ *is a prime ideal.*

Proof. First, note that if μ a non constant fuzzy prime ideal, it is clear that $\mu_{\mu(0)}$ is proper; and it follows from the definition that $\mu_{\mu(0)}$ is a prime ideal.

Conversely, assume that $\mu_{\mu(0)}$ is a prime ideal. Let $t \in [0, 1]$ such that μ_t is nonempty, then $\mu_{\mu(0)} \subseteq \mu_t$. It follows from the extension property of prime ideals that μ_t is a prime ideal provided $\mu_t \neq L$. Therefore, μ is a fuzzy prime ideal of L. □

Our next goal is to establish the extension property for fuzzy ideals.

Theorem 4.7. *(Extention theorem of fuzzy prime ideal)*
Suppose that A, B *are a fuzzy ideals such that* $A \subseteq B$ *and* $A(0) = B(0)$*. If* A *is fuzzy prime, then* B *is also fuzzy prime.*

Proof. To prove that B is a fuzzy prime ideal, it suffices to show that for any $t \in [0, 1]$, the t-level subset $B_t = \{x \in X : B(x) \geq t\}$ is prime ideal of L when $B_t \neq \varnothing$ and $B_t \neq L$. Since for any $t \in [0, 1], A_t \subseteq B_t$, we apply the extension property of prime ideals and obtain the result. □

Corollary 4.8. *Let* μ *be a fuzzy prime ideal of a BL-algebra* L *and* $\alpha \in [0, \mu(0)]$*. Then* $\mu \vee \alpha$ *is a fuzzy prime ideal where for* $x \in L$*,* $(\mu \vee \alpha)(x) = \mu(x) \vee \alpha$*.*

Now, we establish the fuzzy prime ideal theorem for BL-algebras.

Proposition 4.9. *(Fuzzy prime ideal Theorem)*
Let μ *be a non constant fuzzy ideal of a BL-algebra* L*. Then , there is a fuzzy prime ideal* λ *such that* $\mu \leq \lambda$*.*

Proof. Assume that μ is a non constant fuzzy ideal of a BL-algebra L. Then $\mu_{\mu(0)}$ is a proper ideal and there exits a prime ideal P of L containing $\mu_{\mu(0)}$. The characteristic function μ_P is a fuzzy prime ideal of L. Setting $\lambda = \mu_P \vee \alpha$ with $\alpha = \underset{x \in L \setminus P}{Sup}\ \mu(x)$, we obtain the result. □

5. Cosets and fuzzy ideals

In this section, we prove that the set of cosets of a fuzzy ideal is a linearly ordered MV-algebra, not just a BL-algebra as it is the case with fuzzy filters.

Definition 5.1. Let μ be a fuzzy ideal of a BL-algebra L and $x \in L$. The fuzzy set $\mu^x : L \rightarrow [0; 1]$ defined by $\mu^x(y) = min(\mu(\bar{x} \otimes y), \mu(\bar{y} \otimes x))$ for any $y \in L$ is called a coset of the fuzzy ideal μ.

Proposition 5.2. *Let* μ *be a fuzzy ideal of a BL-algebra* L *and* $x, y \in L$*. Then* $\mu^x = \mu^y$ *if and only if* $\mu(\bar{x} \otimes y) = \mu(\bar{y} \otimes x) = \mu(0)$*.*

Proof. Let $x, y \in L$ such that $\mu^x = \mu^y$. We have $\mu^x(x) = \mu^y(x)$, that is $\mu(0) = \mu^x(y) = min(\mu(\bar{x} \otimes y), \mu(\bar{y} \otimes x))$. We apply the fact that μ is a fuzzy ideal and obtain $\mu(\bar{x} \otimes y) = \mu(\bar{y} \otimes x) = \mu(0)$.
Conversely, assume that $\mu(\bar{x} \otimes y) = \mu(\bar{y} \otimes x) = \mu(0)$. For any $z \in L$, since $\overline{(\bar{x} \otimes y)} \otimes$

$(\bar{x} \otimes z) = (\bar{x} \to \bar{y}) \otimes (\bar{x} \otimes z) \leq \bar{y} \otimes z$, we have $\mu(\overline{(\bar{x} \otimes y)} \otimes (\bar{x} \otimes z)) \geq \mu(\bar{y} \otimes z)$. We also have $\mu(\bar{x} \otimes z) \geq min(\mu(\bar{x} \otimes y), \mu(\overline{(\bar{x} \otimes y)} \otimes (\bar{x} \otimes z))) \geq min(\mu(0), \mu(\bar{y} \otimes z)) = \mu(\bar{y} \otimes z)$. We can also prove that $\mu(\bar{z} \otimes x) \geq \mu(\bar{z} \otimes y)$. We conclude that $\mu^x(z) = min(\mu(\bar{x} \otimes z), \mu(\bar{z} \otimes x)) \geq min(\mu(\bar{y} \otimes z), \mu(\bar{z} \otimes y)) = \mu^y(z)$. Similarly, we prove that $\mu^y(z) = min(\mu(\bar{y} \otimes z), \mu(\bar{z} \otimes y)) \geq min(\mu(\bar{x} \otimes z), \mu(\bar{z} \otimes x)) = \mu^x(z)$. Hence $\mu^x = \mu^y$ □

Corollary 5.3. *If μ is a fuzzy ideal of a BL-algebra L, then $\mu^x = \mu^y$ if and only if $x \sim_{\mu_{\mu(0)}} y$ where $x \sim_{\mu_{\mu(0)}} y$ if and only if $\bar{x} \otimes y \in \mu_{\mu(0)}$ and $\bar{y} \otimes x \in \mu_{\mu(0)}$.*

Corollary 5.4. *Let μ be a fuzzy ideal of a BL-algebra L, $\mu^x(z) = min(\mu(\bar{x} \otimes y), \mu(\bar{y} \otimes x))$ for all $z \in [y]_{\mu_{\mu(0)}}$, in particular $\mu^x(z) = \mu(x)$ for all $z \in \mu_{\mu(0)}$ where $[y]_{\mu_{\mu(0)}} = \{z \in L : z \sim_{\mu_{\mu(0)}} y\}$.*

Proposition 5.5. *Let μ be a fuzzy ideal of a BL-algebra L and $x, y, a, b \in L$. If $\mu^x = \mu^a$ and $\mu^y = \mu^b$, then $\mu^{x \wedge y} = \mu^{a \wedge b}$, $\mu^{x \vee y} = \mu^{a \vee b}$, $\mu^{x \otimes y} = \mu^{a \otimes b}$,$\mu^{x \to y} = \mu^{a \to b}$.*

Proof. The proof follows from the Corollary and the fact that $x \sim_{\mu_{\mu(0)}} y$ if and only if $\bar{x} \otimes y \in \mu_{\mu(0)}$ and $\bar{y} \otimes x \in \mu_{\mu(0)}$ is a congruence in the BL-algebra L . □

Proposition 5.6. *Let μ be a fuzzy ideal of a BL-algebra L. Let $L/\mu = \{\mu^x : \mu^x \text{ is a coset of } \mu, x \in L\}$ denotes the set of all cosets of μ. For any $\mu^x, \mu^y \in L/\mu$, if we define $\mu^x \wedge \mu^y = \mu^{x \wedge y}$, $\mu^x \vee \mu^y = \mu^{x \vee y}$, $\mu^x \otimes \mu^y = \mu^{x \otimes y}$,$\mu^x \to \mu^y = \mu^{x \to y}$, then $L/\mu = (L/\mu, \wedge, \vee, \otimes, \to, \mu^0, \mu^1)$ is an MV-algebra.*

Proof. Simple computations prove that $L/\mu = (L/\mu, \wedge, \vee, \otimes, \to, \mu^0, \mu^1)$ satisfies the definition of BL-algebras. To prove that it is an MV-algebra, we need to show that for any $x \in L$, $\mu^{\bar{\bar{x}}} = \mu^x$. Let $y \in L$, we have $\mu^{\bar{\bar{x}}}(y) = min(\mu(\bar{\bar{\bar{x}}} \otimes y), \mu(\bar{y} \otimes \bar{\bar{x}})) = min(\mu(\bar{x} \otimes y), \mu(\overline{\overline{\bar{y} \otimes x}})) = min(\mu(\bar{x} \otimes y), \mu(\bar{y} \otimes x)) = \mu^x(y)$. □

Theorem 5.7. *Let μ be a fuzzy ideal of a BL-algebra L. Define a mapping $\phi : L \to L/\mu$ by for any $x \in L$, $\phi(x) = \mu^x$, we have the following results:*

1. *ϕ is a surjective homomorphism .*
2. *$Ker(\phi) = \mu_{\mu(0)}$.*
3. *$L/\mu \cong L/\mu_{\mu(0)}$.*
4. *μ is a fuzzy prime ideal if and only if L/μ is an MV-chain.*

Proof.

1. It is clear from the construction that ϕ is a surjective homomorphism.
2. Follows from the fact that $\mu^x = \mu^y$ if and only if $x \sim_{\mu_{\mu(0)}} y$.
3. By application of the Main Isomorphism Theorem.
4. Follows from the fact that $\mu_{\mu(0)}$ is a prime ideal if and only if μ is a fuzzy prime ideal of a BL-algebra L .

□

6. Fuzzy ideals versus fuzzy filters

In this section, we continue with the algebraic analysis of BL-algebras using the operators D and N.

Definition 6.1. A fuzzy subset f of a BL-algebra $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is called a *fuzzy filter* if it satisfies:

FF1 : $f(1) \geq f(x)$; for every $x \in L$;
FF2 : For every $x, y \in F$, $f(x \otimes y) \geq min(f(x), f(y))$;
FF3 : For every $x, y \in L$, if $x \leq y$, then $f(x) \leq f(y)$.

Definition 6.2. Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a BL-algebra an μ be any fuzzy subset of L. The fuzzy subsets $N(\mu)$ and $D(\mu)$ from $L \rightarrow [0, 1]$ are defined by: for all $x \in L$, $N(\mu)(x) = \mu(\bar{x})$ and $D(\mu)(x) = \mu(\bar{\bar{x}})$ are called the set of complement and the set of double complement of the fuzzy subset μ.

We establish some properties of the operator N.

Theorem 6.3. *Let f be a fuzzy filter and i a fuzzy ideal of L, we have the following results:*

N1. *$N(i)$ is a fuzzy filter.*
N2. *$N(f)$ is a fuzzy ideal.*
N3. *$i = N(N(i))$.*
N4. *$f \leq N(N(f))$.*
N5. *$N(f) = N(N(N(f)))$.*

Proof. N1. Assume that i is a fuzzy ideal. We have $N(i)(1) = i(\bar{1}) = i(0) \geq i(\bar{x}) = N(i)(x)$ and $N(i)$ satisfies $FF1$.
On the other hand, for any $x, y \in L$, Since $\bar{x} \oslash \bar{y} = \bar{x} \oplus \bar{y} = \overline{x \otimes y}$, we have $N(i)(x \otimes y) = i(\overline{x \otimes y}) = i(\bar{x} \oplus \bar{y}) = i(\bar{x} \oslash \bar{y}) \geq min(i(\bar{x}), i(\bar{y})) = min(N(i)(x), N(i)(y))$ and therefore $N(i)$ satisfies $FF2$. Lastly, $N(i)$ satisfies $FF3$ because $\bar{y} \leq \bar{x}$ when $x \leq y$.

N2. Assume that f is a fuzzy filter of L. We have $N(f)(0) = f(\bar{0}) = f(1) \geq f(\bar{x}) = N(f)(x)$. In addition $N(f)$ satisfies $FI2$ because $\bar{y} \leq \bar{x}$ when $x \leq y$. For $FI1$, let $x, y \in L$, we observe that $\bar{x} \otimes \bar{y} = \overline{\overline{\bar{x} \otimes \bar{y}}} = \overline{\bar{\bar{x}} \rightarrow \bar{\bar{y}}} = \overline{\bar{\bar{x}} \rightarrow y} = \overline{x \oslash y}$. Thus, $N(f)(x \oslash y) = f(\overline{x \oslash y}) = f(\bar{x} \otimes \bar{y}) \geq min(f(\bar{x}), f(\bar{y})) = min(N(f)(x), N(f)(y))$. Hence $N(f)$ is a fuzzy ideal.

N3. $N3$ follows from the fact that $i(\bar{\bar{x}}) = i(x)$ for any $x \in L$.

N4. $N4$ comes from the fact that $x \leq \bar{\bar{x}}$ for any $x \in L$.

N5. To obtain $N5$, we use $N3$. by setting $i = N(f)$.

□

Proposition 6.4. *Let f be a fuzzy filter of L. We have the following results:*

D1. *$D(f)$ is a fuzzy filter.*
D2. *$f \leq D(f)$.*
D3. *$f = D(D(f))$.*

Proof. The proof is similar to the above and is omitted. □

It is clear that the operator D is the identity since $D(i) = i$ for any fuzzy ideal i.

Remark 6.5. By combining the properties of the operator D and the operator N, we have the following: $N(f) = D(N(f))$.

7. Conclusion

We have introduced the concept pseudo-addition in BL-algebras that enabled us to analyze some important algebraic properties of BL-algebras. Moreover, we gave a concrete description of the fuzzy ideal generated by a fuzzy subset and construct the quotient BL-algebra via a fuzzy ideal which turned out to be an MV-algebra. For future work, we could use the pseudo addition $\oslash$ to study others related structures such as residuated lattices, MTL-algebras, Pseudo-BL-algebras and some logic consequences.

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