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On the similarity degree of the sequences of fuzzy numbers

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Abstract. In this paper, we have defined some new concepts related to sequences of fuzzy numbers and given some theorems about similarity degree (or degree of equality) of sequence of fuzzy numbers. We think that, this new idea generates new concepts about the sequence of fuzzy numbers.

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1. Introduction and Preliminaries

In the theory of sequences of fuzzy numbers, there is no built-in notion of degree of equality (similarity) among sequences of fuzzy numbers. But we know that the similarity degree is used in pattern recognition, region extraction, decision making, coding theory, image processing and in many other areas, for example [18], [15].

Therefore, we think that, it is important apply to the sequences of fuzzy numbers of this idea.

The similarity degree of two fuzzy sets has been studied and some methods have been proposed for fuzzy sets in [3], [4], [5], [8], [9], [11], [10] and [12]. In addition to these, one can suggest a new similarity degree for fuzzy sets. The methods for similarity degree for fuzzy sets, which is given in [9], [3], [4] are more important but these methods can not measure the similarity degree of any two sequence of fuzzy numbers. Whereas, we know that, in image processing, in pattern recognition and in many other areas, information may be represented by data packs which may be thought as a sequence of datas. These thoughts might help to scientists in the related areas.

Now we will give some definitions and terminology related to this topic.

A fuzzy number u_θ is a fuzzy subset of $\mathbb{R}$ that is bounded, convex, normal and has a compact support, in other word, a fuzzy number is characterized by a membership function $u_\theta : \mathbb{R} \to [0,1]$ and satisfies the following properties:

`FN1.` u_θ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u_\theta(x_0) = 1$,

`FN2.` u_θ is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\mu \in [0,1]$,
$u_\theta[\mu x + (1-\mu)y] \geq \min\{u_\theta(x), u_\theta(y)\}$,

`FN3.` u_θ is upper semi-continuous,

`FN4.` The closure of $\{x \in \mathbb{R} : u_\theta(x) \geq 0\}$, denoted by u^0, is compact, [14].

Although the fuzzy numbers expressed by membership functions, in many resources (for example, [14]), but we will denote a generalized fuzzy number(trapezoid fuzzy number) with $u = (u_1, u_2, u_3, u_4)$. This notation was used in [6], [12]. If $u_2 = u_3$ then the fuzzy number u is called triangular fuzzy number and triangular fuzzy numbers will be denoted shortly $u = (u^1, u^2, u^3)$ in this paper. We denote the set of all triangular fuzzy numbers by E^1.

Let us define the function S as follows,

$$S : E^1 \times E^1 \to \mathbb{R}, S(u,v) = 1 - \frac{\sum_{i=1}^{3} |u^i - v^i|}{3}. \tag{1.1}$$

The function S is called similarity degree between the fuzzy numbers u and v, where, $u = (u^1, u^2, u^3)$, $v = (v^1, v^2, v^3)$, $u^1 < u^2 < u^3$ and $v^1 < v^2 < v^3$. If $S(u,v) = 1$ then we say that u is completely similar to v or vice versa, we say that v is completely similar to u, if $0 < S(u,v) < 1$ then we say that the fuzzy number u is S- similar to the fuzzy number v (or the fuzzy number v is S- similar to the fuzzy number u), if $S(u,v) > 1$ or $S(u,v) \leq 0$ we say that, u is not similar to v. Similar definitions can be found in [11].

The function $f : \mathbb{N} \to E^1, k \to f(k) = (u_k) = ((u_k^1, u_k^2, u_k^3))$ is called sequences of fuzzy numbers. And sequence of fuzzy numbers is represented by (u_k).

Let us denote the set of all sequences of fuzzy numbers by $w(E^1)$ that is

$$w(E^1) := \{(u_k) = ((u_k^1, u_k^2, u_k^3)) : u : \mathbb{N} \to E^1, u(k) = (u_k^1, u_k^2, u_k^3)\}. \tag{1.2}$$

Here u_k^1, u_k^2 and u_k^3 represent first, middle and end points of general term of sequences of triangular fuzzy numbers, respectively and $u_k^1 \leq u_k^2 \leq u_k^3$, for every $k \in \mathbb{N}$. If degree of membership at u_k^2 is equal to 1 then (u_k) is a fuzzy number, if it is not then the sequence (u_k) is a sequence of the fuzzy sets.

Definition 1.1. Let us define the function $\mathbb{S}$ as,

$$\mathbb{S} : w(E^1) \times w(E^1) \to \mathbb{R}, \mathbb{S}(u_, v) = \lim_k (1 - \frac{\sum_{i=1}^{3} |u_k^i - v_k^i|}{3}) = \lambda. \tag{1.3}$$

The function $\mathbb{S}$ is called similarity degree between sequences of fuzzy numbers (u_k) and (v_k). If $\mathbb{S}(u,v) = 1$ then we say that (u_k) is completely similar to the sequence (v_k), if $0 < \mathbb{S}(u,v) = \lambda < 1$ then we say that the sequence (u_k) is λ- similar to the sequence (v_k), if $\lambda > 1$ or $\lambda \leq 0$ we say that, (u_k) is not similar to (v_k).

The set

$$w(E^*, v) = \{(u_k) \in w(E^1) : 0 < \mathbb{S}(u,v) \leq 1, v \in w(E^1)\} \tag{1.4}$$

is called set of similar sequences to the sequence v of fuzzy numbers.

Definition 1.2. A sequence $u = (u_k)$ of fuzzy numbers is said to be convergent to the fuzzy number u_0 if for each $\epsilon > 0$ there exists a positive integer n_0 such that $d(u_k, u_0) < \epsilon$ for all $k \geq n_0$, and we denote it by writing $\lim_k u_k = u_0$, where $d(u_k, u_0) = \sup_k \max\{|u_k^1 - u_0^1|, |u_k^2 - u_0^2|, |u_k^3 - u_0^3|\}$.

Let us denote the spaces of convergent and null sequences spaces of fuzzy numbers by $c(E^1)$ and $c_0(E^1)$, (for more, [1]), respectively.

Proposition 1.3. *Let us suppose that* $(u_k), (v_k) \in c(E^1)$ *and* $\lim_k u_k = u_0$, $\lim_k v_k = v_0$. *If* $0 \leq \frac{\sum_{i=1}^{3} |u_0^i - v_0^i|}{3} < 1$ *then the sequences of fuzzy numbers* (u_k) *and* (v_k) *are similar to each other and the another condition cannot be satisfied.*

Now, in order to understood, we give some examples for similarity degree.

Example 1.4. Given the sequences of generalized fuzzy number $(u_k) = ((2, 3, 4, 5))$ and $(v_k) = ((-5, -4, -3, -2))$. The similarity degree $\mathbb{S}(u, v)$ between (u_k) and (v_k) is equal to -6, in other words, (u_k) and (v_k) are not similar to each other. In fact

$$\lim_{k\to\infty}(1 - \frac{\sum_{i=1}^{4} |u_k^i - v_k^i|}{4}) = \lim_{k\to\infty}(1 - \frac{|2+5| + |3+4| + |4+3| + |5+2|}{4}) = \lim_{k\to\infty}(1 - \frac{28}{4}) = -6 < 0. \tag{1.5}$$

If we consider the Definition1.1, the right-hand side of (1.5) says us, the sequences $(u_k) = ((2, 3, 4, 5))$ and $(v_k) = ((-5, -4, -3, -2))$ are not similar to each other.

Example 1.5. Let us consider the sequences of fuzzy numbers $(u_k) = (\frac{0.8}{k}, \frac{1}{k}, \frac{1}{k}, \frac{1.2}{k})$ and $(v_k) = (\frac{0.6}{k}, \frac{0.8}{k}, \frac{0.8}{k}, \frac{1}{k})$. The similarity degree $\mathbb{S}(u, v)$ between the sequences $(u_k) = (\frac{0.8}{k}, \frac{1}{k}, \frac{1}{k}, \frac{1.2}{k})$ and $(v_k) = (\frac{0.6}{k}, \frac{0.8}{k}, \frac{0.8}{k}, \frac{1}{k})$ is

$$\begin{aligned}\lim_k S(u, v) &= \lim_k(1 - \frac{\sum_{i=1}^{4} |u_i^k - v_i^k|}{4}) \\ &= \lim_k(1 - \frac{(|\frac{0.8-0.6}{k}| + |\frac{1-0.8}{k}| + |\frac{1-0.8}{k}| + |\frac{1.2-1}{k}|)}{4}) \\ &= \lim_k(1 - \frac{|\frac{0.8}{k}|}{4}) = \lim_k(1 - \frac{1}{5k}) = 1.\end{aligned} \tag{1.6}$$

The right-hand side of (1.6) says to us, the sequences $(u_k) = (\frac{0.8}{k}, \frac{1}{k}, \frac{1}{k}, \frac{1.2}{k})$ and $(v_k) = (\frac{0.6}{k}, \frac{0.8}{k}, \frac{0.8}{k}, \frac{1}{k})$ are completely similar to each other and the similarity degree $\mathbb{S}(u, v) = 1$.

A defuzzification is a process to get a non- fuzzy control action that best represents the possibility distribution of an inferred fuzzy control action. Defuzzification has the result of reducing a fuzzy set to a crisp set; of converting a fuzzy matrix to a crisp matrix; or of making a fuzzy number a crisp number. Furthermore the primary focus of the defuzzification method has been used to explain the process of converting from fuzzy membership function to crisp version. Defuzzification is the conversion of a fuzzy quantity to a precise quantity, just as fuzzification is the conversion, [13]. The more datail about defuzzification can be found in [7]. Sometimes defuzzification is called gravity center of a fuzzy set. Defuzzification is a natural and required process.

In fact, there is an analogous form of defuzzification in mathematics where we solve a complicated problem in the complex plane, find the real and imaginary parts of the solution, then decomplexify the imaginary solution back to the real numbers space [2]. There are numerous other methods for defuzzification that have not been presented here. A review of the literature will provide the details on some of these (see, for example, [6], [16]).

The gravity center of a generalized fuzzy number $u = (u_1, u_2, u_3, u_4)$ is defined by $(r^u, \frac{1}{3})$, where

$$r^u = \frac{\frac{1}{3}(u_3 + u_2) + \frac{2}{3}(u_4 + u_1)}{2}, \tag{1.7}$$

[3], [4].

If u is a triangular fuzzy number then the (1.7) reduces to $r^u = \frac{u_1+u_2+u_3}{3}$.

The gravity center of the sequences of fuzzy numbers has been studied by Zararsız and Şengönül in [17].

2. Main Results

Theorem 2.1. *Let us suppose that $u = (u_k), v = (v_k) \in w(E^1)$. If (u_k) and (v_k) are completely similar to each other, that is $\lim_k \mathbb{S}(u, v) = 1$ then $\lim_k(r_k^u - r_k^v) = 0$, where r_k^u and r_k^v are apsis of the gravity centers of the terms of the sequences $u = (u_k), v = (v_k)$, for every $k \in \mathbb{N}$.*

Proof. If $(r_k^u, \frac{1}{3})$ and $(r_k^v, \frac{1}{3})$ be the gravity centers of the sequences of fuzzy numbers $(u_k) = ((u_k^1, u_k^2, u_k^3))$ and $(v_k) = ((v_k^1, v_k^2, v_k^3))$, respectively; then we have

$$\begin{aligned}\mathbb{S}(u_k, v_k) &= \lim_{k\to\infty} (1 - \frac{\sum_{i=1}^3 |u_k^i - v_k^i|}{3}) \\ &= \lim_{k\to\infty} (1 - \frac{|u_k^1 - v_k^1| + |u_k^2 - v_k^2| + |u_k^3 - v_k^3|}{3}) = 1.\end{aligned}$$

If we consider hypothesis of the theorem we deduce that $\lim_{k\to\infty}(|u_k^1 - v_k^1| + |u_k^2 - v_k^2| + |u_k^3 - v_k^3|) = 0$. Therefore we can write $\lim_{k\to\infty}(u_k^1 - v_k^1) = 0$, $\lim_{k\to\infty}(u_k^2 - v_k^2) = 0$ and $\lim_{k\to\infty}(u_k^3 - v_k^3) = 0$. From the equalities $(r_k^u) = (\frac{u_k^1+u_k^2+u_k^3}{3})$ and $(r_k^v) = (\frac{v_k^1+v_k^2+v_k^3}{3})$ we see that

$$\lim_{k\to\infty} (r_k^u - r_k^v) = 0 \tag{2.1}$$

and this completes the proof. □

Theorem 2.2. *Let us suppose that the sequence of fuzzy numbers (u_k) be λ_1- similar to the sequence of fuzzy numbers (v_k) and the sequence of fuzzy number (t_k) be λ_2 - similar to (v_k) and the sequence of fuzzy numbers $(t_k) \in c_0(E^1)$. Then the similarity degree of the sequence $(u_k + t_k)$ to the sequence (v_k) is equal to or bigger than λ_1.*

Proof. If the sequence of fuzzy numbers (u_k), λ_1 - similar to (v_k) then we have $\lim_k(1 - \frac{\sum_{i=1}^3 |u_i^k - v_i^k|}{3}) = \lambda_1$ and if (t_k), λ_2 - similar to (v_k) also we can write

$\lim_k(1-\frac{\sum_{i=1}^{3}|t_i^k-v_i^k|}{3})=\lambda_2$. Then we write

$$\begin{aligned}\lim_k(1-\frac{\sum_{i=1}^{3}|(u_i^k+t_i^k)-v_i^k|}{3}) &= \lim_k(1-\frac{\sum_{i=1}^{3}|(u_i^k-v_i^k)+t_i^k|}{3})\\ &\geq \lim_k(1-\frac{\sum_{i=1}^{3}(|u_i^k-v_i^k|+|t_i^k|)}{3})\\ &= \lim_k(1-\frac{\sum_{i=1}^{3}|u_i^k-v_i^k|}{3}-\frac{\sum_{i=1}^{3}|t_i^k|}{3})\\ &= \lim_k(1-\frac{\sum_{i=1}^{3}|u_i^k-v_i^k|}{3})-\lim_k(\frac{\sum_{i=1}^{3}|t_i^k|}{3})\\ &= \lambda_1-\frac{1}{3}\lim_k(|t_1^k|+|t_2^k|+|t_3^k|)\\ &= \lambda_1-\frac{1}{3}.0=\lambda_1\end{aligned}$$

and this completes the proof. □

Theorem 2.3. *Let $(u_k),(v_k)\in w(E^1)$ and $\mathbb{S}(u,v)=\lambda$.*

(1) *For the sequences (u_k) and (v_k), if $u_k=v_k$, for all $k\in\mathbb{N}$ then $\lambda=1$*
(2) *Let suppose that $(u_k),(v_k)\in c(E)$ and $u_k\to u_0$, $v_k\to v_0$ for $k\to\infty$. If $u_0\neq v_0$ then $\lambda\neq 1$.*

Proof. (1) Let us suppose that $u_k=v_k$ for the sequences of fuzzy numbers (u_k) and (v_k),$\forall k\in\mathbb{N}$. Then we immediately see that

$$\mathbb{S}(u,v)=\lim_k(1-\frac{\sum_{i=1}^{3}|u_k^i-u_k^i|}{3})=1. \tag{2.2}$$

(2) Let suppose that $(u_k),(v_k)\in c(E)$ and $u_k\to u_0$, $v_k\to v_0$ for $k\to\infty$ and $u_0\neq v_0$. Let us consider to the

$$1-\frac{\sum_{i=1}^{3}|u_k^i-v_k^i|}{3}. \tag{2.3}$$

If we pass to limit for $k\to\infty$ in the (2.3) we see that $\mathbb{S}(u,v)\neq 1$ dir.

□

Theorem 2.4. *The sequence of fuzzy numbers (u_k) be λ_1 - similar to (v_k) and the sequence of fuzzy numbers (t_k) be λ_2 - similar to (v_k) too. Then the similarity degree of the sequence (u_k) to (t_k) is $\lambda_1+\lambda_2-1$.*

Proof. Under the hypothesis of the present Theorem, if

$$\mathbb{S}(u,v)=\lim_k(1-\frac{\sum_{i=1}^{3}|u_i^k-v_i^k|}{3})=\lambda_1 \text{ and } \mathbb{S}(t,v)=\lim_k(1-\frac{\sum_{i=1}^{3}|t_i^k-v_i^k|}{3})=\lambda_2$$

then we write

$$\begin{aligned}\lim_k(1-\frac{\sum_{i=1}^3|u_i^k-t_i^k|}{3}) &\geq \lim_k(1-\frac{\sum_{i=1}^3|u_i^k-v_i^k|+\sum_{i=1}^3|v_i^k-t_i^k|}{3})\\ &=\lim_k(1-\frac{\sum_{i=1}^3|u_i^k-v_i^k|}{3})-\lim_k(\frac{\sum_{i=1}^3|v_i^k-t_i^k|}{3})\\ &=\lambda_1-(1-\lambda_2)\\ &=\lambda_1+\lambda_2-1\end{aligned}$$

which completes the proof. □

Now we will give a theorem about inclusion. Let us denote the set of bounded sequences set of the sequence of fuzzy numbers by $\ell_\infty(E^1)$,(for more, [1]).

Theorem 2.5. *The following statements are hold:*

(1) *If $u=(u_k)\notin\ell_\infty(E^1)$ then the inclusion $\ell_\infty(E^1)\subset w(E^*,u)$ holds,*
(2) *If $u=(u_k)\in\ell_\infty(E^1)$ then the inclusion $\ell_\infty(E^1)\supset w(E^*,u)$ holds,*
(3) *If $u=(u_k)\in c(E^1)$ then the inclusion $c(E^1)\subset w(E^*,u)$ holds.*

Proof. The proof is easily obtained by the definition of the sets $\ell_\infty(E^1)$, $c(E^1)$ and $w(E^*,u)$. □

Proposition 2.6. *Let $w(E^2)=w(E^*,u)\cap\ell_\infty(E^1)$. Then the set $w(E^2)$ is complete metric space with the metric defined by*

$$\overline{d}(u,v)=\sup_k d(u_k,v_k). \tag{2.4}$$

Similarly to above proposition, we have:

Proposition 2.7. *Let $w_c(E^2)=w(E^*,u)\cap c(E^1)$. Then the set $w_c(E^2)$ is complete metric space with the metric defined by (2.4).*

The proof of Proposition 2.6 and Proposition 2.7 are clear, so we omit it.

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