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Representation of trapezoidal fuzzy numbers with shape function

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Abstract. In this paper, the representation of fuzzy numbers with shape function in α-cut has been proved. Also, we shows how to treat the defuzzification, ranking and distance of fuzzy numbers with shape function by modified concept of Graded Mean Integration Representation method.

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1. Introduction

In 1965, Zadeh [14] introduced the concept of fuzzy set theory to meet those problems. In 1978, Dubois and Prade [4] defined any of the fuzzy numbers as a fuzzy subset of the real line. The graded mean integration representation of generalized fuzzy number was introduced, by Chen and Hsieh [1], it also had been compared with some other different methods for representation with several different representation methods. He find that the graded mean representation not only can treat n generalized fuzzy numbers, but also do not change the results of representation values after increase (or decrease) a generalized fuzzy number into (from) original generalized fuzzy numbers group. Chen and Chen [2] presented a method for ranking generalized trapezoidal fuzzy numbers. Rezvani [5]-[12] introduced ranking of fuzzy numbers and Yong Sik Yun [13] presented a method for generalized triangular fuzzy sets. The method for representation of multiplication operation on fuzzy numbers was proposed by Ch-Ch Chou [3]. We would like to counter their argument by proving that α-cut method is general enough to deal with different type of fuzzy arithmetic including exponentiation, extracting nth root, taking logarithm. In fact we illustrate with examples to show that α-cut method is simpler than their proposed method. However we do acknowledge that the proposed method has more mathematical beauty than the existing alpha-cut method.

In this paper, the representation of fuzzy numbers with shape function in α-cut has been proved. Also, we shows how to treat the defuzzification, ranking and distance of fuzzy numbers with shape function by modified concept of Graded Mean Integration Representation method.

2. Preliminaries

Definition 2.1. Generally, a generalized fuzzy number A is described as any fuzzy subset of the real line R, whose membership function $\mu_A(x)$ satisfies the following conditions,

(i) $\mu_A(x)$ is a continuous mapping from R to the closed interval $[0, w]$, $0 < w \leq 1$,
(ii) $\mu_A(x) = 0$, for all $x \in (-\infty, a]$,
(iii) $\mu_L(x) = L(x)$ is strictly increasing on $[a, b]$,
(iv) $\mu_A(x) = w$, for all $[b, c]$, as w is a constant and $0 < w \leq 1$,
(v) $\mu_R(x) = R(x)$ is strictly decreasing on $[c, d]$,
(vi) $\mu_A(x) = 0$, for all $x \in [d, \infty)$,
where a,b,c,d are real numbers such that $a < b \leq c < d$.

Definition 2.2. A normal fuzzy number A with shape function

$$\mu_A = \begin{cases} (\frac{x-a}{b-a})^n & when \;\; x \in [a, b), \\ w & when \;\; x \in [b, c] \\ (\frac{d-x}{d-c})^n & when \;\; x \in (c, d] \\ 0 & otherwise \end{cases} \tag{2.1}$$

where $n > 0$, will be denoted by

$$A = (a, b, c, d)_n \;. \tag{2.2}$$

If A be non-normal fuzzy number, it will be denoted by

$$A = (a, b, c, d; w)_n \;. \tag{2.3}$$

If $n = 1$, we simply write $A = (a, b, c, d)$, which is known as a normal trapezoidal fuzzy number.

Definition 2.3. The set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is said to be the α-cut of a fuzzy set A.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its α-cut according to the formula

$$\mu_A(x) = sup_{\alpha \in (0,1]} \; min(\alpha, \mu_{A_\alpha}(x)), \tag{2.4}$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & if \quad x \in A_\alpha \\ 0 & if \quad otherwise. \end{cases} \tag{2.5}$$

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number and A_α, B_α are the α-cuts of A and B, respectively, we have

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \;, \tag{2.6}$$

and

$$B_\alpha = [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)] \ , \tag{2.7}$$

2.1. **Arithmetic Operations.**

In this section, addition and subtraction between two trapezoidal fuzzy numbers, defined on universal set of real numbers R.

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number, then

i) $A \oplus B = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$
ii) $A \ominus B = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$.

3. Addition of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \;\; x \in [a_1, b_1), \\ 1 & when \;\; x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \;\; x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{3.1}$$

$$\mu_B = \begin{cases} (\frac{x-a_2}{b_2-a_2})^n & when \;\; x \in [a_2, b_2), \\ 1 & when \;\; x \in [b_2, c_2] \\ (\frac{d_2-x}{d_2-c_2})^n & when \;\; x \in (c_2, d_2] \\ 0 & otherwise \end{cases} \tag{3.2}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{3.3}$$

and

$$B_\alpha = [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)] \ . \tag{3.4}$$

Therefore A_α, B_α are the α-cuts of A and B, respectively. To calculate addition of fuzzy numbers A and B we first add the α-cuts of A and B using interval arithmetic.

$$\begin{aligned} A_\alpha + B_\alpha = & [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] + \\ & [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)] \\ = & [(a_1 + a_2) + \alpha^{\frac{1}{n}}(b_1 + b_2 - a_1 - a_2), \\ & (d_1 + d_2) - \alpha^{\frac{1}{n}}(d_1 + d_2 - c_1 - c_2)]. \end{aligned} \tag{3.5}$$

Now, we find the shape function $\mu_{A+B}(x)$

$$x = (a_1 + a_2) + \alpha^{\frac{1}{n}}(b_1 + b_2 - a_1 - a_2)$$

$$\Rightarrow \alpha = \frac{(x-(a_1+a_2))^n}{(b_1+b_2)-(a_1+a_2)} \quad a_1 + a_2 \leq x \leq b_1 + b_2 \tag{3.6}$$

and

$$x = ((d_1 + d_2) - \alpha^{\frac{1}{n}}(d_1 + d_2 - c_1 - c_2)$$

$$\Rightarrow \alpha = \frac{((d_1+d_2)-x)^n}{(d_1+d_2)-(c_1+c_2)} \quad c_1 + c_2 \leq x \leq d_1 + d_2 \tag{3.7}$$

Therefore shape function $\mu_{A+B}(x)$ is

$$\mu_{A+B} = \begin{cases} \frac{(x-(a_1+a_2))^n}{(b_1+b_2)-(a_1+a_2)} & a_1 + a_2 \leq x \leq b_1 + b_2 \\ 1 & b_1 + b_2 \leq x \leq c_1 + c_2 \\ \frac{((d_1+d_2)-x)^n}{(d_1+d_2)-(c_1+c_2)} & c_1 + c_2 \leq x \leq d_1 + d_2 \\ 0 & otherwise \end{cases} \tag{3.8}$$

Example 3.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ and $B = (0.15, 0.2, 0.3, 0.4)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when \;\; 0.2 \leq x \leq 0.35, \\ 1 & when \;\; 0.33 \leq x \leq 0.45 \\ (\frac{0.5-x}{0.05})^n & when \;\; 0.45 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$

$$\mu_B = \begin{cases} (\frac{x-0.15}{0.05})^n & when \;\; 0.15 \leq x \leq 0.2, \\ 1 & when \;\; 0.2 \leq x \leq 0.3 \\ (\frac{0.4-x}{0.1})^n & when \;\; 0.3 \leq x \leq 0.4 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \; ,$$

and

$$B_\alpha = [0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}] \; .$$

Therefore

$$A_\alpha + B_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] + [0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}]$$

$$= [0.35 + 0.18\alpha^{\frac{1}{n}}, 0.9 - 0.15\alpha^{\frac{1}{n}}]$$

Now, we get

$$x = 0.35 + 0.18\alpha^{\frac{1}{n}} \Rightarrow \alpha = \frac{(x-0.35)^n}{0.18} \quad 0.35 \leq x \leq 0.53$$

and

$$x = 0.9 - 0.15\alpha^{\frac{1}{n}} \Rightarrow \alpha = \frac{(0.9-x)^n}{0.15} \quad 0.75 \leq x \leq 0.9$$

which gives

$$\mu_{A+B} = \begin{cases} \frac{(x-0.35)^n}{0.18} & when \;\; 0.35 \leq x \leq 0.53, \\ 1 & when \;\; 0.53 \leq x \leq 0.75 \\ \frac{(0.9-x)^n}{0.15} & when \;\; 0.75 \leq x \leq 0.9 \\ 0 & otherwise \end{cases}$$

4. Subtraction of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \;\; x \in [a_1, b_1), \\ 1 & when \;\; x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \;\; x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{4.1}$$

$$\mu_B = \begin{cases} (\frac{x-a_2}{b_2-a_2})^n & when \;\; x \in [a_2, b_2), \\ 1 & when \;\; x \in [b_2, c_2] \\ (\frac{d_2-x}{d_2-c_2})^n & when \;\; x \in (c_2, d_2] \\ 0 & otherwise \end{cases} \tag{4.2}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \; , \tag{4.3}$$

and

$$B_\alpha = [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)] \; . \tag{4.4}$$

Therefore A_α, B_α are the α-cuts of A and B, respectively. To calculate subtraction of fuzzy numbers A and B we first add the α-cuts of A and B using interval arithmetic.

$$A_\alpha - B_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] -$$
$$[a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)]$$
$$(4.5) \quad = [(a_1 - d_2) + \alpha^{\frac{1}{n}}(b_1 - a_1 + d_2 - c_2),$$
$$(d_1 - a_2) - \alpha^{\frac{1}{n}}(d_1 - c_1 + b_2 - a_2)].$$

Now, we find the shape function $\mu_{A-B}(x)$

$$x = (a_1 - d_2) + \alpha^{\frac{1}{n}}(b_1 - a_1 + d_2 - c_2)$$

$$(4.6) \qquad \Rightarrow \alpha = \frac{(x-(a_1-d_2))^n}{(b_1-a_1+d_2-c_2)} \quad a_1 - d_2 \le x \le b_1 - c_2$$

and

$$x = (d_1 - a_2) - \alpha^{\frac{1}{n}}(d_1 - c_1 + b_2 - a_2)$$

$$(4.7) \qquad \Rightarrow \alpha = \frac{((d_1-a_2)-x)^n}{(d_1-c_1+b_2-a_2)} \quad c_1 - b_2 \le x \le d_1 - a_2$$

Therefore shape function $\mu_{A-B}(x)$ is

$$(4.8) \qquad \mu_{A-B} = \begin{cases} \frac{(x-(a_1-d_2))^n}{(b_1-a_1+d_2-c_2)} & a_1 - d_2 \le x \le b_1 - c_2 \\ 1 & b_1 - c_2 \le x \le c_1 - b_2 \\ \frac{((d_1-a_2)-x)^n}{(d_1-c_1+b_2-a_2)} & c_1 - b_2 \le x \le d_1 - a_2 \\ 0 & otherwise \end{cases}$$

Example 4.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ and $B = (0.15, 0.2, 0.3, 0.4)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when \ \ 0.2 \le x \le 0.35, \\ 1 & when \ \ 0.33 \le x \le 0.45 \\ (\frac{0.5-x}{0.05})^n & when \ \ 0.45 \le x \le 0.5 \\ 0 & otherwise \end{cases}$$

$$\mu_B = \begin{cases} (\frac{x-0.15}{0.05})^n & when \ \ 0.15 \le x \le 0.2, \\ 1 & when \ \ 0.2 \le x \le 0.3 \\ (\frac{0.4-x}{0.1})^n & when \ \ 0.3 \le x \le 0.4 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \ ,$$

and

$$B_\alpha = [0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}] \ .$$

Therefore

$$A_\alpha - B_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] - [0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}]$$
$$= [0.23\alpha^{\frac{1}{n}} - 0.2, 0.35 - 0.1\alpha^{\frac{1}{n}}]$$

Now, we get

$$x = 0.23\alpha^{\frac{1}{n}} - 0.2 \Rightarrow \alpha = \tfrac{(x-0.2)^n}{0.23} \qquad -0.2 \leq x \leq 0.03$$

and

$$x = 0.35 - 0.1\alpha^{\frac{1}{n}} \Rightarrow \alpha = \tfrac{(0.0.35-x)^n}{0.1} \qquad 0.25 \leq x \leq 0.35$$

which gives

$$\mu_{A-B} = \begin{cases} \frac{(x-0.35)^n}{0.18} & when \ -0.2 \leq x \leq 0.03, \\ 1 & when \ 0.03 \leq x \leq 0.25 \\ \frac{(0.9-x)^n}{0.15} & when \ 0.25 \leq x \leq 0.35 \\ 0 & otherwise \end{cases}$$

5. Multiplication of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \ x \in [a_1, b_1), \\ 1 & when \ x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \ x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{5.1}$$

$$\mu_B = \begin{cases} (\frac{x-a_2}{b_2-a_2})^n & when \ x \in [a_2, b_2), \\ 1 & when \ x \in [b_2, c_2] \\ (\frac{d_2-x}{d_2-c_2})^n & when \ x \in (c_2, d_2] \\ 0 & otherwise \end{cases} \tag{5.2}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{5.3}$$

and

(5.4) $$B_\alpha = [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)]\ .$$

Therefore A_α, B_α are the α-cuts of A and B, respectively. To calculate multiplication of fuzzy numbers A and B we first add the α-cuts of A and B using interval arithmetic.

$$A_\alpha.B_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)].[a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)]$$

$$= [\alpha^{\frac{2}{n}}(b_1 - a_1)(b_2 - a_2) + \alpha^{\frac{1}{n}}(a_1(b_2 - a_2) + a_2(b_1 - a_1)) + a_1a_2,$$

(5.5) $$\alpha^{\frac{2}{n}}(d_1 - c_1)(d_2 - c_2) - \alpha^{\frac{1}{n}}(d_1(d_2 - c_2) + d_2(d_1 - c_1)) + d_1d_2]\ .$$

Now, we find the shape function $\mu_{A.B}(x)$

$$x = \alpha^{\frac{2}{n}}(b_1 - a_1)(b_2 - a_2) + \alpha^{\frac{1}{n}}(a_1(b_2 - a_2) + a_2(b_1 - a_1)) + a_1a_2$$

$$\Rightarrow \alpha = \left[\frac{-(a_1(b_2-a_2)+a_2(b_1-a_1))+\sqrt{(a_1(b_2-a_2)+a_2(b_1-a_1))^2-4(b_1-a_1)(b_2-a_2)(a_1a_2-x)}}{2(b_1-a_1)(b_2-a_2)}\right]^n$$

(5.6) $$a_1a_2 \le x \le b_1b_2$$

and

$$x = \alpha^{\frac{2}{n}}(d_1 - c_1)(d_2 - c_2) - \alpha^{\frac{1}{n}}(d_1(d_2 - c_2) + d_2(d_1 - c_1)) + d_1d_2$$

$$\Rightarrow \alpha =$$

$$\left[\frac{(d_1(d_2-c_2)+d_2(d_1-c_1))+\sqrt{(d_1(d_2-c_2)+d_2(d_1-c_1))^2-4(d_1-c_1)(d_2-c_2)(d_1d_2-x)}}{2(d_1-c_1)(d_2-c_2)}\right]^n$$

(5.7) $$c_1c_2 \le x \le d_1d_2$$

Therefore shape function $\mu_{A.B}(x)$ is

(5.8)

$$\mu_{A.B} = \begin{cases} \left[\frac{-(a_1(b_2-a_2)+a_2(b_1-a_1))+\sqrt{(a_1(b_2-a_2)+a_2(b_1-a_1))^2-4(b_1-a_1)(b_2-a_2)(a_1a_2-x)}}{2(b_1-a_1)(b_2-a_2)}\right]^n \\ \qquad\qquad when\ \ a_1a_2 \le x \le b_1b_2 \\ 1 \qquad\qquad b_1b_2 \le x \le c_1c_2 \\ \left[\frac{(d_1(d_2-c_2)+d_2(d_1-c_1))+\sqrt{(d_1(d_2-c_2)+d_2(d_1-c_1))^2-4(d_1-c_1)(d_2-c_2)(d_1d_2-x)}}{2(d_1-c_1)(d_2-c_2)}\right]^n \\ \qquad\qquad when\ \ c_1c_2 \le x \le d_1d_2 \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

Example 5.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ and $B = (0.15, 0.2, 0.3, 0.4)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when \;\; 0.2 \le x \le 0.35, \\ 1 & when \;\; 0.33 \le x \le 0.45 \\ (\frac{0.5-x}{0.05})^n & when \;\; 0.45 \le x \le 0.5 \\ 0 & otherwise \end{cases}$$

$$\mu_B = \begin{cases} (\frac{x-0.15}{0.05})^n & when \;\; 0.15 \le x \le 0.2, \\ 1 & when \;\; 0.2 \le x \le 0.3 \\ (\frac{0.4-x}{0.1})^n & when \;\; 0.3 \le x \le 0.4 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \;,$$

and

$$B_\alpha = [0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}] \;.$$

Therefore

$$A_\alpha.B_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}].[0.15 + 0.05\alpha^{\frac{1}{n}}, 0.4 - 0.1\alpha^{\frac{1}{n}}]$$

$$= [0.03 + 0.0295\alpha^{\frac{1}{n}} + 0.0065\alpha^{\frac{2}{n}}, 0.2 - 0.07\alpha^{\frac{1}{n}} + 0.005\alpha^{\frac{2}{n}}]$$

Now, we get

$$x = 0.03 + 0.0295\alpha^{\frac{1}{n}} + 0.0065\alpha^{\frac{2}{n}} \Rightarrow \alpha = [\frac{-0.0295+\sqrt{0.00009025+0.026x}}{0.013}]^n$$

$$0.03 \le x \le 0.066$$

and

$$x = 0.2 - 0.07\alpha^{\frac{1}{n}} + 0.005\alpha^{\frac{2}{n}} \Rightarrow \alpha = [\frac{0.07+\sqrt{0.0009+0.02x}}{0.01}]^n$$

$$0.135 \le x \le 0.2$$

which gives

$$\mu_{A.B} = \begin{cases} [\frac{-0.0295+\sqrt{0.00009025+0.026x}}{0.013}]^n & when \;\; 0.03 \le x \le 0.066, \\ 1 & when \;\; 0.066 \le x \le 0.135 \\ [\frac{0.07+\sqrt{0.0009+0.02x}}{0.01}]^n & when \;\; 0.135 \le x \le 0.2 \\ 0 & otherwise \end{cases}$$

6. Division of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \;\; x \in [a_1, b_1), \\ 1 & when \;\; x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \;\; x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{6.1}$$

$$\mu_B = \begin{cases} (\frac{x-a_2}{b_2-a_2})^n & when \;\; x \in [a_2, b_2), \\ 1 & when \;\; x \in [b_2, c_2] \\ (\frac{d_2-x}{d_2-c_2})^n & when \;\; x \in (c_2, d_2] \\ 0 & otherwise \end{cases} \tag{6.2}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \;, \tag{6.3}$$

and

$$B_\alpha = [a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)] \;. \tag{6.4}$$

Therefore A_α, B_α are the α-cuts of A and B, respectively. To calculate division of fuzzy numbers A and B we first add the α-cuts of A and B using interval arithmetic.

$$A_\alpha/B_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)]/[a_2 + \alpha^{\frac{1}{n}}(b_2 - a_2), d_2 - \alpha^{\frac{1}{n}}(d_2 - c_2)]$$

$$= [\frac{a_1+\alpha^{\frac{1}{n}}(b_1-a_1)}{d_2-\alpha^{\frac{1}{n}}(d_2-c_2)}, \frac{d_1-\alpha^{\frac{1}{n}}(d_1-c_1)}{a_2+\alpha^{\frac{1}{n}}(b_2-a_2)}] \;. \tag{6.5}$$

Now, we find the shape function $\mu_{A/B}(x)$

$$x = \frac{a_1+\alpha^{\frac{1}{n}}(b_1-a_1)}{d_2-\alpha^{\frac{1}{n}}(d_2-c_2)}$$

$$\Rightarrow \alpha = [\frac{d_2x-a_1}{(b_1-a_1)+x(d_2-c_2)}]^n \qquad a_1/d_2 \le x \le b_1/c_2 \tag{6.6}$$

and

$$x = \frac{d_1-\alpha^{\frac{1}{n}}(d_1-c_1)}{a_2+\alpha^{\frac{1}{n}}(b_2-a_2)}$$

$$\Rightarrow \alpha = [\frac{d_1-a_2x}{x(b_2-a_2)+(d_1-c_1)}]^n \qquad c_1/b_2 \le x \le d_1/a_2 \tag{6.7}$$

Therefore shape function $\mu_{A/B}(x)$ is

$$\mu_{A/B}=\begin{cases}[\frac{d_2x-a_1}{(b_1-a_1)+x(d_2-c_2)}]^n & a_1/d_2\leq x\leq b_1/c_2\\ 1 & b_1/c_2\leq x\leq c_1/b_2\\ [\frac{d_1-a_2x}{x(b_2-a_2)+(d_1-c_1)}]^n & c_1/b_2\leq x\leq d_1/a_2\\ 0 & otherwise\end{cases}\tag{6.8}$$

Example 6.1. Let $A=(0.2,0.33,0.45,0.5)$ and $B=(0.15,0.2,0.3,0.4)$ be two trapezoidal fuzzy number, Suppose the normal shape function of A,B is

$$\mu_A=\begin{cases}(\frac{x-0.2}{0.13})^n & when\ \ 0.2\leq x\leq 0.35,\\ 1 & when\ \ 0.33\leq x\leq 0.45\\ (\frac{0.5-x}{0.05})^n & when\ \ 0.45\leq x\leq 0.5\\ 0 & otherwise\end{cases}$$

$$\mu_B=\begin{cases}(\frac{x-0.15}{0.05})^n & when\ \ 0.15\leq x\leq 0.2,\\ 1 & when\ \ 0.2\leq x\leq 0.3\\ (\frac{0.4-x}{0.1})^n & when\ \ 0.3\leq x\leq 0.4\\ 0 & otherwise\end{cases}$$

$$A_\alpha=[0.2+0.13\alpha^{\frac{1}{n}},0.5-0.05\alpha^{\frac{1}{n}}]\ ,$$

and

$$B_\alpha=[0.15+0.05\alpha^{\frac{1}{n}},0.4-0.1\alpha^{\frac{1}{n}}]\ .$$

Therefore

$$A_\alpha/B_\alpha=[0.2+0.13\alpha^{\frac{1}{n}},0.5-0.05\alpha^{\frac{1}{n}}]/[0.15+0.05\alpha^{\frac{1}{n}},0.4-0.1\alpha^{\frac{1}{n}}]$$

$$=[\frac{0.2+0.13\alpha^{\frac{1}{n}}}{0.4-0.1\alpha^{\frac{1}{n}}},\frac{0.5-0.05\alpha^{\frac{1}{n}}}{0.15+0.05\alpha^{\frac{1}{n}}}]$$

Now, we get

$$x=\frac{0.2+0.13\alpha^{\frac{1}{n}}}{0.4-0.1\alpha^{\frac{1}{n}}}\Rightarrow\alpha=[\frac{0.4x-0.2}{0.13+0.1x}]^n\quad 0.5\leq x\leq 1.1$$

and

$$x=\frac{0.5-0.05\alpha^{\frac{1}{n}}}{0.15+0.05\alpha^{\frac{1}{n}}}\Rightarrow\alpha=[\frac{10-3x}{x+1}]^n\quad 2.25\leq x\leq 3.33$$

which gives

$$\mu_{A/B} = \begin{cases} [\frac{0.4x-0.2}{0.13+0.1x}]^n & when \ \ 0.5 \leq x \leq 1.1, \\ 1 & when \ \ 1.1 \leq x \leq 2.25 \\ [\frac{10-3x}{x+1}]^n & when \ \ 2.25 \leq x \leq 3.33 \\ 0 & otherwise \end{cases}$$

7. Inverse of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

(7.1) $$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \ \ x \in [a_1, b_1), \\ 1 & when \ \ x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \ \ x \in (c_1, d_1] \\ 0 & otherwise \end{cases}$$

Then

(7.2) $$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ ,$$

Therefore A_α, is the α-cut of A . To calculate inverse of fuzzy numbers A we first add the α-cut of A using interval arithmetic.

$$1/A_\alpha = [\frac{1}{a_1+\alpha^{\frac{1}{n}}(b_1-a_1), d_1-\alpha^{\frac{1}{n}}(d_1-c_1)}]$$

(7.3) $$= [\frac{1}{d_1-\alpha^{\frac{1}{n}}(d_1-c_1)}, \frac{1}{a_1+\alpha^{\frac{1}{n}}(b_1-a_1)}] \ .$$

Now, we find the shape function $\mu_{1/A}(x)$

$$x = \frac{1}{d_1-\alpha^{\frac{1}{n}}(d_1-c_1)}$$

(7.4) $$\Rightarrow \alpha = [\frac{xd_1-1}{x(d_1-c_1)}]^n \quad 1/d_1 \leq x \leq 1/c_1$$

and

$$x = \frac{1}{a_1+\alpha^{\frac{1}{n}}(b_1-a_1)}$$

(7.5) $$\Rightarrow \alpha = [\frac{1-xa_1}{x(b_1-a_1)}]^n \quad 1/b_1 \leq x \leq 1/a_1$$

Therefore shape function $\mu_{1/A}(x)$ is

$$\mu_{1/A} = \begin{cases} [\frac{xd_1-1}{x(d_1-c_1)}]^n & 1/d_1 \le x \le 1/c_1 \\ 1 & 1/c_1 \le x \le 1/b_1 \\ [\frac{1-xa_1}{x(b_1-a_1)}]^n & 1/b_1 \le x \le 1/a_1 \\ 0 & otherwise \end{cases} \tag{7.6}$$

Example 7.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when\ \ 0.2 \le x \le 0.35, \\ 1 & when\ \ 0.33 \le x \le 0.45 \\ (\frac{0.5-x}{0.05})^n & when\ \ 0.45 \le x \le 0.5 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \ ,$$

Therefore

$$1/A_\alpha = [\frac{1}{0.2+0.13\alpha^{\frac{1}{n}}, 0.5-0.05\alpha^{\frac{1}{n}}}]$$

$$= [\frac{1}{0.5-0.05\alpha^{\frac{1}{n}}}, \frac{1}{0.2+0.13\alpha^{\frac{1}{n}}}]$$

Now, we get

$$x = \frac{1}{0.5-0.05\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\frac{0.5x-1}{0.05x}]^n \quad\quad 2 \le x \le 2.22$$

and

$$x = \frac{1}{0.2+0.13\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\frac{1-0.2x}{0.13x}]^n \quad\quad 2.86 \le x \le 5$$

which gives

$$\mu_{1/A} = \begin{cases} [\frac{0.5x-1}{0.05x}]^n & when\ \ 2 \le x \le 2.22 \\ 1 & when\ \ 2.22 \le x \le 2.86 \\ [\frac{1-0.2x}{0.13x}]^n & when\ \ 2.86 \le x \le 5 \\ 0 & otherwise \end{cases}$$

8. Exponential of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \ \ x \in [a_1, b_1), \\ 1 & when \ \ x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \ \ x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{8.1}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{8.2}$$

Therefore A_α, is the α-cut of A . To calculate exponential of fuzzy numbers A we first add the α-cut of A using interval arithmetic.

$$exp(A_\alpha) = exp([a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)])$$

$$= [exp(a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1)), exp(d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))] \ . \tag{8.3}$$

Now, we find the shape function $\mu_{1/A}(x)$

$$x = exp(a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1))$$

$$\Rightarrow \alpha = [\frac{\ln x - a_1}{b_1 - a_1}]^n \quad exp(a_1) \leq x \leq exp(b_1) \tag{8.4}$$

and

$$x = exp(d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))$$

$$\Rightarrow \alpha = [\frac{d_1 - \ln x}{d_1 - c_1}]^n \quad exp(c_1) \leq x \leq exp(d_1) \tag{8.5}$$

Therefore shape function $\mu_{exp(A)}(x)$ is

$$\mu_{exp(A)} = \begin{cases} [\frac{\ln x - a_1}{b_1 - a_1}]^n & exp(a_1) \leq x \leq exp(b_1) \\ 1 & exp(b_1) \leq x \leq exp(c_1) \\ [\frac{d_1 - \ln x}{d_1 - c_1}]^n & exp(c_1) \leq x \leq exp(d_1) \\ 0 & otherwise \end{cases} \tag{8.6}$$

Example 8.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when \ \ 0.2 \leq x \leq 0.35, \\ 1 & when \ \ 0.33 \leq x \leq 0.45 \\ (\frac{0.5-x}{0.05})^n & when \ \ 0.45 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \ ,$$

Therefore

$$exp(A_\alpha) = [exp(0.2 + 0.13\alpha^{\frac{1}{n}}), exp(0.5 - 0.05\alpha^{\frac{1}{n}})]$$

Now, we get

$$x = exp(0.2 + 0.13\alpha^{\frac{1}{n}}) \Rightarrow \alpha = [\tfrac{\ln x - 0.2}{0.13}]^n \quad exp(0.2) \leq x \leq exp(0.33)$$

and

$$x = exp(0.5 - 0.05\alpha^{\frac{1}{n}}) \Rightarrow \alpha = [\tfrac{0.5 - \ln x}{0.05}]^n \quad exp(0.45) \leq x \leq exp(0.5)$$

which gives

$$\mu_{exp(A)} = \begin{cases} [\frac{\ln x-0.2}{0.13}]^n & when \ \ exp(0.2) \leq x \leq exp(0.33) \\ 1 & when \ \ exp(0.33) \leq x \leq exp(0.45) \\ [\frac{0.5-\ln x}{0.05}]^n & when \ \ exp(0.45) \leq x \leq exp(0.5) \\ 0 & otherwise \end{cases}$$

9. Logarithm of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \ \ x \in [a_1, b_1), \\ 1 & when \ \ x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \ \ x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{9.1}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{9.2}$$

Therefore A_α, is the α-cut of A . To calculate logarithm of fuzzy numbers A we first add the α-cut of A using interval arithmetic.

$$\ln(A_\alpha) = \ln([a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)])$$

(9.3) $$= [\ln(a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1)), \ln(d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))] \ .$$

Now, we find the shape function $\mu_{1/A}(x)$

$$x = \ln(a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1))$$

(9.4) $$\Rightarrow \alpha = [\tfrac{exp\ x - a_1}{b_1 - a_1}]^n \qquad \ln(a_1) \leq x \leq \ln(b_1)$$

and

$$x = \ln(d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))$$

(9.5) $$\Rightarrow \alpha = [\tfrac{d_1 - exp\ x}{d_1 - c_1}]^n \qquad \ln(c_1) \leq x \leq \ln(d_1)$$

Therefore shape function $\mu_{\ln(A)}(x)$ is

(9.6) $$\mu_{\ln(A)} = \begin{cases} [\tfrac{exp\ x - a_1}{b_1 - a_1}]^n & \ln(a_1) \leq x \leq \ln(b_1) \\ 1 & \ln(b_1) \leq x \leq \ln(c_1) \\ [\tfrac{d_1 - exp\ x}{d_1 - c_1}]^n & \ln(c_1) \leq x \leq \ln(d_1) \\ 0 & otherwise \end{cases}$$

Example 9.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\tfrac{x - 0.2}{0.13})^n & when\ \ 0.2 \leq x \leq 0.35, \\ 1 & when\ \ 0.33 \leq x \leq 0.45 \\ (\tfrac{0.5 - x}{0.05})^n & when\ \ 0.45 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \ ,$$

Therefore

$$\ln(A_\alpha) = [\ln(0.2 + 0.13\alpha^{\frac{1}{n}}), \ln(0.5 - 0.05\alpha^{\frac{1}{n}})]$$

Now, we get

$$x = \ln(0.2 + 0.13\alpha^{\frac{1}{n}}) \Rightarrow \alpha = [\tfrac{exp\ x - 0.2}{0.13}]^n \qquad \ln(0.2) \leq x \leq \ln(0.33)$$

and

$$x = \ln(0.5 - 0.05\alpha^{\frac{1}{n}}) \Rightarrow \alpha = [\tfrac{0.5 - exp\ x}{0.05}]^n \qquad \ln(0.45) \leq x \leq \ln(0.5)$$

which gives

$$\mu_{\ln(A)} = \begin{cases} [\frac{exp\ x - 0.2}{0.13}]^n & when \quad \ln(0.2) \leq x \leq \ln(0.33) \\ 1 & when \quad \ln(0.33) \leq x \leq \ln(0.45) \\ [\frac{0.5 - exp\ x}{0.05}]^n & when \quad \ln(0.45) \leq x \leq \ln(0.5) \\ 0 & otherwise \end{cases}$$

10. Square root of fuzzy numbers with using α-Cut

Let $A = (a_1, b_1, c_1, d_1)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x - a_1}{b_1 - a_1})^n & when \quad x \in [a_1, b_1), \\ 1 & when \quad x \in [b_1, c_1] \\ (\frac{d_1 - x}{d_1 - c_1})^n & when \quad x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{10.1}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{10.2}$$

Therefore A_α, is the α-cut of A . To calculate square root of fuzzy numbers A we first add the α-cut of A using interval arithmetic.

$$\sqrt{(A_\alpha)} = \sqrt{[a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)]}$$

$$= [\sqrt{a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1)}, \sqrt{d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)}] \ . \tag{10.3}$$

Now, we find the shape function $\mu_{1/A}(x)$

$$x = \sqrt{a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1)}$$

$$\Rightarrow \alpha = [\frac{x^2 - a_1}{b_1 - a_1}]^n \qquad \sqrt{a_1} \leq x \leq \sqrt{b_1} \tag{10.4}$$

and

$$x = \sqrt{d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)}$$

$$\Rightarrow \alpha = [\frac{d_1 - x^2}{d_1 - c_1}]^n \qquad \sqrt{c_1} \leq x \leq \sqrt{d_1} \tag{10.5}$$

Therefore shape function $\mu_{\sqrt{A}}(x)$ is

$$\mu_{\sqrt{A}} = \begin{cases} [\frac{x^2-a_1}{b_1-a_1}]^n & \sqrt{a_1} \leq x \leq \sqrt{b_1} \\ 1 & \sqrt{b_1} \leq x \leq \sqrt{c_1} \\ [\frac{d_1-x^2}{d_1-c_1}]^n & \sqrt{c_1} \leq x \leq \sqrt{d_1} \\ 0 & otherwise \end{cases} \tag{10.6}$$

Example 10.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when\ \ 0.2 \leq x \leq 0.35, \\ 1 & when\ \ 0.33 \leq x \leq 0.45 \\ (\frac{0.5-x}{0.05})^n & when\ \ 0.45 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}]\ ,$$

Therefore

$$\sqrt{A_\alpha} = [\sqrt{0.2 + 0.13\alpha^{\frac{1}{n}}}, \sqrt{0.5 - 0.05\alpha^{\frac{1}{n}}}]$$

Now, we get

$$x = \sqrt{0.2 + 0.13\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\tfrac{x^2-0.2}{0.13}]^n \quad \sqrt{0.2} \leq x \leq \sqrt{0.33}$$

and

$$x = \sqrt{0.5 - 0.05\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\tfrac{0.5-x^2}{0.05}]^n \quad \sqrt{0.45} \leq x \leq \sqrt{0.5}$$

which gives

$$\mu_{\sqrt{A}} = \begin{cases} [\frac{x^2-0.2}{0.13}]^n & when\ \ \sqrt{0.2} \leq x \leq \sqrt{0.33} \\ 1 & when\ \ \sqrt{0.33} \leq x \leq \sqrt{0.5} \\ [\frac{0.5-x^2}{0.05}]^n & when\ \ \sqrt{0.45} \leq x \leq \sqrt{0.5} \\ 0 & otherwise \end{cases}$$

11. n^{th} ROOT OF FUZZY NUMBERS WITH USING α-CUT

Let $A = (a_1, b_1, c_1, d_1)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-a_1}{b_1-a_1})^n & when \ \ x \in [a_1, b_1), \\ 1 & when \ \ x \in [b_1, c_1] \\ (\frac{d_1-x}{d_1-c_1})^n & when \ \ x \in (c_1, d_1] \\ 0 & otherwise \end{cases} \tag{11.1}$$

Then

$$A_\alpha = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)] \ , \tag{11.2}$$

Therefore A_α, is the α-cut of A . To calculate n^{th} root of fuzzy numbers A we first add the α-cut of A using interval arithmetic.

$$(A_\alpha)^{\frac{1}{n}} = [a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1), d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1)]^{\frac{1}{n}}$$

$$= [(a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1))^{\frac{1}{n}}, (d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))^{\frac{1}{n}}] \ . \tag{11.3}$$

Now, we find the shape function $\mu_{A^{\frac{1}{n}}}(x)$

$$x = (a_1 + \alpha^{\frac{1}{n}}(b_1 - a_1))^{\frac{1}{n}}$$

$$\Rightarrow \alpha = [\frac{x^n-a_1}{b_1-a_1}]^n \qquad \sqrt[n]{a_1} \leq x \leq \sqrt[n]{b_1} \tag{11.4}$$

and

$$x = (d_1 - \alpha^{\frac{1}{n}}(d_1 - c_1))^{\frac{1}{n}}$$

$$\Rightarrow \alpha = [\frac{d_1-x^n}{d_1-c_1}]^n \qquad \sqrt[n]{c_1} \leq x \leq \sqrt[n]{d_1} \tag{11.5}$$

Therefore shape function $\mu_{\sqrt[n]{A}}(x)$ is

$$\mu_{\sqrt{A}} = \begin{cases} [\frac{x^n-a_1}{b_1-a_1}]^n & \sqrt[n]{a_1} \leq x \leq \sqrt[n]{b_1} \\ 1 & \sqrt[n]{b_1} \leq x \leq \sqrt[n]{c_1} \\ [\frac{d_1-x^n}{d_1-c_1}]^n & \sqrt[n]{c_1} \leq x \leq \sqrt[n]{d_1} \\ 0 & otherwise \end{cases} \tag{11.6}$$

Example 11.1. Let $A = (0.2, 0.33, 0.45, 0.5)$ be a trapezoidal fuzzy number, Suppose the normal shape function of A is

$$\mu_A = \begin{cases} (\frac{x-0.2}{0.13})^n & when \;\; 0.2 \leq x \leq 0.35, \\ 1 & when \;\; 0.33 \leq x \leq 0.45 \\ (\frac{0.5-x}{0.05})^n & when \;\; 0.45 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$

$$A_\alpha = [0.2 + 0.13\alpha^{\frac{1}{n}}, 0.5 - 0.05\alpha^{\frac{1}{n}}] \;,$$

Therefore

$$\sqrt[n]{A_\alpha} = [\sqrt[n]{0.2 + 0.13\alpha^{\frac{1}{n}}}, \sqrt[n]{0.5 - 0.05\alpha^{\frac{1}{n}}}]$$

Now, we get

$$x = \sqrt[n]{0.2 + 0.13\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\tfrac{x^n - 0.2}{0.13}]^n \qquad \sqrt[n]{0.2} \leq x \leq \sqrt[n]{0.33}$$

and

$$x = \sqrt[n]{0.5 - 0.05\alpha^{\frac{1}{n}}} \Rightarrow \alpha = [\tfrac{0.5 - x^n}{0.05}]^n \qquad \sqrt[n]{0.45} \leq x \leq \sqrt[n]{0.5}$$

which gives

$$\mu_{\sqrt[n]{A}} = \begin{cases} [\frac{x^n-0.2}{0.13}]^n & when \;\; \sqrt[n]{0.2} \leq x \leq \sqrt[n]{0.33} \\ 1 & when \;\; \sqrt[n]{0.33} \leq x \leq \sqrt[n]{0.5} \\ [\frac{0.5-x^n}{0.05}]^n & when \;\; \sqrt[n]{0.45} \leq x \leq \sqrt[n]{0.5} \\ 0 & otherwise \end{cases}$$

12. Representation of fuzzy number with normal shape function

Definition 12.1. Let $A = (a_1, b_1, c_1, d_1; w_a)$ be a trapezoidal fuzzy number, then the graded mean integration representation of A is defined by

$$P(A) = \int_0^{w_a} h(\tfrac{L^{-1}(h) + R^{-1}(h)}{2}) \; dh / \int_0^{w_a} h \; dh \; .$$

Theorem 12.2. *Let $A = (a, b, c, d)$ be a trapezoidal fuzzy number with normal shape function, where a,b,c,d are real numbers such that $a < b \leq c < d$. Then the graded mean integration representation of A is*

$$P(A) = \tfrac{(a+d)}{2} + \tfrac{n}{2n+1}(b - a - d + c)$$

Proof.

$$L(x) = (\tfrac{x-a}{b-a})^n$$

$$R(x) = (\tfrac{d-x}{d-c})^n$$

So

$$L^{-1}(h) = a + (b-a)h^{\frac{1}{n}}$$

$$R^{-1}(h) = d - (d-c)h^{\frac{1}{n}}$$

$$P(A) = \tfrac{1}{2}\int_0^1 h[(a+(b-a)h^{\frac{1}{n}}) + (d-(d-c)h^{\frac{1}{n}})]\ dh/\int_0^1 h\ dh$$

$$= \tfrac{1}{2}\int_0^1 h[(a+d) + (b-a-d+c)h^{\frac{1}{n}}]\ dh/\int_0^1 h\ dh$$

$$= \tfrac{1}{2}\int_0^1 [(a+d)h + (b-a-d+c)h^{\frac{n+1}{n}}]\ dh/\int_0^1 h\ dh$$

$$= \tfrac{1}{2}[\tfrac{(a+d)}{2} + \tfrac{n}{2n+1}(b-a-d+c)]/\tfrac{1}{2} = \tfrac{(a+d)}{2} + \tfrac{n}{2n+1}(b-a-d+c)\ .$$

□

Theorem 12.3. *Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy number with normal shape function and $P(A)$ and $P(B)$ are the representation of A and B respectively. Then*
i) $P(m \oplus A) = mP(A)$,
ii) $P(A \oplus B) = P(A) \oplus P(B)$,
iii) $P(A \ominus B) = P(A) - P(B)$.

Proof. i) Since $m \oplus A = (ma_1, mb_1, mc_1, md_1)$, Then

$$P(m \oplus A) = \tfrac{(ma+md)}{2} + \tfrac{n}{2n+1}(mb - ma - md + mc) = mP(A).$$

ii) We have $A \oplus B = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$, Then
$P(A \oplus B)$

$$= \tfrac{((a_1+a_2)+(d_1+d_2))}{2} + \tfrac{n}{2n+1}((b_1+b_2) - (a_1+a_2) - (d_1+d_2) + (c_1+c_2))$$

$$= [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] + [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)]$$

$= P(A) \oplus P(B)$.

iii) We have $A \ominus B = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$, Then
$P(A \ominus B)$

$$= \tfrac{((a_1-a_2)+(d_1-d_2))}{2} + \tfrac{n}{2n+1}((b_1-b_2) - (a_1-a_2) - (d_1-d_2) + (c_1-c_2))$$

$P(A \ominus B)$

$$= [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)]$$

$= P(A) \ominus P(B)$. □

Definition 12.4. Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then $A > B$ if and only if $P(A) > P(B)$.

Definition 12.5. Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then $A = B$ if and only if $P(A) = P(B)$.

Theorem 12.6. *Suppose there are* n *alternative fuzzy numbers of modified fuzzy numbers and* $E = \{A_i \mid i = 1, 2, ..., n\}$. *We can prove that*

i) *every* $A_i, A_j \in E$, *then* $A_i \geq A_j$ *or* $A_j \geq A_i$,

ii) *if* $A_i \geq A_j$ *and* $A_j \geq A_i$, *then* $A_i = A_j$, $\forall A_i, A_j \in E$,

iii) *if* $A_i \geq A_j$ *and* $A_j \geq A_k$, *then* $A_i > A_k$, $\forall A_i, A_j, A_k \in E$,

Now, we define distance of trapezoidal fuzzy numbers.

Definition 12.7. Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, then the distance measure of A and B is

$$d(A, B) = \mid P(A) - P(B) \mid \ . \tag{12.1}$$

Theorem 12.8. *Let* $A = (a_1, b_1, c_1, d_1)$ *and* $B = (a_2, b_2, c_2, d_2)$ *and* $C = (a_3, b_3, c_3, d_3)$ *be three trapezoidal fuzzy numbers, then their distance measure satisfies the following relations*

i) $d(A, A) = 0$,

ii) $d(A, B) = d(B, A)$,

iii) *if* $A \leq B \leq C$, *then* $d(A, B) \leq d(A, C)$,

iv) $d(A, B) \leq d(A, C) + d(C, B)$.

Proof. i) $d(A, A) = \mid P(A) - P(A) \mid$

$= \mid [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] \mid$

$= \mid [\frac{(a_1+d_1)}{2} - \frac{(a_1+d_1)}{2}] + [\frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1) - \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] \mid$

$= 0$

ii) $d(A, B) = \mid P(A) - P(B) \mid$

$= \mid [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$

$= \mid [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] \mid$

$= \mid P(B) - P(A) \mid = d(B, A)$

iii) We have $B \leq C$, So

$P(B) \leq P(C)$

$\Rightarrow [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \leq [\frac{(a_3+d_3)}{2} + \frac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)]$

$\Rightarrow [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)]$

$\leq [\frac{(a_3+d_3)}{2} + \frac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)]$

$\Rightarrow \mid [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] \mid$

$\leq \mid [\frac{(a_3+d_3)}{2} + \frac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] - [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] \mid$

Of (ii), we have

$\Rightarrow \mid [\frac{(a_1+d_1)}{2} + \frac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\frac{(a_2+d_2)}{2} + \frac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$

$$= \mid P(A) - P(B) \mid$$

$$\leq \mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_3+d_3)}{2} + \tfrac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] \mid$$

$$= \mid P(A) - P(C) \mid$$

$$\Rightarrow \mid P(A) - P(B) \mid \leq \mid P(A) - P(C) \mid \Rightarrow d(A,B) \leq d(A,C)\ .$$

iv) $d(A,B) \leq d(A,B) = \mid P(A) - P(B) \mid \leq \mid P(A) - P(B) \mid \Rightarrow$

$$\mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$$

$$\leq \mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$$

$$\Rightarrow \mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$$

$$\leq \mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)]$$
$$\pm [\tfrac{(a_3+d_3)}{2} + \tfrac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] \mid$$

$$\leq \mid [\tfrac{(a_1+d_1)}{2} + \tfrac{n}{2n+1}(b_1 - a_1 - d_1 + c_1)] - [\tfrac{(a_3+d_3)}{2} + \tfrac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] \mid$$

$$+ \mid [\tfrac{(a_3+d_3)}{2} + \tfrac{n}{2n+1}(b_3 - a_3 - d_3 + c_3)] - [\tfrac{(a_2+d_2)}{2} + \tfrac{n}{2n+1}(b_2 - a_2 - d_2 + c_2)] \mid$$

$$\leq \mid P(A) - P(A) \mid + \mid P(C) - P(B) \mid$$
$$= d(A,C) + d(C,B) \Rightarrow d(A,B) \leq d(A,C) + d(C,B).$$ □

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