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Lattice theoretical aspects of fuzzy Mealy machine

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Abstract. In this paper, lattice theoretic aspects of fuzzy Mealy machines are discussed. We show that the classes of subsystems and dual subsystems forms 0,1- sublattices (in fact conjugate sublattices)of the power set of the state set of the fuzzy Mealy machine. Principal elements, completely join and meet irreducible elements of these sublattices are characterized. An algorithm for finding principal filters of M in terms of successor and predecessors is established. Further, in the last section of this paper direct sum and direct sum decomposition of fuzzy Mealy machines is also discussed.

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1. Introduction

Recall that in [10], we have studied the approach of fuzzification of Mealy machine by Mordeson and Nair [8, 9] with the help of the topology induced by the successor function. It is shown that various kinds of fuzzy Mealy machines such as cyclic, retrievable, strongly connected and connected can be studied with the help of this topology. In this paper we use the class of all $\tau-$ closed sets as the class of subsystems as well as the class of dual subsystems to study lattice theoretic aspects of fuzzy Mealy machines.

For lattice theoretic notions in classical automata we refer to [3, 4, 5] and for fuzzy automata we refer to [2, 7, 11, 12]

Here we have discussed lattice of subsystems and dual subsystems of fuzzy Mealy machines. We have discussed principal elements, completely join and meet irreducible elements, atoms and dual atoms of the lattice of subsystems of given fuzzy Mealy machine. We have characterized filters, centers, atoms, direct-sum decompositions and strongly connected fuzzy Mealy machines with the help of above discussed concepts throughout this paper. Following results are established i) the class of all strongly connected subsets of a fuzzy Mealy machine M is an order ideal in the power set of the state set of M. ii) Strongly connected (dual) subsystems of M are atoms of the lattice of (dual) subsystems of M. iii) The set of all filters of M is the center of the lattice of subsystems of M. iv) The atoms of the lattice of filters of M are precisely the principal filters of M.

2. Preliminaries

In this section we introduce few concepts and theorems of lattice theory that are needed throughout the paper. We refer to [1, 4, 6] for details on lattice theory.

Here, in this section, L will stands for a complete lattice with 0 and 1. Also, $\wedge$ and $\vee$ respectively stands for the infimum and the supremum in L.

Definition 2.1. A subset K of L is called a 0-sublattice (respectively 1-sublattice) of L, if $0 \in K$ (respectively $1 \in K$). It is a $0, 1-$sublattice of L, if both 0 and $1 \in K$.

Definition 2.2. A subset K of L is called a complete meet-subsemilattice (respectively complete join-subsemilattice) of L, if it contains the meet (respectively the join) of each its non-empty subset, and it is complete sublattice of L, if it is both complete meet and join subsemilattice of L.

Definition 2.3. A subset K of L will be called an order ideal of L, if for all $a, b \in L, b \in K$ and $a \leq b \Rightarrow a \in K$. Dually, we define dual order ideal.

Definition 2.4. An element $a \in L$ is called an atom of L, if $0 < a$ and there exists no $x \in L$ such that $0 < x < a$.

Definition 2.5. An element $a \in L$ is called completely join irreducible, if $a \neq 0$, and for every subset K of L $a = \bigvee K \Rightarrow a \in K$. Dually we define completely meet irreducible element of L

The set of all completely join irreducible elements of L will be denoted by $CJI(L)$, and the set of all completely meet irreducible elements of L will be denoted by $CMI(L)$.

Definition 2.6. A mapping $\psi : L \to L$ is called extensive, if $a \leq \psi(a)$, for any $a \in L$, isotone (antitone), if for all $a, b \in L, a \leq b \Rightarrow \psi(a) \leq \psi(b)(\psi(b) \leq \psi(a))$ and idempotent, if $\psi^2 = \psi$, that is if $\psi(\psi(a)) = \psi(a)$, for each $a \in L$.

Definition 2.7. A mapping $\psi : L \longrightarrow L$ is an order isomorphism, (dual order isomorphism) if it is isotone(antitone) bijective with isotone(antitone) inverse.

Definition 2.8. An extensive, isotone and idempotent mapping of L into L is called closure operator on L.

Definition 2.9. An element $a \in L$ is called closed (with respect to ψ), if $\psi(a) = a$.

The set of all closed elements with respect to a closure operator ψ on L is a closure system on L, where closure system in L is the complete meet-subsemilattice of L containing the unity of L. Conversely, closure system on L determines a unique closure operator on L

Definition 2.10. A complemented distributive lattice is a Boolean Algebra.

Definition 2.11. A subset K of a complete Boolean algebra B is called a subalgebra of B, if it is both a Boolean subalgebra and a complete sublattice of B.

The set of all complete Boolean subalgebras of L, partially ordered by set inclusion, is a complete lattice, and it is denoted by $CB(L)$.

Definition 2.12. A complete Boolean algebra B is called atomic if any non-zero element of B is the join of some family of atoms of B.

Notation 2.13. *Let B be a Boolean algebra. For a subset L of B, L' will denote the set of complements of elements of L, i.e. $L' = \{a' | a \in L\}$.*

We see that the mapping $a \to a'$ is a dual order isomorphism of the poset L onto the poset L', so L is a sublattice of B if and only if L' is a sublattice, and if B is a complete Boolean algebra, then L is a complete sublattice of B if and only if L' is a complete sublattice of B. In both these cases L and L' are dually isomorphic as lattices, and we will say that L and L' are conjugated sublattices of B. For $a \in L, a'$ will be called the dual of a. If $L = L'$, then we say that L is a self-conjugated sublattices of B. Clearly, self-conjugated sublattices of B are exactly its Boolean subalgebras.

Let U be a non-empty set and ψ be a closure system $\mathcal{L}$ on U. For a non-empty $H \subseteq U$, the intersection of all elements of $\mathcal{L}$ containing H, denoted by $\mathcal{L}(H)$, is the smallest element of $\mathcal{L}$ containing H, and it is called the element of $\mathcal{L}$ generated by H. For $a \in U, \mathcal{L}(a)$ is called the principal element of $\mathcal{L}$ generated by a. The set of all principal elements of $\mathcal{L}$ is called the principal part of $\mathcal{L}$.

Theorem 2.14. *For a complete sublattice L of $\wp(U)$, the principal part of L coincides to the set $CJI(L)$ of all completely join irreducible elements of L.*

Definition 2.15. An element a of a lattice L is called neutral, if $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$, for all $x, y \in L$.

Definition 2.16. If L is lattice with 0 and 1, then the set of all neutral complemented elements of L is called a center of L, and it is denoted by $C(L)$.

Theorem 2.17. *$C(L)$ is a sublattice of L and a Boolean subalgebra of L. Further if L is distributive lattice, then $C(L)$ consists of all complemented elements of L.*

Theorem 2.18. *If L is a $0,1-$sublattice of some Boolean algebra, then $C(L) = L \cap L'$.*

3. Lattice of subsystems of a fuzzy Mealy machines

In this section we introduced lattice structure on the set of subsystems of the given fuzzy Mealy machine. Various lattice theoretical properties of the set of subsystems and dual subsystems are also discussed. We begin by introducing the definition of fuzzy Mealy machines, subsystems and dual subsystems.

Definition 3.1. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $H \subseteq Q$. Then H is called subsystem of M, if for any $h \in H, x \in X^*, y \in Y^*, \mu(h, x, p, y) > 0$, for some $p \in Q$ implies that $p \in H$. And H is called dual subsystem of M, if for any $h \in H, x \in X^*, y \in Y^*, \mu(q, x, h, y) > 0$, for some $q \in Q$ implies that $q \in H$.

Remark 3.2. Subsystems of fuzzy Mealy machine M are $\tau-$ closed subsets of Q, discussed in [10].

Theorem 3.3. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then*

(1) *Intersection and union of two subsystems of M is a subsystem of M.*

(2) *Intersection and union of two dual subsystems of M is a dual subsystem of M.*

Proof. 1. Let $h \in H_1 \cap H_2, x \in X^*$ and $y \in Y^*$. Let $\mu(h, x, p, y) > 0$, for $p \in Q$. Since $h \in H_1$ and $\mu(h, x, p, y) > 0$, we have $p \in H_1$. Similarly, $h \in H_2 \Rightarrow p \in H_2$. Therefore, $p \in H_1 \cap H_2$. This proves that the intersection of two subsystems of M is a subsystem of M.
Let $h \in H_1 \cup H_2, x \in X^*$ and $y \in Y^*$. Let $\mu(h, x, p, y) > 0$, for $p \in Q$.

case(i) Let $h \in H_1$. Since $\mu(h, x, p, y) > 0$, we have $p \in H_1$. Thus, $p \in H_1 \cup H_2$.

case(ii) Let $h \in H_2$. Since $\mu(h, x, p, y) > 0$, we have $p \in H_2$. Thus, $p \in H_1 \cup H_2$

Therefore, $H_1 \cup H_2$ is also subsystem M. □

The complement of a subsystem (a dual subsystem) of a given fuzzy Mealy machine M need not be a subsystem (dual subsystem). The following example depict this

Example 3.4. Let $Q = \{p_1, p_2, p_3, p_4\}, X = \{a, b\}, Y = \{y_1, y_2, y_3\}$ and μ be defined as follows:
$\mu(p_1, a, p_2, y_2) = 0.3, \mu(p_1, b, p_3, y_1) = 0.4, \mu(p_2, a, p_3, y_1) = 0.5, \mu(p_2, b, p_1, y_1) = 0.8, \mu(p_3, a, p_3, y_1) = 0.4, \mu(p_3, b, p_3, y_2) = 0.1, \mu(p_4, a, p_3, y_2) = 0.7, \mu(p_4, b, p_4, y_3) = 1.0$ and $\mu(p, x, q, y) = 0$ for all other $p, q \in Q$ and $x \in X$, $y \in Y$. Take $H_1 = \{p_1, p_2, p_3\}$. Then H_1 is a subsystem of M. But $H_1^c = \{p_4\}$ is not a subsystem of M as $\mu(p_4, a, p_3, y_2) > 0$ and $p_3 \notin H_1^c$. Similarly, if $H_2 = \{p_4\}$ then H_2 is a dual subsystem of M. But $H_2^c = \{p_1, p_2, p_3\}$ is not a dual subsystem of M as $\mu(p_4, a, p_3, y_2) > 0$ and $p_4 \notin H_2^c$.

Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. The set of all subsystem of M is denoted by $Sub(M)$ and the set of all dual-subsystem of M is denoted by $DSub(M)$.

Theorem 3.5. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $H \subseteq Q$. Then $H \in Sub(M)$ if and only if $H^c \in DSub(M)$.*

Proof. Suppose $H \in Sub(M)$. Let $p \in H^c, x \in X^*, y \in Y^*$ and $\mu(q, x, p, y) > 0$. Suppose $q \notin H^c$. Then $q \in H$. But then $p \in H$, as H is a subsystem of M, which is contradiction. Therefore, $q \in H^c$. Hence, H^c is a dual subsystem of M. The converse is similar. □

These $Sub(M)$ and $DSub(M)$ have lattice structure on them, due to Theorem (3.3).

Theorem 3.6. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then $Sub(M)$ and $DSub(M)$ are complete 0,1-sublattices of $\wp(Q)$. Moreover, $Sub(M)$ and $DSub(M)$ are conjugated sublattices of $\wp(Q)$.*

$M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Denote

$$S(q) = \{p \in Q | \; \exists \; (u \in X^*, v \in Y^*) \text{ such that } \mu(q, u, p, v) > 0\}$$

$$P(q) = \{p \in Q | \ \exists \ (u \in X^*, v \in Y^*) \text{ such that } \mu(p, u, q, v) > 0\}.$$

The set $S(q)$ is called the set of all successors of q and $P(q)$ the set of all predecessors of q. We shall see that these $S(q)$ and $P(q)$ have special lattice theoretic importance in the further study.

Lemma 3.7. $S(q) = \bigcap_{B \in Sub(M)} \{B | S(q) \subseteq B\}$ *and* $P(q) = \bigcap_{D \in DSub(M)} \{D | P(q) \subseteq D\}$.

Proof. Clearly, $S(q) \subseteq \bigcap_{B \in Sub(M)} \{B | S(q) \subseteq B\}$. For converse let $p \in B$, whenever $B \in Sub(M)$ such that $S(q) \subseteq B$. Since $S(q) \in Sub(M)$ contained in $S(q)$, we have $p \in S(q)$. Therefore, $\bigcap_{B \in Sub(M)} \{B | S(q) \subseteq B\} \subseteq S(q)$. □

Theorem 3.8. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $q \in Q$ be an any state. Then*

(1) *$S(q)$ is the principal element of $Sub(M)$ generated by q and*
(2) *$P(q)$ is the principal element of $DSub(M)$ generated by q.*

Proof. 1. Clearly, $S(q) \in Sub(M)$. Also by above Lemma (3.7),

$$S(q) = \bigcap_{B \in Sub(M)} \{B | \ S(q) \subseteq B\}.$$

Therefore, $S(q)$ is the principal element of $Sub(M)$ generated by q. □

We elaborate the principal element in the $Sub(M)$ for a given fuzzy Mealy machine $M = (Q, X, Y, \mu)$ with an example as follows:
Let $Q = \{q_0, q_1, q_2, q_3\}, X = \{a, b\}, Y = \{y_1, y_2\}$ and μ be defined as follows:
$\mu(q_0, a, q_1, y_2) = 0.2,\ \mu(q_1, b, q_1, y_1) = 0.1,\ \mu(q_1, a, q_2, y_2) = 0.4,\ \mu(q_2, a, q_2, y_2) = 0.1,\ \mu(q_2, b, q_1, y_2) = 0.5,\ \mu(q_0, b, q_3, y_2) = 0.4,\ \mu(q_3, a, q_3, y_1) = 0.1\ \mu(q_3, b, q_3, y_1) = 0.2$ and $\mu(p, x, q, y) = 0$ for all other $p, q \in Q$, $x \in X$ and $y \in Y$.
In this example $Sub(M) = \{\phi,\ \{q_3\},\ \{q_1, q_2\},\ \{q_1, q_2, q_3\},\ Q\}$ and $S(q_3) = \{q_3\}$ which is a principal element of $Sub(M)$ generated by q_3, because $\{\{q_3\},\ \{q_1, q_2, q_3\},\ Q\}$ forms a closure system. Also, $S(q_1) = S(q_2) = \{q_1, q_2\}$ are principal element of $Sub(M)$ generated by q_1 and q_2 respectively, because $\{\{q_1, q_2\},\ \{q_1, q_2, q_3\},\ Q\}$ forms a closure system.

Theorem 3.9. *Let $M = (Q, X, Y, \mu)$ be fuzzy Mealy machine. Then*

(1) $p \in S(q) \Rightarrow S(p) \subseteq S(q)$
(2) $p \in P(q) \Rightarrow P(p) \subseteq P(q)$
(3) $p \in S(q) \Leftrightarrow q \in P(p)$

Definition 3.10. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. For a subset D of Q the sets $S(D) = \bigcup_{p \in D} S(p)$ and $P(D) = \bigcup_{p \in D} P(p)$ are respectively called subsystem and dual subsystem of M generated by D.

If $D = \{q\},\ q \in Q$, then $S(q)$ and $P(q)$ are respectively designated as principal subsystem and principal dual subsystem of M generated by q. Following theorems give more information about these elements.

Theorem 3.11. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then*

(1) *completely join irreducible elements of $Sub(M)$ are exactly the principal subsystem of M*
(2) *completely meet irreducible elements of $Sub(M)$ are exactly the duals of principal dual-subsystem of M.*

Proof. 1. Any $q \in Q$, $S(q)$ is a principal element of $Sub(M)$ generated by q. Therefore, principal part of $Sub(M)$ is the set $\{S(q)|q \in Q\}$. But principal part of $Sub(M)$ coincides to the set $CJI(Sub(M))$ of all completely join irreducible elements. Hence, completely join irreducible elements of $Sub(M)$ are exactly the principal subsystem of M. □

Following theorem follows due to (3) of Theorem (3.9).

Theorem 3.12. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Let p and $q \in Q$. Then $S(p) = S(q) \Leftrightarrow P(p) = P(q)$.*

In view of this theorem we define an equivalence relation γ on M as $q\gamma p \Leftrightarrow S(q) = S(p)$. For $q \in Q$, we denote G_q by the equivalence class of q w.r.t. γ. The following theorem gives more insight into these classes.

Theorem 3.13. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $q \in Q$. Then $G_q = S(q) \cap P(q)$.*

Proof. Let $r \in G_q$. Then $r\gamma q \Rightarrow S(r) = S(q)$ and $P(r) = P(q)$. Since $r \in S(r) = S(q)$ and $r \in P(r) = P(q)$ we have $r \in S(q)$ and $r \in P(q)$. Thus, $r \in S(q) \cap P(q)$. Therefore, $G_q \subseteq S(q) \cap P(q)$.
Conversely, let $r \in S(q) \cap P(q)$. Then $r \in S(q)$ and $r \in P(q)$ by (3) of Theorem (3.9), $q \in S(r)$ and $r \in S(q)$. Therefore, $S(q) \subseteq S(r)$ and $S(r) \subseteq S(q)$. Thus, $S(q) = S(r)$. i.e. $q\gamma r$. Hence, $r \in G(q)$. □

Definition 3.14. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. A subset D of Q is called strongly connected, if for each pair $q, p \in D$ there exists $u \in X^*, v \in Y^*$ such that $\mu(p, u, q, v) > 0$. The set of all strongly connected subsets of M is denoted by $Scs(M)$.

Consider a fuzzy Mealy machine M with $\mu(p, a_1, q, u_1) = 0.3 = \mu(q, a_2, r, u_2) = \mu(r, a_3, p, u_3)$ and $\mu(p, a_4, s, u_4) = 0.5 = \mu(t, a_5, p, u_5)$. Then $A = \{p, q, r\}$ is strongly connected set, but it is neither subsystem nor dual subsystem of M. On the other hand $H = \{p, q, r, s\}$ and $K = \{p, q, r, t\}$ are respectively subsystem and dual subsystem of M, but none of them is strongly connected set. The set $Scs(M)$ has special importance with reference to the power set lattice.

Theorem 3.15. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then $Scs(M)$ is an order ideal of $\wp(Q)$.*

Proof. Let $A, B \in \wp(Q)$ with $A \subseteq B$ and $B \in Scs(M)$. Since subset of strongly connected subset is strongly connected, we have $A \in Scs(M)$. Hence $Scs(M)$ is order ideal. □

(Dual) Subsystem which is strongly connected subsets of M is called strongly connected (dual) subsystem of M. These strongly connected subsystems and dual subsystems of M have special importance in the lattice namely $Sub(M)$.

Theorem 3.16. *(a) Every strongly connected subsystem of a fuzzy Mealy machine $M = (Q, X, Y, \mu)$ is an atom of $Sub(M)$.*
(b)Complement of every strongly connected dual subsystem of a fuzzy Mealy machine $M = (Q, X, Y, \mu)$ is a dual atom of $Sub(M)$.

Proof. (a) Let H be strongly connected subsystem of M which is not an atom of $Sub(M)$. Then there exists subsystem H_1 such that $\phi \subset H_1 \subset H$. Thus, there exists $q \in H$ such that $q \notin H_1$. Let $p \in H_1$. Then $q, p \in H$. Since H is strongly connected subsystem there exist $u_2 \in X^*, v_2 \in Y^*$ such that $\mu(p, u_2, q, v_2) > 0$. Now H_1 is a subsystem and $p \in H_1$, This gives $q \in H_1$. This contradicts to the fact that $q \notin H_1$. Hence H necessarily be an atom of $Sub(M)$. □

We now characterize atoms (and dual atoms) of $Sub(M)$ in terms of principal (dual) subsystems and equivalence classes of the equivalence relation γ.

Theorem 3.17. *Let $B \in Sub(M)$. Then the following are equivalent*
(1) *B is an atom in $Sub(M)$.*
(2) *$B = S(q)$, for any $q \in B$.*
(3) *$B = G_q$, for some $q \in Q$.*

Proof. **(1) ⇒ (2):** Let $q \in B$. Then, $S(q)$ is subsystem of B and $q \in S(q)$ implies that $S(q) \neq \phi$. Since B is an atom, we must have $S(q) = B$.
(2) ⇒ (3): Suppose $B = S(q)$, for any $q \in B$. By Theorem (3.13), we have $G_q = S(q) \cap P(q)$. Thus, $G_q \subseteq S(q) = B$ i.e. $G_q \subseteq B$. On the other hand, we have for any $r \in B = S(r)$. Therefore, $S(q) = S(r)$. That is $q\gamma r$. Hence, $r \in G_q$. This proves $B \subseteq G_q$. Therefore, $B = G_q$, for some $q \in Q$.
(3) ⇒ (1): Let $B = G_q$ for some $q \in Q$. Assume that B is not an atom in $Sub(M)$. Then there exists subsystem N of $Sub(M)$ such that $\phi \neq N \subset B$. So there exists $p \in B$ such that $p \notin N$. Since $B = G_q$, for some $q \in Q$, we have by definition $p \in G_q$. Thus $S(q) = S(p)$. For any $r \in N$ we have $B = G_q$. Thus $r \in S(r) = S(q)$. Now $p \in S(q) \Rightarrow p \in S(r) \subset N$ (since N is subsystem), which is contradiction. Hence, B is an atom in $Sub(M)$. □

Similarly,

Theorem 3.18. *Let $B' \in Sub(M)$. Then the following are equivalent*
(1) *B' is a dual atom in $Sub(M)$.*
(2) *$B' = P(q)$, for any $q \in B'$.*
(3) *$B' = G_q$, for some $q \in Q$.*

4. The center of the lattice of fuzzy Mealy machines

In this section, filters of $Sub(M)$ are introduced and their properties are discussed. Principal filters of M are characterized in terms of the sequences of n^{th} predecessors of successors and n^{th} successors of predecessors. We first define filters in $Sub(M)$.

Definition 4.1. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. A subset H of Q is called *filter* of M, if it is both subsystem and dual-subsystem of M.
The set of all filters of M is denoted by $Filt(M)$. Hence, $Filt(M) = Sub(M) \cap DSub(M)$.

We define principal filters of M

Definition 4.2. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $q \in Q$. Let $Filt(q)$ denote the principal element of $Filt(M)$ generated by q. The filter $Filt(q)$ is called the principal filter of M generated by q

Consider a fuzzy Mealy machine M with $\mu(p, a_1, q, u_1) = 0.6 = \mu(q, a_2, r, u_2) = \mu(r, a_3, p, u_3)$. Then $F = \{p, q, r\}$ is a filter of M. The vary definition of the filter, theorems (3.3) and (3.5) immediately leads us to

Theorem 4.3. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then*

(1) *union and intersection of two filters is again a filter of M*
(2) *$F \in Filt(M)$ if and only if $F^c \in Filt(M)$*
(3) *$Filt(M)$ is $0, 1$-sublattice of M and*
(4) *Every filter is union of principal filter of M. (Filter Decomposition Theorem)*

We have $Filt(M)$ is not only sublattice of M, but it is even more.

Theorem 4.4. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then*

(1) *$Filt(M)$ is the center of $Sub(M)$*
(2) *$Filt(M)$ is a complete sublattice $Sub(M)$ and*
(3) *$Filt(M)$ is a complete atomic Boolean algebra.*

Proof. 1. Since $Sub(M)$ is $0, 1-$ sublattice of the Boolean algebra $\wp(Q)$, we have center of $Sub(M) = Sub(M) \cap DSub(M)$. i.e. center of $Sub(M)$ is $Filt(M)$.
2. Obvious.
3. by Filter Decomposition Theorem, $Filt(M)$ is a complete atomic Boolean algebra. □

We now characterize filters of M in term sequences of predecessors of successors and successors of predecessors.

Theorem 4.5. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then atoms of $Filt(M)$ are exactly the principal filters of M.*

Proof. For any $q \in Q$ we know that $Filt(q)$ be a principal element of $Filt(M)$ generated by q. Therefore, $Filt(q)$ is a principal filter of M. Then by (3) of Theorem (4.4), $Filt(M)$ is the complete atomic Boolean algebra of $Sub(M)$. □

Definition 4.6. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine Define $P_1(q) = P(S(q))$, $P_{n+1}(q) = P(S(P_n(q)))$, for $n \geq 1$ and $S_1(q) = S(P(q)), S_{n+1}(q) = S(P(S_n(q)))$, for $n \geq 1$. Then $\{P_n(q)\}_{n\in\mathbb{N}}$ and $\{S_n(q)\}_{n\in\mathbb{N}}$ are respectively called the sequences of predecessors of successors of q and successors of predecessors q.

Theorem 4.7. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then following assertions hold.*

(1) *The sequences $\{P_n(q)\}_{n\in\mathbb{N}}$ and $\{S_n(q)\}_{n\in\mathbb{N}}$ are increasing.*
(2) *There exists $m, n \in \mathbb{N}$ such that $P_m(q) = P_{m+k}(q)$ and $S_n(q) = S_{n+k}(q)$, $\forall\, k \geq 1$.*
(3) *$Filt(q) = \bigcup_{n\in\mathbb{N}} P_n(q) = \bigcup_{n\in\mathbb{N}} S_n(q)$.*

Proof. 1. For any $n \in \mathbb{N}$, we have $P_n(q) \subseteq S(P_n(q)) \subseteq P(S(P_n(q))) = P_{n+1}(q)$. Therefore, $P_n(q) \subseteq P_{n+1}(q)\ \forall\ n \in \mathbb{N}$. Therefore, $\{P_n(q)\}_{n\in\mathbb{N}}$ is an increasing sequence.
2. Obvious.
3. Let $P = \bigcup_{n\in\mathbb{N}} P_n(q)$, since each $P_n(q)$ is a dual-subsystem of M, P is also dual-subsystem. Now we show that P is subsystem of M. Let $p \in P$ then $p \in P_n(q)$ for some $n \in \mathbb{N}$ with $\mu(p, u, r, v) > 0$ for $u \in X^*, v \in Y^*$ for some $r \in S(P_n(q))$. Since, $r \in S(P_n(q)) \subseteq P(S(P_n(q))) = P_{n+1}(q)$. i.e. $r \in P$. Thus P is subsystem as well as dual-subsystem, hence P becomes filter of M containing q, hence $Filt(q) \subseteq P$.
Conversely, to prove $P \subseteq Filt(q)$, it is enough to prove that $P_n(q) \subseteq Filt(q)$ for any $n \in \mathbb{N}$. This will be proved by induction. First, we know that $S(q) \subseteq Filt(q)$ since $Filt(q)$ is a subsystem of M containing q, and now $P_1(q) = P(S(q)) \subseteq Filt(q)$ since $Filt(q)$ is a dual-subsystem of M. Suppose $P_n(q) \subseteq Filt(q)$ for some $n \in \mathbb{N}$. Then $S(P_n(q)) \subseteq Filt(q)$ since $Filt(q)$ is subsystem of M containing $P_n(q)$ and $P_{n+1}(q) = P(S(P_n(q))) \subseteq Filt(q)$, since $Filt(q)$ is dual-subsystem. Hence, $P_n(q) \subseteq Filt(q)$ for any $n \in \mathbb{N}$. Hence, $Filt(q) = \bigcup_{n\in\mathbb{N}} P_n(q)$.
Now, we will prove $Filt(q) = \bigcup_{n\in\mathbb{N}} S_n(q)$
We have $q \in S(q) \Rightarrow P(q) \subseteq P(S(q)) = P_1(q)$. Also $S_1(q) = S(P(q)) \subseteq S(P(S(q))) \subseteq P(S(P(S(q)))) = P_2(q)$. Thus $S_1(q) \subseteq P_2(q)$.
Suppose, $S_n(q) \subseteq P_{n+1}(q)$ for some $n \in \mathbb{N}$. Then we have $P(S_n(q)) \subseteq P(P_{n+1}(q)) = P_{n+1}(q)$ (since P_{n+1} is dual-subsystem of M). So, $S_{n+1}(q) = S(P(S_n(q))) \subseteq S(P_{n+1}(q)) \subseteq P(S(P_{n+1}(q))) = P_{n+2}(q)$. Thus, $S_{n+1}(q) \subseteq P_{n+2}(q)$. Therefore, by induction method, we have $S_n(q) \subseteq P_{n+1}(q)\ \forall\ n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} S_n(q) \subseteq \bigcup_{n\in\mathbb{N}} P_{n+1}(q)$ $= Filt(q)$. i.e. $\bigcup_{n\in\mathbb{N}} S_n(q) \subseteq Filt(q)$.
On the other hand, since $q \in P(q) \Rightarrow S(q) \subseteq S(P(q)) = S_1(q)$. Now, $P_1(q) = P(S(q)) \subseteq P(S_1(q)) \subseteq S(P(S_1(q))) \subseteq S_2(q)$. That is $P_1(q) \subseteq S_2(q)$. Suppose $P_n(q) \subseteq S_{n+1}(q)$ for some $n \in \mathbb{N}$. Then $S(P_n(q)) \subseteq S(S_{n+1}(q)) = S_{n+1}(q)$. (since S_{n+1} is subsystem of M) Hence, $P_{n+1}(q) = P(S(P_n(q))) \subseteq P(S_{n+1}(q)) \subseteq S(P(S_{n+1}(q))) = S_{n+2}(q)$. Thus, $P_{n+1}(q) \subseteq S_{n+2}(q)$. Therefore, by induction method, we have $P_n(q) \subseteq S_{n+1}(q)\ \forall\ n \in \mathbb{N} \Rightarrow Filt(q) = \bigcup_{n\in\mathbb{N}} P_n(q) \subseteq \bigcup_{n\in\mathbb{N}} S_{n+1}(q)$. That is $Filt(q) \subseteq \bigcup_{n\in\mathbb{N}} S_n(q)$. Hence, $Filt(q) = \bigcup_{n\in\mathbb{N}} S_n(q)$. □

Corollary 4.8. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then there exist*
$m = min\ \{k \in \mathbb{N} |(\ \forall\ q \in Q)\ P_k(q) = P_{k+1}(q)\}$ *and*
$n = min\ \{k \in \mathbb{N} |(\ \forall\ q \in Q)\ S_k(q) = S_{k+1}(q)\}$
for which also holds, $n, m \leq |Q|$ and $Filt(q) = P_m(q) = S_n(q)$, for any $q \in Q$

Proof. We have $P_n(q) \subseteq Q,\ \forall\ q \in Q$. Thus $\bigcup_{n\in\mathbb{N}} P_n(q) \subseteq Q$ and $\{P_n(q)\}$ is an increasing sequence. Therefore, $\bigcup_{n=1}^{k} P_n(q) \subseteq P_{k+1}(q)$. Since Q is finite, this chain of $P_n(q)$ will be constant after finitely many steps. Therefore, we can find
$m = min\{k \in \mathbb{N} |(\ \forall\ q \in Q) P_k(q) = P_{k+1}(q)\}$
If $P_i \neq P_{i+1}$ then $i \leq |P_i|$ and $P_m \subseteq Q \Rightarrow m \leq |P_m| \leq |Q| \Rightarrow m \leq |Q|$. □

Algorithm to find a principal filter: Let M be given fuzzy Mealy machine and $q \in Q$

Step I. $P_1(q) = P(S(q)); P_{n+1}(q) = P(S(P_n(q))),\ \forall\ n \geq 1$ and

$S_1(q) = S(P(q)); S_{n+1}(q) = S(P(S_n(q))),\ \forall\, n \geq 1$
Step II. $m = min\{k \in N : P_k(q) = P_{k+1}(q)\}$ and
$n = min\{k \in N : S_k(q) = S_{k+1}(q)\}$
Step III. $Filt(q) = P_m(q) = S_n(q)$.

5. Direct sum decomposition of fuzzy Mealy machines

Throughout this section, we use subsystem H of a fuzzy Mealy machine $M = (Q, X, Y, \mu)$ and its associated fuzzy Mealy machine $(H, X, Y, \mu_{|_H})$ interchangeably. This natural identification allows us to talked about union and intersection of subsystems of a given fuzzy Mealy machine. Also, the following make sense.

Theorem 5.1. *Let H be a subsystem (dual subsystem, filter) of a fuzzy Mealy machine K and K is a subsystem (dual subsystem, filter) of a fuzzy Mealy machine M. Then H be a subsystem (dual subsystem, filter) of a fuzzy Mealy machine M.*

Definition 5.2. A fuzzy Mealy machine M is called direct sum of strongly connected subsystems $H_1, H_2, ..., H_n$ of M, if $M = \bigcup_{\alpha=1}^{n} H_\alpha$ and $H_\alpha \cap H_\beta = \phi,\ \forall\, \alpha \neq \beta$.

Definition 5.3. Let M be a fuzzy Mealy machine. An equivalence relation $\sim$ on M (i.e. on Q) is called direct sum relation on M, if M is a direct sum of equivalence classes of $\sim$. In this case the equivalence classes, which are subsystems also, are called direct summands of M.

The following is obvious.

Theorem 5.4. *For every direct sum decomposition of fuzzy Mealy machine M there is a direct sum equivalence relation on M, where classes are direct summand of M. Conversely for each direct sum equivalence relation of M, there is a direct sum decomposition of M.*

A fuzzy Mealy machine M which has at least two non-empty direct summands is said to have a non-trivial direct sum decomposition. The Theorem (3.5) defines a non-trivial direct sum decomposition of fuzzy Mealy machine.

Definition 5.5. A fuzzy Mealy machine M is called direct sum indecomposable, if it has no non-trivial direct sum decomposition.

Theorem 5.6. *Following statements are equivalent:*

1. *a fuzzy Mealy machine M is direct sum indecomposable*
2. *universal relation is the only direct sum equivalence relation on M*
3. *M has no proper filter.*

To discuss more about direct sum equivalence relation, we introduce the following definitions.

Definition 5.7. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. An equivalence relation $\sim$ on Q is called congruence relation if for $p, q \in Q, x \in X^*, y \in Y^*$ if $p \sim q$ and $\mu(p, x, s, y) \wedge \mu(q, x, r, y) > 0$, then $r \sim s$.

Definition 5.8. Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $\sim$ a direct sum equivalence relation on M. Then the fuzzy Mealy machine $M/\sim = (Q/\sim, X, Y, \mu^{\sim})$, where $\mu^{\sim}([p]_{\sim}, x, [q]_{\sim}, y) = \bigvee_{s \in [p]_{\sim}, t \in [q]_{\sim}} \mu(s, x, t, y)$, is called quotient fuzzy Mealy machine of $M/\sim$.

Definition 5.9. A fuzzy Mealy machine $M = (Q, X, Y, \mu)$is called identity if $\mu(p, x, q, y) > 0$ for $p, q \in Q, x \in X^*, y \in Y^*$ implies that $p = q$.

Theorem 5.10. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine and $\sim$ be an equivalence relation on M. Then following statements are equivalent:*

(1) *$\sim$ is a direct sum equivalence relation on M*
(2) *for all $p, q \in Q, x \in X^*, y \in Y^*$ if $\mu(p, x, q, y) > 0$ then $p \sim q$*
(3) *$\sim$ is a congruence relation on M and $M/\sim$ is an identity fuzzy Mealy machine.*

Proof. **(1) $\Rightarrow$ (2):** Let $p, q \in Q, x \in X^*, y \in Y^*$ and $\mu(p, x, q, y) > 0$. Since $[p]_{\sim}$ is a subsystem of M, we have $q \in [p]_{\sim}$ i.e. $p \sim q$.
(2) $\Rightarrow$ (3): $p \sim q$ and $\mu(p, x, r, y) \wedge \mu(q, x, s, y) > 0$ implies that $p \sim r$ and $q \sim s$. Thus $\sim$ is a congruence relation on M. Let $\mu^{\sim}([p], x, [q], y) > 0$. Then $\mu(s, x, t, y) > 0$ for some $s \in [p]$ and $t \in [q]$. But then $s \sim t, s \in [p]$ and $t \in [q]$ implies that $[s] = [t]$ and $[p] = [q]$.
(3) $\Rightarrow$ (1): To prove that $\sim$ is a direct sum equivalence relation, it is sufficient to prove that $[p]_{\sim}$, for $p \in Q$ is a subsystem of M. Let $q \in [p]_{\sim}, x \in X^*, y \in Y^*$ and $\mu(q, x, t, y) > 0$ for some $t \in Q$. Then $\mu^{\sim}([p], x, [t], y) > 0$. By assumption $[p] = [t]$. Hence $t \in [p]_{\sim}$. □

The following theorem characterize strongly connected fuzzy Mealy machines in terms of the relation γ defined in the section 3.

Theorem 5.11. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then the following statements are equivalent:*

(1) *M is strongly connected*
(2) *$S(q) = Q, \ \forall \, q \in Q$*
(3) *$P(q) = Q, \ \forall \, q \in Q$*
(4) *γ is the universal relation on M.*

Proof. **(1) $\Rightarrow$ (2):** and **(2) $\Rightarrow$ (3):** are trivial.
(3) $\Rightarrow$ (4): For all $p, q \in Q$ we have $P(p) = P(q)$. Therefore, $p \gamma q, \ \forall \, p, q$. This prove that γ is the universal relation on M.
(4) $\Rightarrow$ (1): γ is the universal relation on M implies that $\mu(p, x, q, y) > 0$, for all $p, q \in Q, x \in X^*, y \in Y^*$. Thus by definition M is strongly connected. □

We now prove that $DSC(M)$, the lattice of all direct sum congruence relations on M, is the principal dual ideal of the lattice $E(M)$, of all equivalence relations of M. To describe the generator of the principal dual ideal, we first introduce the relation $\frown$ (called *disjoint successor relation*) on Q as $p \frown q$ if and only if $S(p) \cap S(q) \neq \phi$. Clearly $\frown$ is reflexive and symmetric but not necessarily transitive. We denote $\frown^*$ by the transitive closure of the relation $\frown$ on Q. This $\frown^*$ serve as the generator for the principal dual ideal of $DSC(M)$ of $E(M)$.

Theorem 5.12. *$DSC(M)$ is a principal dual ideal of $E(M)$ generated by $\frown^*$.*

Proof. Clearly by Theorem (5.10) the relation $\frown^*$ is a direct sum congruence on M. Let $I(\frown^*)$ denotes the principal dual ideal of $E(M)$ generated by $\frown^*$. If $\# \in I(\frown^*)$, then $\frown^* \leq \#$ and $\# \in DSC(M)$. On the other hand if $\# \in DSC(M)$ and $p \frown^* q$ for $p, q \in Q$, then there exists $r \in S(p) \cap S(q)$. This proves $p\#r$ and $q\#r$, by Theorem (5.10). But then $p\#q$. Thus, $\frown^* \leq \#$. Hence $\# \in I(\frown^*)$. □

This theorem reduces to:

Theorem 5.13. *The set of all direct sum decompositions of M is a principal ideal of the partition lattice of M.*

Theorem 5.14. *The lattice of direct sum decomposition of M is isomorphic to the lattice of complete Boolean subalgebra of $Filt(M)$.*

Proof. To prove the above isomorphism, we find an order isomorphism between these two lattices. Let B be a complete Boolean subalgebra of $Filt(M)$. For $q \in Q$, let $B(q)$ denotes principal element of B generated by q. Since B is a complete sublattice of $Filt(M)$, by Theorem (4.4), B is atomic, and the atoms of B are exactly its principal elements. Set $\mathcal{D}_B = \{B(q)|q \in Q\}$. It is clear that $\mathcal{D}_B$ is a direct sum decomposition of M, whose summands are exactly the atoms of B. We will prove that the mapping $B \longrightarrow \mathcal{D}_B$ is an order isomorphism of the lattice of complete Boolean subalgebra of $Filt(M)$ onto the lattice of direct sum decomposition of M.

Let B and E be two complete Boolean subalgebras of $Filt(M)$. If $B \subseteq E$, then $E(q) \subseteq B(q)$, for any $q \in Q$, hence $\mathcal{D}_B \subseteq \mathcal{D}_E$ in $Part(M)$. Conversely, Let $\mathcal{D}_B \subseteq \mathcal{D}_E$ in $Part(M)$. Then for any $q \in Q$, $\exists\, p \in Q$ such that $E(q) \subseteq B(p)$ and $q \in B(q) \Rightarrow q \in B(p)$. Hence, $B(q) \subseteq B(p) \Rightarrow B(q) = B(p)$. Since $B(p)$ is atom in B. Therefore, $E(q) \subseteq B(q)$ for any $q \in Q$ which means that $B \subseteq E$. Hence, $B \subseteq E \Leftrightarrow \mathcal{D}_B \subseteq \mathcal{D}_E$.

Now we prove that the mapping $B \longrightarrow \mathcal{D}_B$ is onto. Let $\mathcal{D} = \{M_\alpha|\alpha \in Z\}$ be an arbitrary direct sum decomposition of M. By Theorem (4.4), $M_\alpha(\alpha \in Z)$ are filters of M. Set $B = \{F \in Filt(M)|(I \subseteq Z)F = \bigcup_{\alpha \in I} M_\alpha\}$. Note that $\phi \in B$ since we can assume $I\phi$. Then B is a complete Boolean subalgebra of $Filt(M)$, and so is a complete atomic Boolean algebra whose atoms are exactly $M_\alpha(\alpha \in Z)$. In the other words, for $\alpha \in Z$ and $\alpha M_\alpha, M_\alpha = B(q)$. Now we have $\mathcal{D} = \mathcal{D}_B$, which proves that $B \Leftrightarrow \mathcal{D}_B$ is onto. Hence the theorem. □

Theorem 5.15. *A fuzzy Mealy machine $M = (Q, X, Y, \mu)$ can be represented as a direct sum of direct sum indecomposable fuzzy Mealy machines. This is the greatest direct sum decomposition of M and its summands are the atoms of $Filt(M)$.*

Proof. The existence of the greatest direct sum decomposition of M follows by the Theorem (5.13), and this decomposition corresponds to the greatest complete Boolean subalgebra of $Filt(M)$, i.e. to the whole Boolean algebra $Filt(M)$, and its summands are exactly the atoms of $Filt(M)$. Let B be an arbitrary summand in the greatest direct decomposition of M. If B is not direct decomposable, then B has a proper filter C. Since, if B subset of M is filter of M if and only if B is a direct summand of M. Therefore, C is also a filter of M, which contradicts the fact that B is an atom of $Filt(M)$. Therefore, any summand in the greatest direct sum decomposition of M must be direct sum indecomposable. □

Theorem 5.16. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine. Then following statements are equivalent:*

(1) *M is a direct sum of strongly connected fuzzy Mealy machine;*
(2) *γ is a direct sum congruence on M;*
(3) *$Sub(M) \subseteq DSub(M)$;*
(4) *$DSub(M) \subseteq Sub(M)$;*
(5) *$S(q) = P(q)$, for any $q \in Q$;*
(6) *$Sub(M)$ is a Boolean algebra.*

Proof. **(1) ⇒ (2):** Suppose M is direct sum of strongly connected subsystem. We have prove that γ is a direct sum congruence on M. Let G_q be a γ-class containing q. Then $G_q = S(q) \cap P(q)$. Now we have to prove G_q is subsystem. $M = \sum M_\alpha$ where each M_α is strongly connected subsystem. We have $q \in M_\alpha$. Then, $S(q) = M_\alpha$. Similarly, $P(q) = M_\alpha$.(since M_α is strongly connected) Therefore, $S(q) \cap P(q) = M_\alpha$ that is $G_q = M_\alpha$. Thus, G_α is subsystem and hence,γ is a direct sum congruence on M.

(2) ⇒ (1): Given γ is direct sum congruence $\gamma = \frown^*$, which means that $Filt(M) = Sub(M)$. Hence, M is direct sum indecomposable and since γ is direct sum congruence. Therefore, M is direct sum of strongly connected fuzzy Mealy machine.

(2) ⇒ (3): Suppose γ is a direct sum congruence on M. Therefore, G_q is subsystem of M. Let $B \in Sub(M)$. Let $\mu(q, u, v, p) > 0$ with $p \in B$ We prove that $q \in B$. We have $p\gamma q$. (since G_q is a subsystem of M.) $\Rightarrow S(p) = S(q)$. Since $p \in B$ and B is subsystem. Therefore, $S(p) \subseteq B$ hence $S(q) \subseteq B$. But $q \in S(q) \subseteq B$. Therefore, $B \in Sub(M)$. Hence, $Sub(M) \subseteq DSub(M)$

(3) ⇒ (4): Suppose, $Sub(M) \subseteq DSub(M)$ Then we know that $Sub(M) = Filt(M)$ and by Theorem (4.4), $Filt(M)$ is Boolean algebra and hence, $DSub(M) \subseteq Sub(M)$

(4) ⇒ (5): Suppose $DSub(M) \subseteq Sub(M)$ We have to prove $S(q) = P(q)$ for any $q \in Q$. let $p \in S(q)$ $\exists\, u \in X^*, v \in Y^*$ such that $\mu(q, u, v, p) > 0$. Now $q \in P(q)$ and $P(q) \in Dsub(M) \subseteq Sub(M) \Rightarrow P(q)$ is subsystem. Therefore, $p \in P(q)$. Therefore, $S(q) \subseteq P(q)$. Now we prove $P(q) \subseteq S(q)$. Let $p \in P(q) \Rightarrow q \in S(p) \subset P(q) \Rightarrow p \in P(q) \Rightarrow P(q) \subseteq S(q)$. Hence, $S(q) = P(q)$ for any $q \in Q$

(5) ⇒ (6): Suppose, $S(q) = P(q)$ for any $q \in Q$. To prove $Sub(M)$ is a Boolean algebra it enough to prove $Sub(M) \subseteq DSub(M)$. Let $S(p) \in Sub(M)$ and $S(q) = P(q)$ Therefore, $S(p) \in DSub(M) \Rightarrow Sub(M) \subseteq DSub(M) \Rightarrow Sub(M) = Filt(M)$ and $Filt(M)$ is Boolean algebra.

(6) ⇒ (2): Given $Sub(M)$ is Boolean algebra. Therefore, $Sub(M)$ is $Filt(M)$, hence, $Filt(q) = Sub(q)\ \forall\, q \in Q$ and $\gamma = \frown^*$. Therefore, γ is direct sum congruence on M. □

Finally we conclude that

Theorem 5.17. *Let $M = (Q, X, Y, \mu)$ be a fuzzy Mealy machine is strongly connected if and only if $Sub(M)$ is a two-element Boolean algebra.*

6. Conclusion

This paper shows that the set of all subsystems, $Sub(M)$, and dual subsystems, $Dsub(M)$, of a given fuzzy Mealy machine M are endowed with a lattice structure (with respect to inclusion relation) on them. Hence, we attempt to discuss usual concepts of fuzzy Mealy machine such as successor, predecessor (of elements and subsets), strongly connected subsets, strongly connected subsystems and filters with their lattice theoretic meaning (such as principal element, completely irreducible element, center, atom etc.) in $Sub(M)$ and $Dsub(M)$. This leads us to discuss direct sum decomposition of a Fuzzy Mealy machine M and its characterizations through atoms, dual atoms and universal relation on M. We establish that the $DSC(M)$ -lattice of all direct sum congruence relations on M, is the principal dual ideal of the lattice $E(M)$- of all equivalence relations of M. Finally, we conclude that strongly connected fuzzy Mealy machine M have $Sub(M)$ as a two-element Boolean algebra and conversely.

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