

Intuitionistic fuzzy incline matrix and determinant

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ABSTRACT. In this paper, we introduce intuitionistic fuzzy incline (IFI), intuitionistic fuzzy incline matrix (IFIM) and its determinant. Also the transitive closure, power of convergent, nilpotence of IFIM and adjoint of an IFIM are considered here. Some properties of determinant of IFIM and triangular IFIM are also introduced here.

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1. INTRODUCTION

Inclines are the additively idempotent semirings in which products are less than or equal to factors. Thus inclines are generalized Boolean algebra, fuzzy algebra and distributive lattice. The Boolean matrices, the fuzzy matrices and the lattice matrices are the prototypical examples of the incline matrices. Boolean algebra and fuzzy algebra are applied to automata theory, design of switching circuits, logic of binary relations, medical diagnosis, etc. Marcov chain, information system and clustering are instances in which inclines can be applied. Also, inclines are applied to nervous system, probable reasoning, finite state machines, psychological measurement, dynamical programming, decision theory, etc.

In 1965, Zadeh [21] developed fuzzy set first, then in 1984 Cao et al. [4] developed incline algebra and its applications. After that several researchers [8, 9, 13, 14, 17] work on this topics. In 1986, Atanassov [1] introduced intuitionistic fuzzy sets (IFS) which becomes a popular topics for investigation in the fuzzy sets community. With max-min operation the fuzzy algebra and its matrix theory are considered by many authors [3, 6, 15, 18, 19]. Determinant theory, powers and nilpotent conditions of matrices over a distributive lattice are considered by Zhang [22] and Tan [20] and the transitivity of matrices over path algebra (i.e., additively idempotent semiring) is discussed by Hashimoto [10, 11, 12]. Generalized fuzzy matrices, matrices over an

incline and some results about the transitive closer, determinant, adjoint matrices, convergence of powers and conditions for nilpotency are considered by Duan [5] and Lur et al. [16].

2. PRELIMINARIES

In this section we recall some basic notions of incline and intuitionistic fuzzy sets.

Definition 2.1 (Semiring). A semiring is a set R equipped with two binary operations $+$ and \cdot , called addition and multiplication, such that

- (1) $(R, +)$ is a commutative monoid with identity 0.
- (2) (R, \cdot) is monoid with identity 1.
- (3) Multiplication distributes over addition.
- (4) 0 annihilates R , with respect to multiplication.
i.e., $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$.

An idempotent semiring (dioid) is one whose addition is idempotent: $a + a = a$ for all $a \in R$ that is $(R, +, \cdot)$ is a join-semilattice with zero.

Example 2.2. Any bounded distributive lattice is a commutative idempotent semiring under join and meet.

Definition 2.3 (Incline). An incline (algebra) is a set R on which two binary operations, denoted by $+$ and \cdot are defined, satisfying the following axioms. Let $a, b, c \in R$

- (A1). $+$ is commutative: $a + b = b + a$.
- (A2). $+$ and \cdot are associative: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$.
- (A3). \cdot distributed over $+$: $a(b + c) = ab + ac$, $(b + c)a = ba + ca$.
- (A4). $+$ is idempotent: $a + a = a$.
- (A5). The incline property holds: $a + ac = a$, $c + ac = c$.

Thus an incline is a semiring with idempotent addition in which the product with a suitable ordering is less than or equal to either factor. Products reduce the value of quantities and make them go down, which is why these structures were named inclines.

Example 2.4. Let $K = \{0, a, b, c, d, 1\}$ be a lattice ordered by the Hass graph shown in Figure 1. Define $R \times R \rightarrow R$ by $x \cdot y = d$ for all $x, y \in \{1, b, c, d\}$ and 0 otherwise. Then (R, \vee, \cdot) is an incline which is not a distributive lattice, where $x \vee y = \max\{x, y\}$.

The Hass diagram Fig(1) shows that, it is a lattice. Incline property also holds since

$$\begin{aligned} a \vee (a \cdot c) &= a \vee 0 = a \\ \text{and } c \vee (a \cdot c) &= c \vee 0 = c \end{aligned}$$

Also $b \vee (c \cdot d) = b \vee d = b$ but $(b \vee c) \cdot (b \vee d) = 1 \cdot b = d$.

That is, $b \vee (c \cdot d) \neq (b \vee c) \cdot (b \vee d)$.

Hence it is not a distributive lattice.

Note 1. An incline algebra R is said to be commutative if $xy = yx$ for all $x, y \in R$.

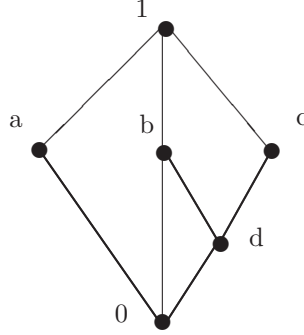


FIGURE 1. Intuitionistic fuzzy incline

Definition 2.5 (Intuitionistic fuzzy set). An intuitionistic fuzzy set (IFS) A in E (universe of discourse) is defined as an object of the following form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle | x \in E \},$$

where the functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ in A , respectively and for every $x \in E$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let \mathcal{I} be the set of all real numbers lying between 0 and 1, i.e., $\mathcal{I} = \{x : 0 \leq x \leq 1\}$. Also let $\langle F \rangle$ be the set of tuples $\langle a, b \rangle$, where $a, b \in \mathcal{I}$ and $0 \leq a + b \leq 1$ i.e.,

$$\langle F \rangle = \{ \langle a, b \rangle : 0 \leq a + b \leq 1, a, b \in \mathcal{I} \}.$$

The addition and multiplication between any two elements of $\langle F \rangle$ are defined bellow.

Definition 2.6. Let $x = \langle x_\mu, x_\nu \rangle$ and $y = \langle y_\mu, y_\nu \rangle$ are any two elements of $\langle F \rangle$. The addition (+) and multiplication (\cdot) between x and y are defined as

$$\begin{aligned} x + y &= \langle x_\mu, x_\nu \rangle + \langle y_\mu, y_\nu \rangle \\ &= \langle \max(x_\mu, y_\mu), \min(x_\nu, y_\nu) \rangle \\ &= \langle x_\mu \vee y_\mu, x_\nu \wedge y_\nu \rangle \\ \text{and } x \cdot y &= \langle x_\mu, x_\nu \rangle \cdot \langle y_\mu, y_\nu \rangle \\ &= \langle \min(x_\mu, y_\mu), \max(x_\nu, y_\nu) \rangle \\ &= \langle x_\mu \wedge y_\mu, x_\nu \vee y_\nu \rangle \end{aligned}$$

In arithmetic operations (such as addition, multiplication etc.) only the values of membership and nonmembership are needed. So from now we denote IFS as

$$A = \{x = \langle x_\mu, x_\nu \rangle | x \in E\}.$$

3. INTUITIONISTIC FUZZY INCLINE (IFI)

In this section first we proved that an intuitionistic fuzzy set is incline.

To prove this we consider an intuitionistic fuzzy set R in E , and let $x, y, z \in R$ where $x = \langle x_\mu, x_\nu \rangle$, $y = \langle y_\mu, y_\nu \rangle$, $z = \langle z_\mu, z_\nu \rangle$.

$$\begin{aligned}
 (A1). \quad x + y &= \langle x_\mu, x_\nu \rangle + \langle y_\mu, y_\nu \rangle \\
 &= \langle \max(x_\mu, y_\mu), \min(x_\nu, y_\nu) \rangle \\
 &= \langle \max(y_\mu, x_\mu), \min(y_\nu, x_\nu) \rangle \\
 &= \langle y_\mu, y_\nu \rangle + \langle x_\mu, x_\nu \rangle \\
 &= y + x.
 \end{aligned}$$

That is, commutative property under addition holds in R .

$$\begin{aligned}
 (A2). \quad x \cdot (y \cdot z) &= \langle x_\mu, x_\nu \rangle \cdot (\langle y_\mu, y_\nu \rangle \cdot \langle z_\mu, z_\nu \rangle) \\
 &= \langle x_\mu, x_\nu \rangle \cdot \langle (y_\mu \wedge z_\mu), (y_\nu \vee z_\nu) \rangle \\
 &= \langle x_\mu \wedge (y_\mu \wedge z_\mu), x_\nu \vee (y_\nu \vee z_\nu) \rangle \\
 &= \langle (x_\mu \wedge y_\mu) \wedge z_\mu, (x_\nu \vee y_\nu) \vee z_\nu \rangle \\
 &= \langle (x_\mu \wedge y_\mu), (x_\nu \vee y_\nu) \rangle \cdot \langle z_\mu, z_\nu \rangle \\
 &= (x \cdot y) \cdot z.
 \end{aligned}$$

Similarly, we can prove that $(x + y) + z = x + (y + z)$.

That is, associative properties under addition and multiplication hold in R .

$$\begin{aligned}
 (A3). \quad x \cdot (y + z) &= \langle x_\mu, x_\nu \rangle \cdot (\langle y_\mu, y_\nu \rangle + \langle z_\mu, z_\nu \rangle) \\
 &= \langle x_\mu, x_\nu \rangle \cdot \langle (y_\mu \vee z_\mu), (y_\nu \wedge z_\nu) \rangle \\
 &= \langle x_\mu \wedge (y_\mu \vee z_\mu), x_\nu \vee (y_\nu \wedge z_\nu) \rangle \\
 &= \langle (x_\mu \wedge y_\mu) \vee (x_\mu \wedge z_\mu), (x_\nu \vee y_\nu) \wedge (x_\nu \vee z_\nu) \rangle \\
 &= \langle (x_\mu \wedge y_\mu), (x_\nu \vee y_\nu) \rangle + \langle (x_\mu \wedge z_\mu), (x_\nu \vee z_\nu) \rangle \\
 &= \langle x_\mu, x_\nu \rangle \cdot \langle y_\mu, y_\nu \rangle + \langle x_\mu, x_\nu \rangle \cdot \langle z_\mu, z_\nu \rangle \\
 &= x \cdot y + x \cdot z.
 \end{aligned}$$

Similarly, we can prove that $(x + y) \cdot z = x \cdot z + y \cdot z$.

That is, multiplication is distributed over addition in R .

$$\begin{aligned}
 (A4). \quad x + x &= \langle x_\mu, x_\nu \rangle + \langle x_\mu, x_\nu \rangle \\
 &= \langle \max(x_\mu, x_\mu), \min(x_\nu, x_\nu) \rangle \\
 &= \langle x_\mu, x_\nu \rangle \\
 &= x.
 \end{aligned}$$

That is, addition is idempotent.

$$\begin{aligned}
 (A5). \quad x + x \cdot z &= \langle x_\mu, x_\nu \rangle + \langle x_\mu, x_\nu \rangle \cdot \langle z_\mu, z_\nu \rangle \\
 &= \langle x_\mu, x_\nu \rangle + \langle \min(x_\mu, z_\mu), \max(x_\nu, z_\nu) \rangle \\
 &= \langle \max\{x_\mu, \min(x_\mu, z_\mu)\}, \min\{x_\nu, \max(x_\nu, z_\nu)\} \rangle \\
 &= \langle x_\mu, x_\nu \rangle \\
 &= x.
 \end{aligned}$$

Similarly, $z + x \cdot z = z$.

Thus any intuitionistic fuzzy set is an incline. □

$$\begin{aligned}
 \text{Also } x \cdot y &= \langle x_\mu, x_\nu \rangle \cdot \langle y_\mu, y_\nu \rangle \\
 &= \langle x_\mu \wedge y_\mu, x_\nu \vee y_\nu \rangle \\
 &= \langle y_\mu \wedge x_\mu, y_\nu \vee x_\nu \rangle \\
 &= y \cdot x.
 \end{aligned}$$

That is, the commutative property under multiplication is also hold in R .

Hence any intuitionistic fuzzy set is commutative incline.

Definition 3.1. Let R be an IFI and $x, y \in R$ where $x = \langle x_\mu, x_\nu \rangle$ and $y = \langle y_\mu, y_\nu \rangle$ then $x = y$ if and only if $x_\mu = y_\mu$ and $x_\nu = y_\nu$.

Definition 3.2. Let R be an IFI and $x, y \in R$ where $x = \langle x_\mu, x_\nu \rangle$ and $y = \langle y_\mu, y_\nu \rangle$ then $x \leq y$ if and only if $x_\mu \leq y_\mu$ and $y_\nu \leq x_\nu$.

Definition 3.3. Let R be an IFI and $x, y \in R$ where $x = \langle x_\mu, x_\nu \rangle$ and $y = \langle y_\mu, y_\nu \rangle$ then $x < y$ if and only if $x \leq y$ and $x \neq y$.

Theorem 3.4. The relation ' \leq ' is partial order relation in an IFI.

Proof. Let R be an IFI and $x, y, z \in R$ where $x = \langle x_\mu, x_\nu \rangle, y = \langle y_\mu, y_\nu \rangle, z = \langle z_\mu, z_\nu \rangle$.

I. Since $x_\mu \leq x_\mu$ and $x_\nu \leq x_\nu$ then we write that $x \leq x$ for all $x \in R$ i.e., the relation ' \leq ' is reflexive.

II. Let $x \leq y$ and $y \leq x$ for any $x, y \in R$. Then

$$\begin{aligned} x_\mu &\leq y_\mu, y_\nu \leq x_\nu \text{ and } y_\mu \leq x_\mu, x_\nu \leq y_\nu \\ \Rightarrow x_\mu &= y_\mu \text{ and } x_\nu = y_\nu \\ \Rightarrow x &= y. \end{aligned}$$

Thus $x \leq y$ and $y \leq x \Rightarrow x = y$ for any $x, y \in R$.

That is, the relation ' \leq ' is antisymmetric.

III. Let $x \leq y$ and $y \leq z$ for any $x, y, z \in R$. Then

$$\begin{aligned} x_\mu &\leq y_\mu, y_\nu \leq x_\nu \text{ and } y_\mu \leq z_\mu, z_\nu \leq y_\nu \\ \Rightarrow x_\mu &\leq y_\mu \leq z_\mu \text{ and } z_\nu \leq y_\nu \leq x_\nu \\ \Rightarrow x_\mu &\leq z_\mu \text{ and } z_\nu \leq x_\nu \\ \Rightarrow x &\leq z. \end{aligned}$$

Thus $x \leq y$ and $y \leq z \Rightarrow x \leq z$ for any $x, y, z \in R$.

That is, the relation ' \leq ' is transitive. Hence the relation ' \leq ' in an IFI is a partial order relation. \square

Atanassov [2] proposed that an intuitionistic fuzzy set is a lattice with respect to the operation ' \subset ' (inclusion) which is the same with the relation ' \leq ' we defined.

Definition 3.5 (Zero element). The zero element of an IFI is denoted by \emptyset and is define by $\emptyset = \langle 0, 1 \rangle$.

Definition 3.6 (Unit element). The unit element of an IFI is denoted by I and is defined by $I = \langle 1, 0 \rangle$.

Cao et al. [4] and Golan [7] proved some propositions in incline algebra. In IFI these are also true.

Proposition 3.7. Let R be an IFI and $x, y, z \in R$ then

- (a) $\emptyset \leq x \leq I$.
- (b) if $x \leq y$ then $x + z \leq y + z, xz \leq yz$ and $zx \leq zy$.
- (c) $x \leq x + y$ and $x + y$ is the least upper bound of x and y . In other words, if there is an element z satisfying $x \leq z$ and $y \leq z$ then $x + y \leq z$.
- (d) $xy \leq x, xy \leq y$. That is, xy is a lower bound of x and y .
- (e) $xzy \leq xy$.
- (f) $x + y = \emptyset$ if and only if $x = \emptyset = y$.
- (g) $xy = I$ if and only if $x = I = y$.

Proof. (a) Here $x = \langle x_\mu, x_\nu \rangle, y = \langle y_\mu, y_\nu \rangle$ and $z = \langle z_\mu, z_\nu \rangle$. Since $0 \leq x_\mu, x_\nu \leq 1$. Therefore $\emptyset \leq x \leq I$.

(b) Let $x \leq y$ then $x_\mu \leq y_\mu$ and $y_\nu \leq x_\nu$. Therefore $\max(x_\mu, z_\mu) \leq \max(y_\mu, z_\mu)$ and $\min(y_\nu, z_\nu) \leq \min(x_\nu, z_\nu)$. Thus $x + z \leq$

$y + z$.

Also $\min(x_\mu, z_\mu) \leq \min(y_\mu, z_\mu)$ and $\max(y_\nu, z_\nu) \leq \max(x_\nu, z_\nu)$. Thus $xz \leq yz$.

Similarly, $zx \leq zy$.

(c) We know that $x_\mu \leq \max(x_\mu, y_\mu)$ and $\min(x_\nu, y_\nu) \leq x_\nu$. So $x \leq x + y$.

Similarly, $y \leq x + y$.

Thus $x + y$ is the upper bound of x and y .

If possible let $z \neq x + y$ be the least upper bound of x and y then

$$\begin{aligned} x &\leq z \text{ and } y \leq z \\ \text{i.e., } x_\mu &\leq z_\mu, z_\nu \leq x_\nu \text{ and } y_\mu \leq z_\mu, z_\nu \leq y_\nu. \end{aligned}$$

Therefore $\max(x_\mu, y_\mu) \leq z_\mu$ and $z_\nu \leq \min(x_\nu, y_\nu)$.

Thus

$$(3.1) \quad x + y \leq z.$$

Also, since $x + y$ is the upper bound of x, y ; and z is the least upper bound so

$$(3.2) \quad z \leq x + y$$

From equation (3.1) and (3.2) it can be written as $x + y = z$

i.e., $x + y$ is the least upper bound of x and y .

(d) Similarly, we can prove that xy is the greatest lower bound of x and y .

(e) We know that $\min(x_\mu, z_\mu, y_\mu) \leq \min(x_\mu, y_\mu)$ and $\max(x_\mu, y_\mu) \leq \max(x_\nu, z_\nu, y_\nu)$.

Therefore $xzy \leq xy$.

(f) Let $x + y = \emptyset$

i.e., $\langle \max(x_\mu, y_\mu), \min(x_\nu, y_\nu) \rangle = \langle 0, 1 \rangle$.

Therefore, $\max(x_\mu, y_\mu) = 0$ and $\min(x_\nu, y_\nu) = 1$.

Also $0 \leq x_\mu, y_\mu, x_\nu, y_\nu \leq 1$.

Hence $x_\mu = 0 = y_\mu$ and $x_\nu = 1 = y_\nu$.

Therefore $x = \langle x_\mu, x_\nu \rangle = \langle 0, 1 \rangle = \emptyset$ and

$y = \langle y_\mu, y_\nu \rangle = \langle 0, 1 \rangle = \emptyset$.

(g) Let $xy = I$

i.e., $\langle \min(x_\mu, y_\mu), \max(x_\nu, y_\nu) \rangle = \langle 1, 0 \rangle$.

Therefore, $\min(x_\mu, y_\mu) = 1$ and $\max(x_\nu, y_\nu) = 0$.

Also $0 \leq x_\mu, y_\mu, x_\nu, y_\nu \leq 1$.

Hence $x_\mu = 1 = y_\mu$ and $x_\nu = 0 = y_\nu$.

Therefore $x = \langle x_\mu, x_\nu \rangle = \langle 1, 0 \rangle = I$ and $y = \langle y_\mu, y_\nu \rangle = \langle 1, 0 \rangle = I$.

Thus under the relation ' \leq ' an IFI is a partial order set, every pair of which has a least upper bound and greatest lower bound in IFI. So an IFI is a lattice. \square

Definition 3.8 (Inverse element). Let R be an IFI and $x \in R$ where $x = \langle x_\mu, x_\nu \rangle$ the inverse element x is denoted by $x^- \in R$ and defined by

$$\begin{aligned} x + x^- &= \emptyset \quad \text{for addition (max-min)} \\ \text{and } x \cdot x^- &= I \quad \text{for multiplication (min-max)}. \end{aligned}$$

The following theorem says that the inverse of any element of an IFI does not exist under max-min and min-max operations.

Theorem 3.9. *The inverse of the elements of IFI does not exist under '+' and '·', except the unit elements.*

Proof. Let R be an IFI and let $x = \langle x_\mu, x_\nu \rangle \neq \emptyset$ be any element of R . If possible let $y = \langle y_\mu, y_\nu \rangle$ be the inverse of x in R under $+$.

i.e., $x + y = \emptyset$.

Then by using Proposition 3.7(f) we get $x = \emptyset = y$, which contradict that $x \neq \emptyset$.

Thus inverse element does not exist except the additive identity \emptyset .

Similarly, we can prove that the inverse element does not exist except the multiplicative identity I for multiplication. \square

4. INTUITIONISTIC FUZZY INCLINE MATRIX (IFIM)

The set of square matrix of order n over an IFI R is denoted by $M_n(R)$. The zero matrix $O_n = [\langle 0, 1 \rangle]$ and the identity matrix I_n (whose main diagonal elements are all I and all other elements are \emptyset) of order n are defined as if R were a field. For matrix $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ and $B = [b_{ij}] = [\langle b_{ij\mu}, b_{ij\nu} \rangle]$ in $M_n(R)$, define the operations $A + B = [a_{ij} + b_{ij}] = [\langle a_{ij\mu} \vee b_{ij\mu}, a_{ij\nu} \wedge b_{ij\nu} \rangle]$

and $AB = \left[\sum_{k=1}^n a_{ik} b_{kj} \right] = \left[\langle \bigvee_k (a_{ik\mu} \wedge b_{kj\mu}), \bigwedge_k (a_{ik\nu} \vee b_{kj\nu}) \rangle \right]$.

Theorem 4.1. $M_n(R)$ is an additively idempotent semiring under matrix addition and multiplication with additive identity O_n and multiplicative identity I_n . But no longer an incline.

Proof. Let $A, B, C \in M_n(R)$ where $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ also $a_{ij}, b_{ij}, c_{ij} \in R$ for all $i, j \in \{1, 2, 3, \dots, n\}$.

$$\begin{aligned} (A1). \text{ Now } A + B &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \text{ [By commutative property of IFI]} \\ &= B + A. \end{aligned}$$

That is, commutative property holds under addition.

$$\begin{aligned} (A2). \quad A(BC) &= [a_{ij}] \left[\sum_{k=1}^n b_{ik} c_{kj} \right] \\ &= \left[\sum_{l=1}^n a_{il} d_{lj} \right] \text{ [where } d_{ij} = \sum_{k=1}^n b_{ik} c_{kj}] \\ &= \left[\sum_{l=1}^n a_{il} \left(\sum_{k=1}^n b_{lk} c_{kj} \right) \right] \\ &= \left[\sum_{l=1}^n \sum_{k=1}^n a_{il} b_{lk} c_{kj} \right] \\ &= \left[\sum_{k=1}^n \left(\sum_{l=1}^n a_{il} b_{lk} \right) c_{kj} \right] \\ &= \left[\sum_{k=1}^n e_{ik} c_{kj} \right] \text{ [where } e_{ik} = \sum_{l=1}^n a_{il} b_{lk}] \\ &= [e_{ij}] [c_{ij}] \\ &= \left[\sum_{l=1}^n a_{il} b_{lj} \right] [c_{ij}] \\ &= (AB)C. \end{aligned}$$

$$\begin{aligned} \text{Also } A + (B + C) &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \text{ [By associative property of IFI]} \\ &= (A + B) + C. \end{aligned}$$

That is, associative properties hold for the addition and multiplication.

$$\begin{aligned}
 (A3). \quad A(B + C) &= [a_{ij}][b_{ij} + c_{ij}] \\
 &= \left[\sum_{k=1}^n a_{ik}(b_{kj} + c_{kj}) \right] \\
 &= \left[\sum_{k=1}^n (a_{ik}b_{kj} + a_{ik}c_{kj}) \right] \quad [\text{By distributive property of IFI}] \\
 &= \left[\sum_{k=1}^n a_{ik}b_{kj} \right] + \left[\sum_{k=1}^n a_{ik}c_{kj} \right] \\
 &= AB + AC.
 \end{aligned}$$

Similarly, we can prove that $(A + B)C = AC + BC$. That is, multiplication is distributive over addition.

$$\begin{aligned}
 (A4). \quad A + A &= [a_{ij} + a_{ij}] \\
 &= [a_{ij}] \quad [\text{By idempotent property of IFI}] \\
 &= A.
 \end{aligned}$$

That is, addition is idempotent.

Hence the set of square matrices of order n over an IFI R i.e., $M_n(R)$ is a semiring under matrix addition and multiplication.

To show that $M_n(R)$ is not an incline we follow:

$$\begin{aligned}
 (A5). \quad A + AC &= [a_{ij}] + \left[\sum_{k=1}^n a_{ik}c_{kj} \right] \\
 &= \left[a_{ij} + \sum_{k=1}^n a_{ik}c_{kj} \right] \quad \square \\
 &\neq [a_{ij}] = A.
 \end{aligned}$$

Definition 4.2. The partial order relation ' \leq ' over $M_n(R)$ is defined as $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j \in \{1, 2, 3, \dots, n\}$.

That is $A \leq B$ if and only if $A + B = B$. $A < B$ denotes $A \leq B$ and $A \neq B$.

Definition 4.3. Let R be an IFI and $A \in M_n(R)$. The ij -th entry of square matrix A^m is denoted by $a_{ij}^{(m)}$, and obviously

$$(4.1) \quad a_{ij}^{(m)} = \sum_{1 \leq j_1, j_2, \dots, j_{m-1} \leq n} a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{m-1} j}.$$

A matrix A is said to be nilpotent if $A^k = O_n$ for some $k \in N$, A is idempotent if $A^2 = A$.

Lemma 4.4. Let R be an IFI and $A \in M_n(R)$. If $m \geq n$, then

$$A^m \leq \sum_{k=0}^{n-1} A^k. \quad (\text{Here } A^0 = I_n.)$$

Proof. Let $B = \sum_{k=0}^{n-1} A^k$.

Now $a_{ii}^{(m)} \leq I = b_{ii}$. [Since $a_{ii}^{(0)} = I$.]

If $i \neq j$ we consider an arbitrary summand of right hand side of equality (4.1),
 $a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{m-1} j}.$

Since $i, j_1, j_2, j_3, \dots, j_{m-1}, j \in \{1, 2, 3, \dots, n\}$ and $m+1 > n$, there are r, s such that $j_r = j_s$ ($0 \leq r < s \leq m$, $j_0 = i$, $j_n = j$). Deleting $a_{j_r j_{r+1}} a_{j_{r+1} j_{r+2}} a_{j_{r+2} j_{r+3}} \dots a_{j_{s-1} j_s}$ from the summand $a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{m-1} j}$, we obtain

$$a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{m-1} j} \leq a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{r-1} j_r} a_{j_s j_{s+1}} \dots a_{j_{m-1} j}.$$

[By Proposition 3.7(e).] If the number $r + m - s + 2$ of the subscripts in the right hand side of the above inequality still more than n , the same deleting method is used.

Therefore, there is a positive integer $t \leq n - 1$ such that

$$a_{ij_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{m-1} j} \leq a_{il_1} a_{l_1 l_2} a_{l_2 l_3} \dots a_{l_{t-1} j}.$$

Hence by the definition of A^m we have

$$\begin{aligned} a_{ij}^{(m)} &\leq \sum_{k=1}^{n-1} a_{ij}^k = b_{ij} \\ \text{i.e., } A^m &\leq \sum_{k=0}^{n-1} A^k \end{aligned}$$

□

Remark 4.5. From the above lemma we conclude that $A^{m+1} \leq \sum_{k=1}^n A^k$.

Definition 4.6 (Transitive closure). Let R be an IFI and $A, B \in M_n(R)$. Matrix A is said to be transitive, if $A^2 \leq A$. Matrix B is said to be transitive closure of matrix A , if B is transitive, $A \leq B$ and $B \leq C$ for any transitive matrix C , satisfying $A \leq C$. The transitive closure of matrix A is denoted by $t(A)$.

Theorem 4.7. Let R be an IFI and $A \in M_n(R)$. Then the transitive closure of matrix A is given by $t(A) = \sum_{k=1}^n A^k$.

Proof. Let $B = \sum_{k=1}^n A^k$, obviously $A \leq B$.

Since $M_n(R)$ is additively idempotent, we have

$$\begin{aligned} B^2 &= \sum_{k=2}^{2n} A^k \leq \sum_{k=1}^{2n} A^k \\ \text{or } B^2 &\leq B + \sum_{k=n+1}^{2n} A^k. \end{aligned}$$

By the Lemma 4.4, $A^k \leq \sum_{l=1}^n A^l = B$ as $k > n$.

Hence $B^2 \leq B$.

If there is a matrix C such that $A \leq C$ and $C^2 \leq C$, then $A^2 \leq AC \leq C^2 \leq C$, and by induction we have $A^k \leq C^k < C$ for all positive integers k .

Hence $B \leq C$.

Thus by the definition of transitive closure, we obtain $B = t(A) = \sum_{k=1}^n A^k$. □

Example 4.8. Let

$$A = \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix}.$$

That is $A \in M_2(R)$. Now

$$A^2 = \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix} \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix} = \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix}$$

Therefore

$$\begin{aligned} t(A) &= A + A^2 \\ &= \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix} + \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.8, 0.1 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.4 \rangle \end{bmatrix} \\ &= A. \end{aligned}$$

Definition 4.9. Let R be an IFI and $A \in M_n(R)$. A is said to be power-convergent if $A^p = A^{p+1}$ for some positive integer p . If A is power-convergent the least positive integer p such that $A^p = A^{p+1}$ is called the index of A and is denoted by $i(A)$.

Definition 4.10. Let R be an IFI and $A \in M_n(R)$. A is said to be row-diagonally dominant if $a_{ij} \leq a_{ii}$ ($1 \leq i, j \leq n$). A is column-diagonally dominant if $a_{ji} \leq a_{ii}$ ($1 \leq i, j \leq n$).

Bhowmik and Pal [3] introduced the max-min and min-max compositions over intuitionistic fuzzy matrices (IFMs). They also investigated the conditions for convergence of IFM.

Theorem 4.11 ([3]). Let A be an IFM.

- (a) If $A^q \leq A^p$ where $q < p$ then A converges.
- (b) If for all $i, j \leq n$ there exists $k \leq n$ such that $a_{ij\mu} \leq a_{ik\mu}a_{kj\mu}$ and $a_{ij\nu} \geq a_{ik\nu} + a_{kj\nu}$ then A converges to A^c where $c \leq (n - 1)$.

From the above theorems we conclude the following result.

Theorem 4.12. Let R be an IFI and $A \in M_n(R)$. If $A^q \leq A^p$ where $q < p$ and A is row or column diagonally dominant, then A is power convergent and converges to $t(A)$ i.e., transitive closure of A .

Proof. From above theorem if $A^q \leq A^p$ where $q < p$ then A converges, taking $q = 1, p = 2$ we get $A \leq A^2$. Similarly $A^2 \leq A^3 \leq A^4 \leq \dots$.

$$\text{Now } t(A) = \sum_{k=1}^n A^k = A + A^2 + A^3 + \dots + A^n.$$

Also since A is row or column diagonally dominant then A converges to A^c for some $c \leq n - 1$ (see Corollaries 5.3 and 5.4 of [3]), then we get

$$A \leq A^2 \leq A^3 \leq \dots \leq A^c = A^{c+1} = A^{c+2} = \dots = A^n.$$

Therefore $t(A) = A^c$.

Thus A is converge to $t(A)$. □

5. DETERMINANT OF IFIM

The determinant theory of intuitionistic fuzzy matrix (IFM) was introduced by Pal [18] is important in IFI. He define some terms related to determinant theory in IFM and proved some results. Here we introduce these with respect to IFIM.

Definition 5.1. Let R be an IFI and $A \in M_n(R)$. The determinant $|A|$ (or permanent) of the matrix A is defined as follows:

$$|A| = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, 3, \dots, n\}$.

Definition 5.2 (Adjoint of IFIM). Let R be an IFI and $A \in M_n(R)$ ($n \geq 2$). A matrix B is said to be adjoint of the matrix A if $b_{ij} = |A_{ji}|$ ($1 \leq i, j \leq n$) where A_{ji} is the matrix of order $n - 1$ formed by deleting row j and column i from A . The adjoint of the matrix A is denoted by $adj(A)$.

Proposition 5.3. Let R be an IFI and $A, B \in M_n(R)$ then

- (a) $|A'| = |A|$, where A' denote the transpose of A .
- (b) $|rA| = r|A|$, where $r \in (0, 1]$ and $rA = [ra_{ij}] = [\langle ra_{ij\mu}, ra_{ij\nu} \rangle]$.
- (c) $|E_{ij}A| = |AE_{ij}| = |A|$, where E_{ij} (elementary matrix) is the matrix obtain from the identity matrix I_n by interchanging row i and row j .
- (d) $|A| = \sum_{j=1}^n a_{ij} |A_{ij}|$, where A_{ij} is the matrix of order $n - 1$ formed by deleting row i and column j from A .

Proof. The proposition (a) and (c) are proved by Khan and Pal [15].

(b) Let $|A| = \langle x_\mu, y_\nu \rangle$.

$$\begin{aligned} \text{Now } |rA| &= \sum_{\sigma \in S_n} ra_{1\sigma(1)} ra_{2\sigma(2)} ra_{3\sigma(3)} \cdots ra_{n\sigma(n)} \\ &= \sum_{\sigma \in S_n} \langle ra_{1\sigma(1)\mu}, ra_{1\sigma(1)\nu} \rangle \langle ra_{2\sigma(2)\mu}, ra_{2\sigma(2)\nu} \rangle \cdots \langle ra_{n\sigma(n)\mu}, ra_{n\sigma(n)\nu} \rangle. \end{aligned}$$

Also after calculating the value of the above determinant, the value must belongs to $\langle F \rangle$, whose both the membership and non-membership values are multiplied by r . Therefore the value must be $\langle rx_\mu, rx_\nu \rangle$.

$$\begin{aligned} \text{Therefore } |rA| &= \langle rx_\mu, rx_\nu \rangle \\ &= r \langle x_\mu, x_\nu \rangle \\ &= r|A|. \end{aligned}$$

(d) We know that

$$\begin{aligned} |A| &= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} \\ &= \sum_{j=1}^n \sum_{\sigma \in S_n, \sigma(i)=j} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} \\ &= \sum_{j=1}^n a_{ij} \sum_{\Pi \in S_{n_i n_j}} a_{1\Pi(1)} a_{2\Pi(2)} \cdots a_{i-1\Pi(i-1)} a_{i+1\Pi(i+1)} \cdots a_{n\Pi(n)} \end{aligned}$$

where $n_i = \{1, 2, 3, \dots, n\} \setminus \{i\}$ and $S_{n_i n_j}$ is the set of all bijections from the set n_i to the set n_j . Now by the definition of determinant we see that

$$|A_{ij}| = \sum_{\Pi \in S_{n_i n_j}} a_{1\Pi(1)} a_{2\Pi(2)} \cdots a_{i-1\Pi(i-1)} a_{i+1\Pi(i+1)} \cdots a_{n\Pi(n)}.$$

Hence $|A| = \sum_{j=1}^n a_{ij} |A_{ij}|.$

□

Definition 5.4. Let R be an IFI and $A \in M_n(R)$. We define $A(p \Rightarrow q)$ be the matrix obtain from A by replacing row q of A by row p of A .

Proposition 5.5. Let R be an IFI and $A, B \in M_n(R)$, then

- (a) $|A||B| \leq |AB|$
- (b) $|A|^n + |A||adj(A)| \leq |A adj(A)|$
- (c) $|A|^n + |A||adj(A)| \leq |adj(A)A|.$

Proof. (a) We Know that if $A = [a_{ij}]$ and $B = [b_{ij}]$ then $AB = \left[\sum_{k=1}^n a_{ik} b_{kj} \right]$. Therefore,

$$\begin{aligned} |AB| &= \sum_{\sigma \in S_n} \left(\sum_{k=1}^n a_{1k} b_{k\sigma(1)} \sum_{k=1}^n a_{2k} b_{k\sigma(2)} \sum_{k=1}^n a_{3k} b_{k\sigma(3)} \cdots \sum_{k=1}^n a_{nk} b_{k\sigma(n)} \right) \\ &= \sum_{k_1, k_2, \dots, k_n} \left(\sum_{\sigma \in S_n} a_{1k_1} a_{2k_2} a_{3k_3} \cdots a_{nk_n} b_{k_1\sigma(1)} b_{k_2\sigma(2)} b_{k_3\sigma(3)} \cdots b_{k_n\sigma(n)} \right). \end{aligned}$$

Now,

$$\begin{aligned} |A||B| &= \sum_{\Pi \in S_n} (a_{1\Pi(1)} a_{2\Pi(2)} a_{3\Pi(3)} \cdots a_{n\Pi(n)} |B|) \\ &= \sum_{\Pi \in S_n} \left(a_{1\Pi(1)} a_{2\Pi(2)} a_{3\Pi(3)} \cdots a_{n\Pi(n)} \sum_{\sigma \in S_n} (b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)} \cdots b_{n\sigma(n)}) \right). \end{aligned}$$

Also we know that for $x, y, z \in R$ (an IFI) $xzy \leq xy$. So from the above two we conclude that $|A||B| \leq |AB|$.

(b) Let $B = A adj(A)$. Then $b_{ij} = \sum_{k=1}^n a_{ik} |A_{jk}| = |A(i \Rightarrow j)|$.

Thus $|A adj(A)| = \sum_{\sigma \in S_n} |A(1 \Rightarrow \sigma(1))| |A(2 \Rightarrow \sigma(2))| |A(3 \Rightarrow \sigma(3))| \cdots |A(n \Rightarrow \sigma(n))|$
 $\therefore |A|^n = |A(1 \Rightarrow \sigma(1))| |A(2 \Rightarrow \sigma(2))| |A(3 \Rightarrow \sigma(3))| \cdots |A(n \Rightarrow \sigma(n))| \leq |A adj(A)|$.
 Here $|A(i \Rightarrow \sigma(i))| = |E_{i\sigma(i)} A| = |A|$.

Also we have $|A||adj(A)| \leq |A adj(A)|$ [by Proposition 5.5(a)]. Therefore

$$|A|^n + |A||adj(A)| \leq |A adj(A)|.$$

(c) Similarly, we can proved that $|A|^n + |adj(A)||A| \leq |adj(A)A|$. □

Definition 5.6 (Triangular IFIM). Let R be an IFI. A matrix $A \in M_n(R)$ is called an upper triangular if $a_{ij} = \emptyset = \langle 0, 1 \rangle$ for all $j > i$. A is called lower triangular if $a_{ij} = \emptyset = \langle 0, 1 \rangle$ for all $j < i$. The matrix which is either upper triangular or lower triangular is called triangular matrix.

Pal [18] proved that for a triangular IFM A , $|A| = \prod_{i=1}^n \langle a_{ii\mu}, a_{ii\nu} \rangle = \prod_{i=1}^n a_{ii}$.

Proposition 5.7. *Let R be an IFI and $A, B \in M_n(R)$ are either both upper triangular or both lower triangular then $|AB| = |A||B|$.*

Proof. Let A and B be both upper triangular and let $D = AB = [d_{ij}]$. Then

$$d_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

Now for $j > i$, $i > k \Rightarrow j > k$ then $b_{kj} = \emptyset = \langle 0, 1 \rangle$
and if $i < k$ then $a_{ik} = \emptyset = \langle 0, 1 \rangle$.

$$\begin{aligned} \text{Therefore } d_{ij} &= \sum_{k=1}^n a_{ik} b_{kj} \\ &= \sum_{k, i > k} \langle a_{ik\mu}, a_{ik\nu} \rangle \langle 0, 1 \rangle + \sum_{k, i < k} \langle 0, 1 \rangle \langle b_{kj\mu}, b_{kj\nu} \rangle \\ &= \langle 0, 1 \rangle. \end{aligned}$$

Hence $D = AB$ is upper triangular.

$$\begin{aligned} \text{Therefore } |AB| &= |D| = \prod_{i=1}^n d_{ii} \\ &= d_{11} d_{22} d_{33} \cdots d_{nn} \\ &= \sum_{k=1}^n a_{1k} b_{k1} \sum_{k=1}^n a_{2k} b_{k2} \sum_{k=1}^n a_{3k} b_{k3} \cdots \sum_{k=1}^n a_{nk} b_{kn}. \end{aligned}$$

$$\begin{aligned} \text{Now } a_{ik} b_{ki} &= \langle a_{ik\mu}, a_{ik\nu} \rangle \langle b_{ki\mu}, b_{ki\nu} \rangle \\ &= \begin{cases} \langle a_{ik\mu}, a_{ik\nu} \rangle \langle 0, 1 \rangle, & \text{if } i > k \\ \langle 0, 1 \rangle \langle b_{ki\mu}, b_{ki\nu} \rangle, & \text{if } i < k \\ \langle a_{ii\mu}, a_{ii\nu} \rangle \langle b_{ii\mu}, b_{ii\nu} \rangle, & \text{if } i = k \end{cases} \end{aligned}$$

$$\begin{aligned} \text{Therefore } \sum_{k=1}^n a_{ik} b_{ki} &= \langle a_{ii\mu}, a_{ii\nu} \rangle \langle b_{ii\mu}, b_{ii\nu} \rangle \\ &= a_{ii} b_{ii}. \end{aligned}$$

$$\begin{aligned} \text{Hence } |AB| &= a_{11} b_{11} a_{22} b_{22} a_{33} b_{33} \cdots a_{nn} b_{nn} \\ &= (a_{11} a_{22} a_{33} \cdots a_{nn}) (b_{11} b_{22} b_{33} \cdots b_{nn}) \\ &= |A| |B|. \end{aligned}$$

Similarly, we can prove the proposition for lower triangular matrices. \square

Remark 5.8. Let R be an IFI and $A, B \in M_n(R)$ both are lower (or upper) triangular IFIMs, then AB is lower (or upper) triangular IFIM.

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