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More results on highly irregular bipolar fuzzy graphs

Hossein Rashmanlou, Young Bae Jun, R. A. Borzooei

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Abstract. In this paper, weak isomorphism, co-weak isomorphism and isomorphism of neighbourly irregular bipolar fuzzy graphs are defined. Some results on order, size and degrees of the nodes between isomorphic neighbourly irregular and isomorphic highly irregular bipolar fuzzy graphs are discussed. Isomorphisms between neighbourly irregular and highly irregular bipolar fuzzy graphs are proved to be an equivalence relation. Finally, isomorphism properties of μ-complement, self μ-complement and self weak μ-complement of highly irregular bipolar fuzzy graphs are stablished.

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Corresponding Author: Hossein Rashmanlou (rashmanlou@gmail.com)

1. Introduction

Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, transportation, etc. In 1965, Zadeh [25] introduced the notion of a fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artifical intelligence, expert systems, decision making and automata theory. In 1994, Zhang [26] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. A bipolar fuzzy set is an extension of Zadeh's fuzzy set theory whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property.

In 1975, Rosenfeld [16] discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffman [8] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analougs of several graph theoretical concepts. Bhattacharya [6] gave some remarks on fuzzy graphs. The complement of a fuzzy graph was defined by Mordeson and Nair [9]. Alavi, et al., [4] introduced highly irregular graphs and investigated several problems concerning the existence and enumeration of highly irregular graphs. Alavi, et al., [5] introduced k-path irregular graphs and studied some properties on k-path irregular graphs. Given a positive integer n and a partition with distinct parts Gnaana Bhragsam and Ayyaswamy [7] suggested a method to construct a neighbourly irregular graph of order n. Gani and Latha [10] introduced neighbourly irregular fuzzy graphs, highly irregular fuzzy graphs and a comparative study between them was studied. Recently, the bipolar fuzzy graphs have been discussed in [1]-[3].

Talebi and Rashmanlou [21] studied properties of isomorphism and complement on interval-valued fuzzy graphs. Rashmanlou and Jun defined complete interval-valued fuzzy graphs [12]. Talebi, Rashmanlou and Mehdipoor defined isomorphism and some new operations on vague graphs [22, 23].Talebi and Rashmanlou defined product bipolar fuzzy graphs [24]. Samanta and Pal introduced fuzzy tolerance graph [17], irregular bipolar fuzzy graphs [19], fuzzy k-competition graphs and p-competition fuzzy graphs [20], bipolar fuzzy hypergraphs [18] and investigated several properties. Pal and Rashmanlou [11] studied lost of properties of irregular interval-valued fuzzy graphs. Also they defined antipodal interval-valued fuzzy graphs [13], balanced interval-valued fuzzy graphs [14] and isometry on interval-valued fuzzy graphs [15]. In this paper, weak isomorphism, Co-weak isomorphism and isomorphism of neighbourly irregular bipolar fuzzy graphs and highly irregular bipolar fuzzy graphs are defined and isomorphism between neighbourly irregular and highly irregular bipolar fuzzy graphs are proved to be an equivalence relation.

2. Preliminaries

A graph is an order pair $G^* = (V, E)$, where V and E are the sets of vertices and edges of G^*.

A fuzzy graph with a non-empty finite set V as the underlying set is a pair $G^* = (\sigma, \mu)$, where $\sigma : V \to [0, 1]$ is a fuzzy subset of V, $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on the fuzzy subset σ, such that

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \quad \text{for all} \;\; x, y \in V$$

where $\wedge$ stands for minimum. The underlying crisp graph of the fuzzy graph $G = (\sigma, \mu)$ is denoted as $G^* = (\sigma^*, \mu^*)$, where

$$\sigma^* = \{u \in V \mid \sigma(u) > 0\} \;\; \text{and} \;\; \mu^* = \{(u, v) \in V \times V \mid \mu(u, v) > 0\}.$$

We use the notation xy for an element of E.
A fuzzy graph G is said to be a complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in \sigma^*$, it is denoted as $K_\sigma : (\sigma, \mu)$.

Definition 2.1. Let $G = (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex u is $d(u) = \sum_{u \neq v} \mu(u, v)$.

Definition 2.2. Let $G = (\sigma, \mu)$ be a connected fuzzy graph. G is said to be neighbourly irregular fuzzy graph if every two adjacent vertices of G have distinct degree.

Definition 2.3. Let $G = (\sigma, \mu)$ be a connected fuzzy graph. G is said to be a highly irregular fuzzy graph if every vertex of G is adjacent vertices with distinct degrees.

Definition 2.4. The order of a fuzzy graph G is $\circ(G) = \sum_{u \in V} \sigma(u)$. The size of a fuzzy graph G is $S(G) = \sum_{u,v \in V} \mu(u, v)$.

Definition 2.5. The complement of a fuzzy graph $G = (\sigma, \mu)$ is a fuzzy graph $\overline{G} = (\sigma, \overline{\mu})$, where $\overline{\sigma} = \sigma$ and $\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all $u, v \in V$.

Definition 2.6 ([26])**.** Let X be a non-empty set. A bipolar fuzzy set B in X is an object having the form

$$B = \{(x, \mu_{B^P}(x), \mu_{B^N}(x)) \mid x \in X\}$$

where $\mu_{B^P} : X \to [0, 1]$ and $\mu_{B^N} : X \to [-1, 0]$ are mappings.

Definition 2.7 ([26])**.** Let X be a non-empty set. Then we call a mapping $A = (\mu_{A^P}, \mu_{A^N}) : X \times X \to [0, 1] \times [-1, 0]$ a bipolar fuzzy relation on X such that $\mu_{A^P}(x, y) \in [0, 1]$ and $\mu_{A^N}(x, y) \in [-1, 0]$.

Definition 2.8 ([1])**.** Let $A = (\mu_{A^P}, \mu_{A^N})$ and $B = (\mu_{B^P}, \mu_{B^N})$ be bipolar fuzzy sets on a set X. If $A = (\mu_{A^P}, \mu_{A^N})$ is a bipolar fuzzy relation on a set X, then $A = (\mu_{A^P}, \mu_{A^N})$ is called a bipolar fuzzy relation on $B = (\mu_{B^P}, \mu_{B^N})$ if $\mu_{A^P}(x, y) \leq \min(\mu_{B^P}(x), \mu_{B^P}(y))$ and $\mu_{A^N}(x, y) \geq \max(\mu_{B^N}(x), \mu_{B^N}(y))$ for all $x, y \in X$. A bipolar fuzzy relation A on X is called symmetric if $\mu_{A^P}(x, y) = \mu_{A^P}(y, x)$ and $\mu_{A^N}(x, y) = \mu_{A^N}(y, x)$ for all $x, y \in X$.

Definition 2.9 ([1])**.** By a bipolar fuzzy graph $G = (A, B)$ of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = (\mu_{A^P}, \mu_{A^N})$ is a bipolar fuzzy set on V and $B = (\mu_{B^P}, \mu_{B^N})$ is a bipolar fuzzy relation on E such that $\mu_{B^P}(xy) \leq \min(\mu_{A^P}(x), \mu_{A^P}(y))$ and $\mu_{B^N}(xy) \geq \max(\mu_{A^N}(x), \mu_{A^N}(y))$ for all $x, y \in E$.

Throughout this paper, G^* is a crisp graph, and G is a bipolar fuzzy graph.

Definition 2.10. Given a bipolar fuzzy graph $G = (A, B)$, with the underlying set V, the order of G is defined and denoted as $\circ(G) = \Big(\sum_{x \in V} \mu_{A^P}(x), \sum_{x \in V} \mu_{A^N}(x)\Big)$. The size of a bipolar fuzzy graph G is

$$S(G) = (S^P(G), S^N(G)) = \Big(\sum_{\substack{x \neq y \\ x,y \in V}} \mu_{B^P}(xy), \sum_{\substack{x \neq y \\ x,y \in V}} \mu_{B^N}(xy)\Big).$$

Definition 2.11. Let $G = (A, B)$ be a bipolar fuzzy graph on G^*. The open degree of a vertex u is defined as $\deg(u) = (d^P(u), d^N(u))$, where

$$d^P(u) = \sum_{\substack{u \neq v \\ v \in V}} \mu_{B^P}(uv) \quad \text{and} \quad d^N(u) = \sum_{\substack{u \neq v \\ v \in V}} \mu_{B^N}(uv).$$

Definition 2.12. Let $G = (A, B)$ be a connected bipolar fuzzy graph. G is said to be a neighbourly irregular bipolar fuzzy graph if every two adjacent vertices of G have distinct degree.

Definition 2.13. Let G be a connected bipolar fuzzy graph. G is called a highly irregular bipolar fuzzy graph if every vertex of G is adjacent to vertices with distinct degrees.

Definition 2.14. A bipolar fuzzy graph G is called complete if
$\mu_{B^P}(xy) = \min(\mu_{A^P}(x), \mu_{A^P}(y))$, $\mu_{B^N}(xy) = \max(\mu_{A^N}(x), \mu_{A^N}(y))$ for all $xy \in E$.

Definition 2.15. The complement of a highly irregular bipolar fuzzy graph (neighbourly irregular bipolar fuzzy graph) $G = (A, B)$ of a graph $G^* = (V, E)$ is a bipolar fuzzy graph $\overline{G} = (\overline{A}, \overline{B})$ of $\overline{G^*} = (V, V \times V)$, where $\overline{A} = A = [\mu_{A^P}, \mu_{A^N}]$ and $\overline{B} = [\overline{\mu_{B^P}}, \overline{\mu_{B^N}}]$ is defined by
$\overline{\mu_{B^P}}(xy) = \mu_{A^P}(x) \wedge \mu_{A^P}(y) - \mu_{B^P}(xy)$ for all $x, y \in V$,
$\overline{\mu_{B^N}}(xy) = \mu_{A^N}(x) \vee \mu_{A^N}(y) - \mu_{B^N}(xy)$ for all $x, y \in V$.

3. Isomorphic properties of neighbourly irregular and highly irregular bipolar fuzzy graphs

Definition 3.1. A homomorphism h of neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) G_1 and G_2 is a mapping $h : V_1 \to V_2$ which satisfies the following conditions:
(a) $\mu_{A_1^P}(u_1) \leq \mu_{A_2^P}(h(u_1))$, $\mu_{A_1^N}(u_1) \geq \mu_{A_2^N}(h(u_1))$ for all $u_1 \in V_1$,
(b) $\mu_{B_1^P}(u_1v_1) \leq \mu_{B_2^P}(h(u_1)h(v_1))$, $\mu_{B_1^N}(u_1v_1) \geq \mu_{B_2^N}(h(u_1)h(v_1))$ for all $u_1v_1 \in E$.

Example 3.2. Let $V_1 = \{a, b, c, d\}$ and $V_2 = \{u, v, x, w\}$. Consider two neighbourly irregular bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

	a	b	c	d
$\mu_{A_1^P}$	0.3	0.3	0.5	0.4
$\mu_{A_1^N}$	-0.5	-0.5	-0.5	-0.4

	ab	bc	cd	ad
$\mu_{B_1^P}$	0.3	0.1	0.4	0.2
$\mu_{B_1^N}$	-0.3	-0.5	-0.2	-0.4

	u	v	x	w
$\mu_{A_2^P}$	0.3	0.4	0.5	0.4
$\mu_{A_2^N}$	-0.5	-0.6	-0.5	-0.4

	uv	vx	wx	uw
$\mu_{B_2^P}$	0.3	0.3	0.4	0.2
$\mu_{B_2^N}$	-0.3	-0.3	-0.2	-0.4

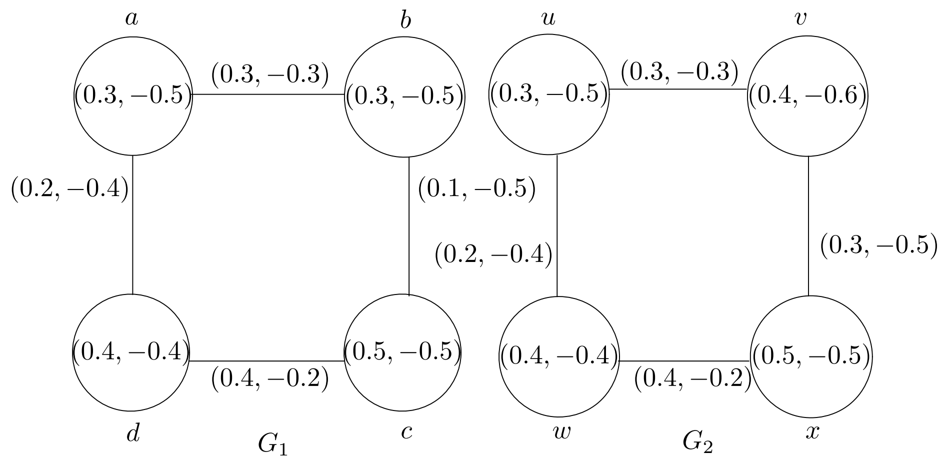


Fig. 1: Homomorphism of neighbourly irregular bipolar fuzzy graphs G_1 and G_2

There is a homomorphism $h : V_1 \to V_2$ such that $h(a) = u$, $h(b) = v$, $h(c) = x$, $h(d) = w$.

Definition 3.3. A weak isomorphism h of neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:
(c) h is homomorphism,
(d) $\mu_{A_1^P}(u_1) = \mu_{A_2^P}(h(u_1))$, $\mu_{A_1^N}(u_1) = \mu_{A_2^N}(h(u_1))$ for all $u_1 \in V_1$.

Example 3.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular bipolar fuzzy graphs defined as follows.

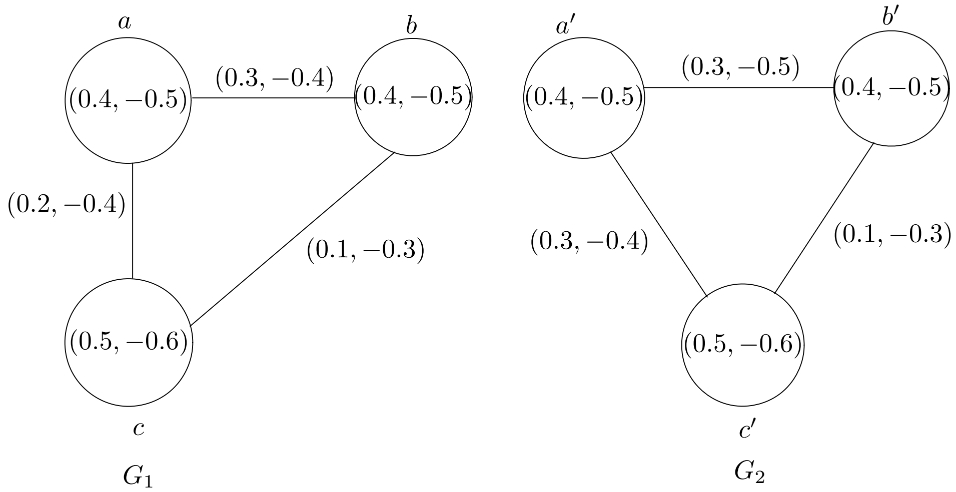


Fig. 2: Weak isomorphism of highly irregular bipolar fuzzy graphs

There is a weak isomorphism $h : V_1 \to V_2$ such that $h(a) = a'$, $h(b) = b'$, $h(c) = c'$.

Definition 3.5. A co-weak isomorphism h of neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:
(e) h is homomorphism,
(f) $\mu_{B_1^P}(u_1v_1) = \mu_{B_2^P}(h(u_1)h(v_1))$, $\mu_{B_1^N}(u_1v_1) = \mu_{B_2^N}(h(u_1)h(v_1))$ for all $u_1v_1 \in E_1$.

Example 3.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighbourly irregular bipolar fuzzy graphs defined as follows.

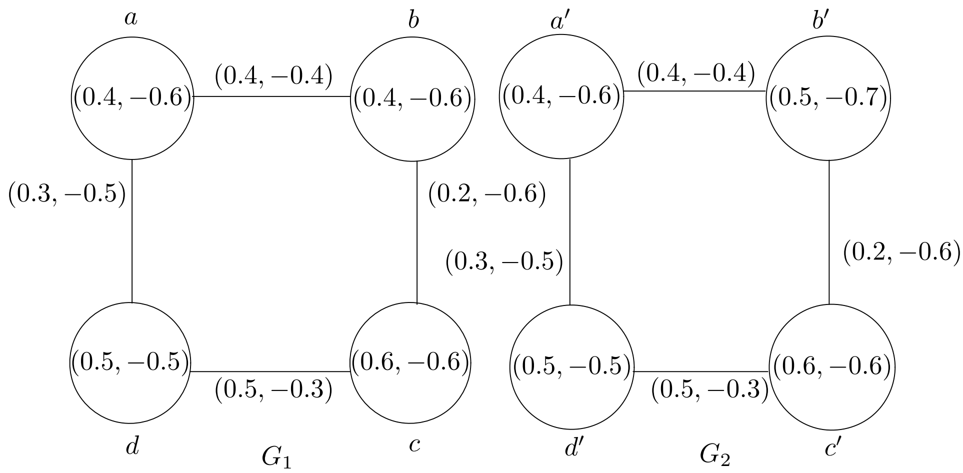


Fig. 3: Co-weak isomorphism of neighbourly irregular bipolar fuzzy graphs

There is a co-weak isomorphism $h : V_1 \to V_2$ such that $h(a) = a'$, $h(b) = b'$, $h(c) = c'$, $h(d) = d'$.

Definition 3.7. An isomorphism h of neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) G_1 and G_2 is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:
(g) $\mu_{A_1^P}(u_1) = \mu_{A_2^P}(h(u_1))$, $\mu_{A_2^N}(u_1) = \mu_{A_2^N}(h(u_1))$
(h) $\mu_{B_1^P}(u_1v_1) = \mu_{B_2^P}(h(u_1)h(v_1))$, $\mu_{B_1^N}(u_1v_1) = \mu_{B_2^N}(h(u_1)h(v_1))$ for all $u_1 \in V_1$, $u_1v_1 \in E_1$.

Theorem 3.8. *For any two isomorphism neighbourly irregular bipolar fuzzy graphs, their order and size are same.*

Proof. If h from G_1 to G_2 be an isomorphism between the neighbourly irregular bipolar fuzzy graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively then,

$$\mu_{A_1^P}(u) = \mu_{A_2^P}(h(u)), \quad \mu_{A_1^N}(u) = \mu_{A_2^N}(h(u)) \quad \text{for all} \quad u \in V,$$

$$\mu_{B_1^P}(uv) = \mu_{B_2^P}(h(u)h(v)), \quad \mu_{B_1^N}(uv) = \mu_{B_2^N}(h(u)h(v)) \quad \text{for all} \quad u, v \in V.$$

So, we have

$$\begin{aligned}
\circ(G_1) &= \Big(\sum_{u_1\in V_1}\mu_{A_1^P}(u_1), \sum_{u_1\in V_1}\mu_{A_1^N}(u_1)\Big)\\
&= \Big(\sum_{u_1\in V_1}\mu_{A_2^P}(h(u_1)), \sum_{u_1\in V_1}\mu_{A_2^N}(h(u_1))\Big)\\
&= \Big(\sum_{u_2\in V_2}\mu_{A_2^P}(u_2), \sum_{u_2\in V_2}\mu_{A_2^N}(u_2)\Big) = \circ(G_2)\\
S(G_1) &= \Big(\sum_{u_1v_1\in E_1}\mu_{B_1^P}(u_1v_1), \sum_{u_1v_1\in E_1}\mu_{B_1^N}(u_1v_1)\Big)\\
&= \Big(\sum_{u_1,v_1\in V_1}\mu_{B_2^P}(h(u_1)h(v_1)), \sum_{u_1,v_1\in V_1}\mu_{B_2^N}(h(u_1)h(v_1))\Big)\\
&= \Big(\sum_{u_2v_2\in E_2}\mu_{B_2^P}(u_2v_2), \sum_{u_2v_2\in E_2}\mu_{B_2^N}(u_2v_2)\Big) = S(G_2).
\end{aligned}$$

□

Remark 3.9. The above theorem is true for highly irregular bipolar fuzzy graphs.

Corollary 3.10. *Converse of the above theorem need not be true for both neighbourly irregular and highly irregular bipolar fuzzy graphs. The converse part is proved in the following example.*

Example 3.11. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular bipolar fuzzy graphs defined as follows.

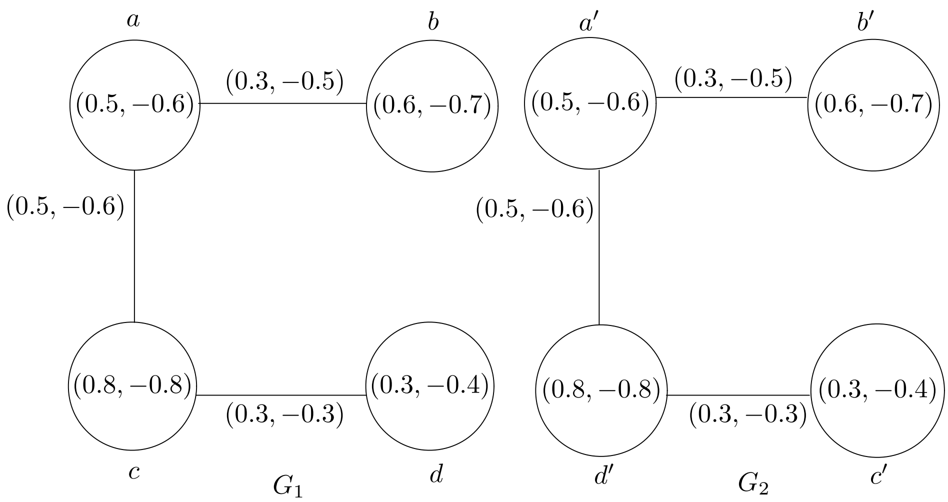


Fig. 4: Highly irregular bipolar fuzzy graphs G_1 and G_2

In both the graphs, $\circ(G_1) = \circ(G_2) = (2.2, -2.5)$ and $Size(G) = (1.1, -1.4)$. But G_1 is not isomorphic to G_2.

Proposition 3.12. *If the neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) are weak isomorphic then their orders are same. But the* *neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) of same order need not be weak isomorphic.*

Example 3.13. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two neighbourly irregular bipolar fuzzy graphs defined as follows.

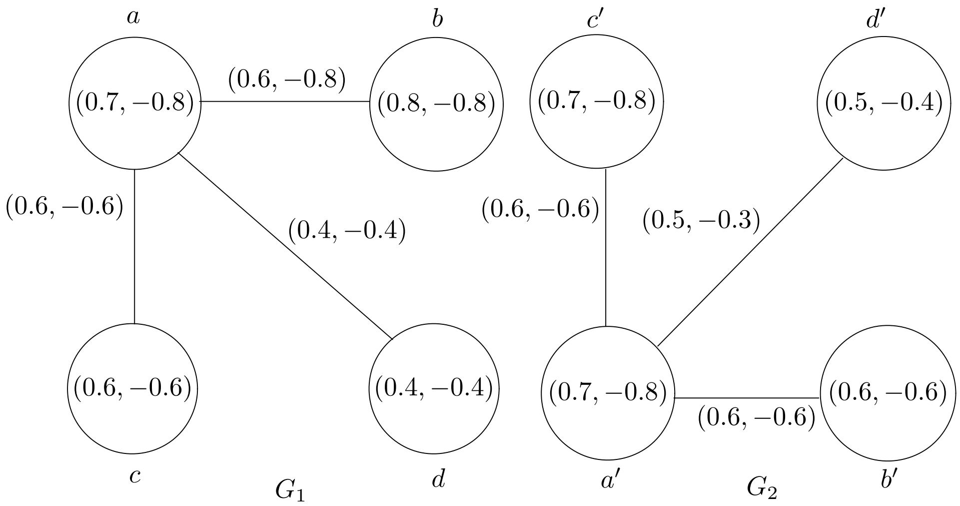


Fig. 5: Neighbourly irregular bipolar fuzzy graphs G_1 and G_2

In both the graphs, $\circ(G_1) = (2.5, -2.6) = \circ(G_2)$, but they are not weak isomorphic.

Proposition 3.14. *If the neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) are co-weak isomorphism their size are same. But the neighbourly irregular bipolar fuzzy graphs (highly irregular bipolar fuzzy graphs) of same size need not be co-weak isomorphic.*

Example 3.15. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular bipolar fuzzy graphs defined as follows.

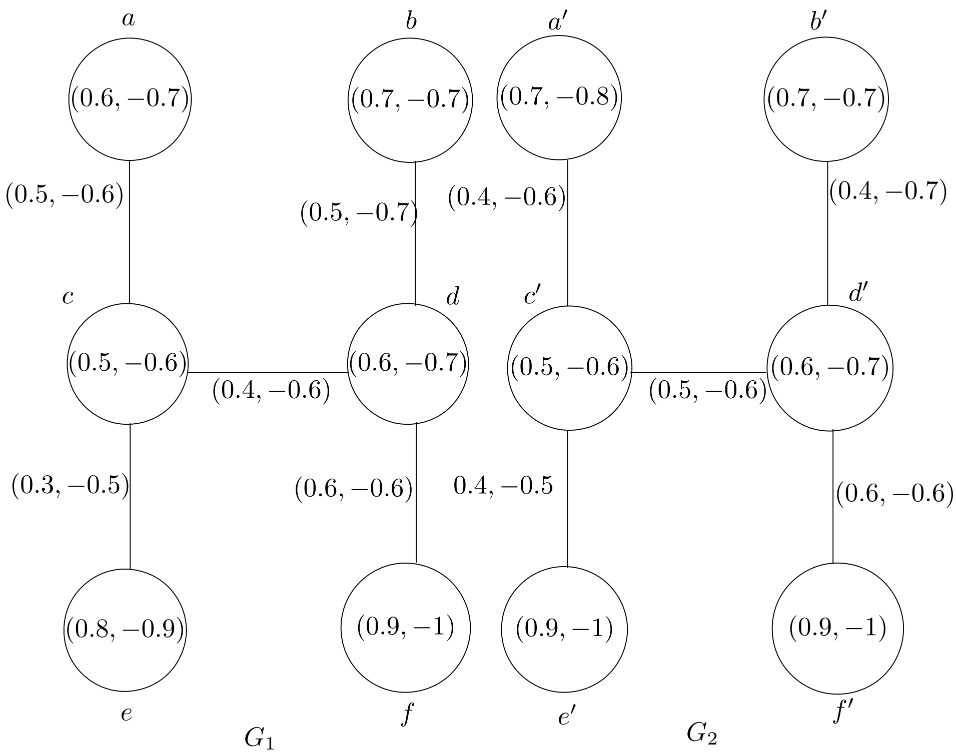


Fig. 6: Highly irregular bipolar fuzzy graphs G_1 and G_2

The size of the above two highly irregular bipolar fuzzy graphs are same

$$(S(G) = (2.3, -3) = S(G_1)).$$

But they are not co-weak isomorphic.

Theorem 3.16. *If G_1 and G_2 are isomorphic neighbourly irregular bipolar fuzzy graphs then, the degrees of the corresponding vertices u and $h(u)$ are preserved.*

Proof. If $h : G_1 \to G_2$ is an isomorphism between the neighbourly irregular bipolar fuzzy graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively then,

$$\mu_{B_1^P}(u_1v_1) = \mu_{B_2^P}(h(u_1)h(v_1)), \quad \mu_{B_1^N}(u_1v_1) = \mu_{B_2^N}(h(u_1)h(v_1)) \text{ for all } u_1, v_1 \in V_1.$$

Therefore,

$$\begin{aligned} d^P(u_1) &= \sum_{u_1,v_1\in V_1} \mu_{B_1^P}(u_1v_1) = \sum_{u_1v_1\in V_1} \mu_{B_2^P}(h(u_1)h(v_1)) = d^P(h(u_1)) \\ d^N(u_1) &= \sum_{u_1,v_1\in V_1} \mu_{B_1^N}(u_1v_1) = \sum_{u_1v_1\in V_1} \mu_{B_2^N}(h(u_1)h(v_1)) = d^N(h(u_1)) \end{aligned}$$

for all $u_1 \in V_1$. That is, the degrees of the corresponding vertices of G_1 and G_2 are the same. □

Remark 3.17. The above theorem is true for highly irregular bipolar fuzzy graphs.

Corollary 3.18. *Converse of the Theorem 3.16 and Remark 3.17 need not be true. This is proved in the following example.*

Example 3.19. Consider two neighbourly irregular bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined as follows.

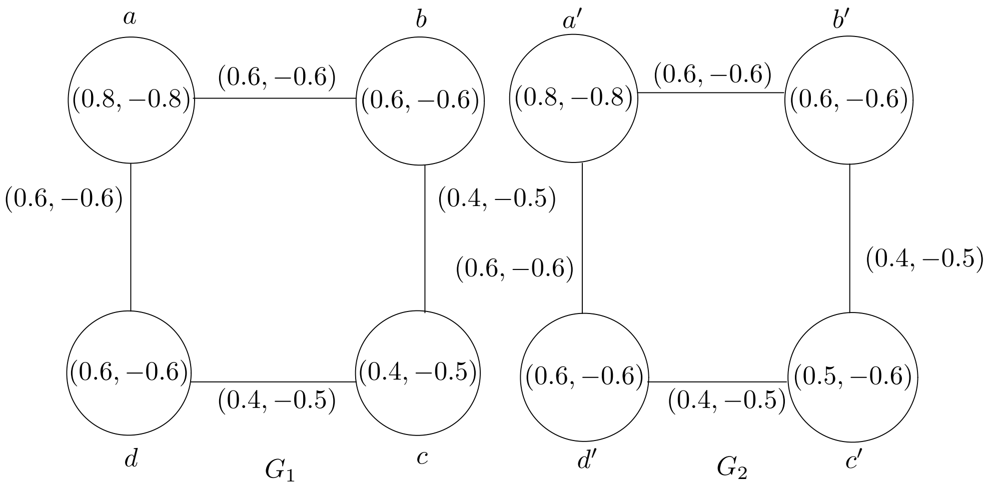


Fig. 7: Neighbourly irregular bipolar fuzzy graphs G_1 and G_2 with same degrees

$$\deg(a) = \deg(a') = (1.2, -1.2), \quad deg(b) = \deg(b') = (1, -1.1),$$
$$\deg(c) = \deg(c') = (0.8, -1), \quad deg(d) = \deg(d') = (1, -1.1).$$

In both the graphs the degrees of the corresponding vertices are the same, but G_1 and G_2 are only co-weak isomorphic but not isomorphic.

Example 3.20. Consider two highly irregular bipolar fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined as follows:

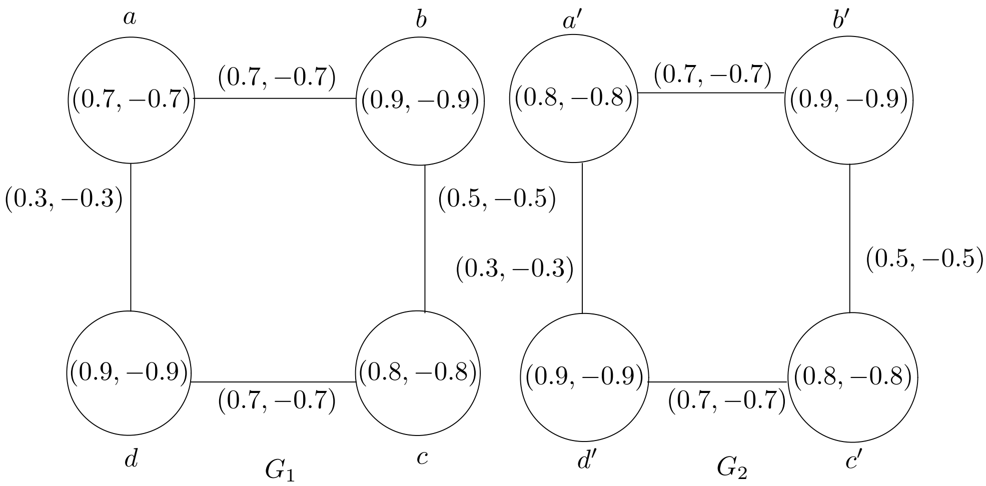


Fig. 8: Highly irregular bipolar fuzzy graphs G_1 and G_2 with same degrees

Here,

$$\deg(a) = \deg(a') = (1, -1), \quad deg(b) = \deg(b') = (1.2, -1.2),$$
$$\deg(c) = \deg(c') = (1.2, -1.2), \quad deg(d) = \deg(d') = (1, -1).$$

From the above two graphs, it is clear that the degrees of the corresponding nodes are the same. G_1 and G_2 are not isomorphic, but only co-weak isomorphic.

Theorem 3.21. *Isomorphism between neighbourly irregular bipolar fuzzy graphs is an equivalence relation.*

Proof. Let $G_1 = (A_1, B_1)$, $G_2 = (A_2, B_2)$ and $G_3 = (A_3, B_3)$ be neighbourly irregular bipolar fuzzy graphs with vertex sets V_1, V_2 and V_3 respectively.
Reflexive: (i.e) To prove $G \sim G$.
Consider the identity map $h : V \to V$ such that $h(u) = u$ for all $u \in V$. Clearly h is a bijective map satisfying $\mu_{A^P}(u) = \mu_{A^P}(h(u))$, $\mu_{A^N}(u) = \mu_{A^N}(h(u))$ for all $u \in V$ and $\mu_{B^P}(uv) = \mu_{B^P}(h(u), h(v))$, $\mu_{B^N}(uv) = \mu_{B^N}(h(u), h(v))$ for all $u, v \in V$.
Therefore h is an isomorphism of the neighbourly irregular bipolar fuzzy graph to itself. Hence h satisfies reflexive relation.
Symmetric: To prove, if $G_1 \sim G_2$ then $G_2 \sim G_1$.
Assume $G_1 \sim G_2$. Let $h : V_1 \to V_2$ be an isomorphism of G_1 onto G_2 such that $h(u_1) = u_2$ for all $u_1 \in V_1$. This h is a bijective map satisfying

$$(3.1) \qquad \mu_{A_1^P}(u_1) = \mu_{A_2^P}(h(u_1)), \quad \mu_{A_1^N}(u_1) = \mu_{A_2^N}(h(u_1)) \text{ for all } u_1 \in V_1$$

and
(3.2)
$$\mu_{B_1^P}(u_1v_1) = \mu_{B_2^P}(h(u_1), h(v_1)), \mu_{B_1^N}(u_1v_1) = \mu_{B_2^N}(h(u_1), h(v_1)) \text{ for all } u_1, v_1 \in V_1.$$

Since h is bijective, inverse exists. (i.e.) $h^{-1}(u_2) = u_1$ for all $u_2 \in V_2$.
From (3.1) and (3.2) we have
$\mu_{A_1^P}(h^{-1}(u_2)) = \mu_{A_2^P}(u_2)$, $\mu_{A_1^N}(h^{-1}(u_2)) = \mu_{A_2^N}(u_2)$ for all $u_2 \in V_2$ and
$\mu_{B_1^P}(h^{-1}(u_2)h^{-1}(v_2)) = \mu_{B_2^P}(u_2v_2)$, $\mu_{B_1^N}(h^{-1}(u_2)h^{-1}(v_2)) = \mu_{B_2^N}(u_2v_2)$
for all $u_2, v_2 \in V_2$.
Thus $h^{-1} : V_2 \to V_1$ is a 1-1, onto map which is an isomorphism from G_2 to G_1. Therefore, isomorphism satisfies symmetric relation.
Transitive: To prove if $G_1 \sim G_2$ and $G_2 \sim G_3$ then $G_1 \sim G_3$.
Assume $G_1 \sim G_2$. Let $h : V_1 \to V_2$ be an isomorphism of G_1 onto G_2 such that $h(u_1) = u_2$ for all $u_1 \in V_1$ satisfying $\mu_{A_1^P}(u_1) = \mu_{A_2^P}(h(u_1))$,
$\mu_{A_1^N}(u_1) = \mu_{A_2^N}(h(u_1))$ for all $u_1 \in V_1$ and
$\mu_{B_1^P}(u_1v_1) = \mu_{B_2^P}(h(u_1)h(v_1)) = \mu_{B_2^P}(u_2v_2)$, $\mu_{B_1^N}(u_1v_1) = \mu_{B_2^N}(h(u_1)h(v_1)) = \mu_{B_2^N}(u_2v_2)$ for all $u_1, v_1 \in V_1$.
Assume $G_2 \sim G_3$. Let $g : V_2 \to V_3$ be an isomorphism of G_2 onto G_3 such that $g(u_2) = u_3$ for all $u_2 \in V_2$ satisfying $\mu_{A_2^P}(u_2) = \mu_{A_3^P}(g(u_2)) = \mu_{A_3^P}(u_3)$,
$\mu_{A_2^N}(u_2) = \mu_{A_3^N}(g(u_2)) = \mu_{A_3^N}(u_3)$ for all $u_2 \in V_2$ and
$\mu_{B_2^P}(u_2v_2) = \mu_{B_3^P}(g(u_2)g(v_2)) = \mu_{B_3^P}(u_3v_3)$, $\mu_{B_2^N}(u_2v_2) = \mu_{B_3^N}(g(u_2)g(v_2)) = \mu_{B_3^N}(u_3v_3)$ for all $u_2, v_2 \in V_2$.
Since $h : V_1 \to V_2$ and $g : V_2 \to V_3$ are isomorphism from G_1 onto G_2 and G_2 onto

G_3 respectively, then $g \circ h$ is a 1-1, onto map from V_1 to V_3. (i.e.) $g \circ h : V_1 \to V_3$ where $(g \circ h)(u) = g(h(u))$ for all $u_1 \in V_1$. Now

$$\mu_{A_1^P}(u_1) = \mu_{A_2^P}(h(u_1)) = \mu_{A_2^P}(u_2) = \mu_{A_3^P}(g(u_2)) = \mu_{A_3^P}(g(h(u_1))),$$

$$\mu_{A_1^N}(u_1) = \mu_{A_2^N}(h(u_1)) = \mu_{A_2^N}(u_2) = \mu_{A_3^N}(g(u_2)) = \mu_{A_3^N}(g(h(u_1)))$$

for all $u_1 \in V_1$ and

$$\begin{aligned}\mu_{B_1^P}(u_1v_1) &= \mu_{B_2^P}(h(u_1)h(v_1)) = \mu_{B_2^P}(u_2v_2)\\ &= \mu_{B_3^P}(g(u_2)g(v_2)) = \mu_{B_3^P}(g(h(u_1)), g(h(v_1))),\end{aligned}$$

$$\begin{aligned}\mu_{B_1^N}(u_1v_1) &= \mu_{B_2^N}(h(u_1)h(v_1)) = \mu_{B_2^N}(u_2v_2)\\ &= \mu_{B_3^N}(g(u_2)g(v_2)) = \mu_{B_3^N}(g(h(u_1)), g(h(v_1)))\end{aligned}$$

for all $u_1, v_1 \in V_1$. Hence $g \circ h$ is an isomorphism between G_1 and G_3. Therefore, isomorphism between neighbourly irregular bipolar fuzzy graphs is an equivalence relation. □

Theorem 3.22. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular bipolar fuzzy graphs. G_1 and G_2 are isomorphic if and only if, their complement are isomorphic. But the complement need not be highly irregular.*

Proof. Assume that G_1 and G_2 are isomorphic, there exists a bijective map $h : V_1 \to V_2$ satisfying $\mu_{A_1^P}(x) = \mu_{A_2^P}(h(x))$, $\mu_{A_1^N}(x) = \mu_{A_2^N}(h(x))$ for all $x \in V_1$, $\mu_{B_1^P}(xy) = \mu_{B_2^P}(h(x)h(y))$, $\mu_{B_1^N}(xy) = \mu_{B_2^N}(h(x)h(y))$ for all $xy \in E_1$.
By the definition of complement, we have

$$\begin{aligned}\overline{\mu_{B_1^P}}(xy) &= \min(\mu_{A_1^P}(x), \mu_{A_1^P}(y)) - \mu_{B_1^P}(xy)\\ &= \min(\mu_{A_2^P}(h(x)), \mu_{A_2^P}(h(y))) - \mu_{B_2^P}(h(x)h(y)),\end{aligned}$$

$$\begin{aligned}\overline{\mu_{B_1^N}}(xy) &= \max(\mu_{A_1^N}(x), \mu_{A_1^N}(y)) - \mu_{B_1^N}(xy)\\ &= \max(\mu_{A_2^N}(h(x)), \mu_{A_2^N}(h(y))) - \mu_{B_2^N}(h(x)h(y))\end{aligned}$$

for all $xy \in E_1$. Hence $\overline{G_1} \cong \overline{G_2}$. The proof of the converse part is straight forward. □

The following example illustrates the above theorem.

Example 3.23. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two highly irregular bipolar fuzzy graphs defined as follows:

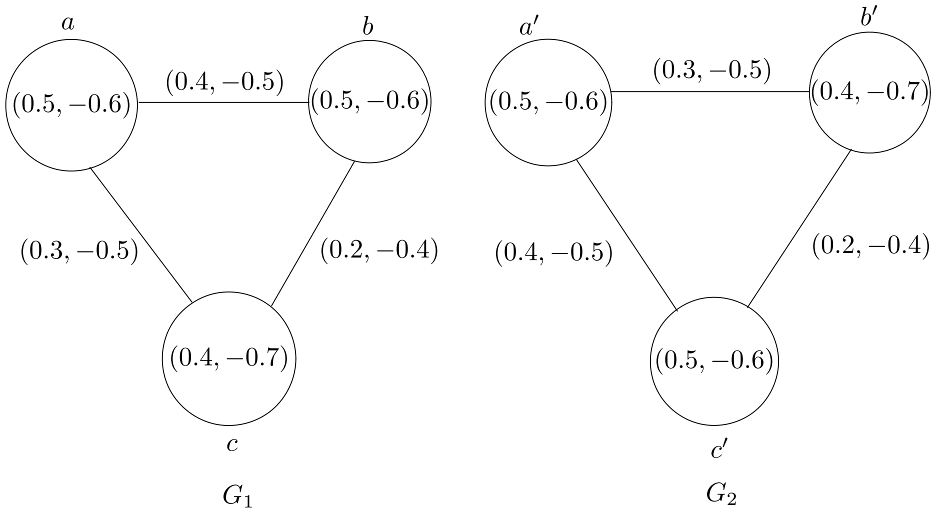
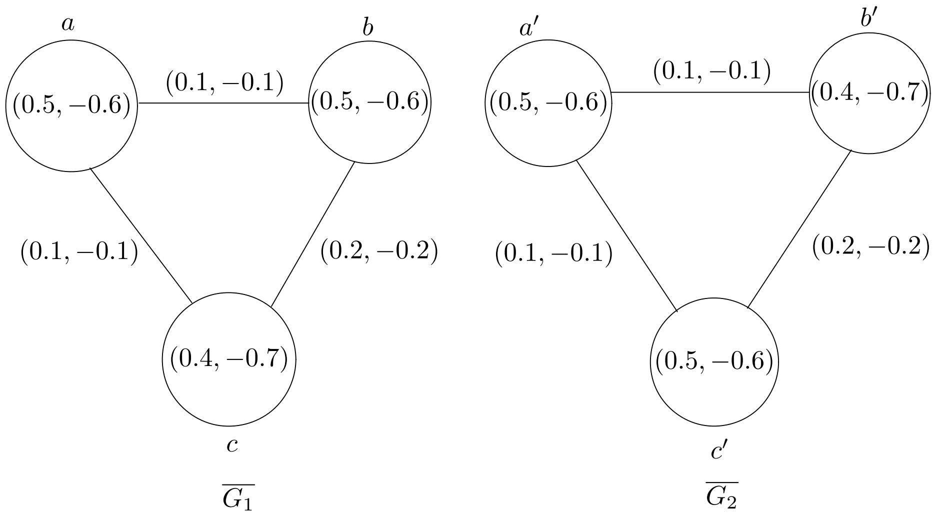


Fig. 9: Highly irregular bipolar fuzzy graphs G_1 and G_2 with same degrees

There is an isomorphism $h : V_1 \to V_2$ such that $h(a) = a'$, $h(b) = c'$, $h(c) = b'$ and $G_1 \cong G_2$ and $\overline{G_1} \cong \overline{G_2}$, but the complements are not highly irregular bipolar fuzzy graphs.



Theorem 3.24. *Let G_1 and G_2 be two highly irregular bipolar fuzzy graphs. If G_1 is weak isomorphism with G_2 then $\overline{G_1}$ is weak isomorphic with $\overline{G_2}$.*

Proof. If h is a weak isomorphism between G_1 and G_2, then $h : V_1 \to V_2$ is a bijective map satisfies

$$\mu_{A_1^P}(x) = \mu_{A_2^P}(h(x)), \mu_{A_1^N}(x) = \mu_{A_2^N}(h(x)) \text{ for all } x \in V_1 \tag{3.3}$$

and

(3.4) $\mu_{B_1^P}(xy) \leq \mu_{B_2^P}(h(x)h(y)), \mu_{B_1^N}(xy) \geq \mu_{B_2^N}(h(x)h(y))$ for all $x, y \in V_1$.

As $h^{-1} : V_2 \to V_1$ is also bijective, for every $x_2 \in V_2$, there is an $x_1 \in V_1$ such that $h^{-1}(x_2) = x_1$, using (3.3), we have
$\mu_{A_2^P}(x_2) = \mu_{A_1^P}(h^{-1}(x_2))$, $\mu_{A_2^N}(x_2) = \mu_{A_1^N}(h^{-1}(x_2))$ for all $x_2 \in V_2$.
Again by (3.3) and (3.4)

$$\overline{\mu_{B_1^P}}(x_1y_1) = \min(\mu_{A_1^P}(x_1), \mu_{A_1^P}(y_1)) - \mu_{B_1^P}(x_1y_1) \quad x_1, y_1 \in V_1,$$

$$\overline{\mu_{B_1^P}}(h^{-1}(x_2)h^{-1}(y_2)) \geq \min(\mu_{A_2^P}(h(x_1)), \mu_{A_2^P}(h(y_1))) - \mu_{B_2^P}(h(x_1)h(y_1)) \quad x_1, y_1 \in V_1,$$

$= \min(\mu_{A_2^P}(x_2), \mu_{A_2^P}(y_2)) - \mu_{B_2^P}(x_2y_2)$
$= \overline{\mu_{B_2^P}}(x_2y_2)$ for all $x_2, y_2 \in V_2$,
i.e. $\overline{\mu_{B_2^P}}(x_2y_2) \leq \overline{\mu_{B_1^P}}(h^{-1}(x_2)h^{-1}(y_2))$ for all $x_2, y_2 \in V_2$.
Also,
$\overline{\mu_{B_1^N}}(x_1y_1) = \max(\mu_{A_1^N}(x_1), \mu_{A_1^N}(y_1)) - \mu_{B_1^N}(x_1y_1)$ for all $x_1, y_1 \in V_1$.
So,

$$\begin{aligned} \overline{\mu_{B_1^N}}(h^{-1}(x_2)h^{-1}(y_2)) &\leq \max(\mu_{A_2^N}(h(x_1)), \mu_{A_2^N}(h(y_1))) - \mu_{B_2^N}(h(x_1)h(y_1)) \\ &= \max(\mu_{A_2^N}(x_2), \mu_{A_2^N}(y_2)) - \mu_{B_2^N}(x_2y_2) \\ &= \overline{\mu_{B_2^N}}(x_2y_2) \text{ for all } x_2, y_2 \in V_2, \end{aligned}$$

i.e. $\overline{\mu_{B_2^N}}(x_2y_2) \geq \overline{\mu_{B_1^N}}(h^{-1}(x_2)h^{-1}(y_2))$ for all $x_2, y_2 \in V_2$.
Therefore, $h^{-1} : V_2 \to V_1$ is a weak isomorphism between $\overline{G_1}$ and $\overline{G_2}$. □

Remark 3.25. The above theorem is true for neighbourly irregular bipolar fuzzy graphs.

Theorem 3.26. *Let G_1 and G_2 be two highly irregular bipolar fuzzy graphs (neighbourly irregular bipolar fuzzy graphs). If G_1 is a co-weak isomorphism with G_2, then there exists a homomorphism between $\overline{G_1}$ and $\overline{G_2}$.*

Definition 3.27. A bipolar fuzzy graph G is said to be a self complementary if $G \cong \overline{G}$.

Definition 3.28. A bipolar fuzzy graph G is said to be a self weak complementary if G is weak isomorphic with $\overline{G}$.

Proposition 3.29. *A highly irregular bipolar fuzzy graph need not be self complementary.*

Example 3.30. Consider the bipolar fuzzy graphs G and $\overline{G}$ defined as follows.

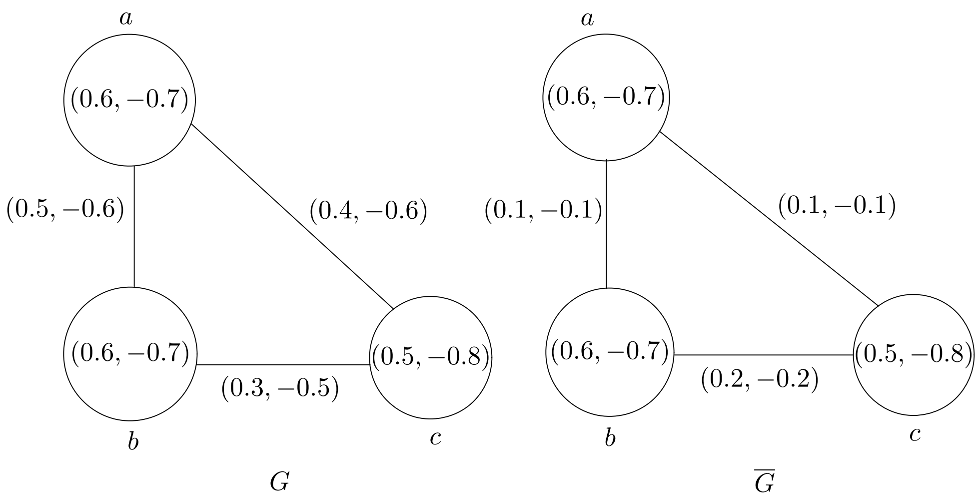


Fig. 10: Bipolar fuzzy graphs G and $\overline{G}$

In this example, G is highly irregular bipolar fuzzy graph but not a self complementary bipolar fuzzy graph.

Theorem 3.31. *Let G be a self weak complementary highly irregular bipolar fuzzy graph, then*
$\sum\limits_{x\neq y}\mu_{B^P}(xy) \leq \frac{1}{2}\sum\limits_{x\neq y}\min(\mu_{A^P}(x), \mu_{A^P}(y))$ *and*
$\sum\limits_{x\neq y}\mu_{B^N}(xy) \geq \frac{1}{2}\sum\limits_{x\neq y}\max(\mu_{A^N}(x), \mu_{A^N}(y))$.

Proof. Let $G = (A, B)$ be a self weak complementary highly irregular bipolar fuzzy graph of $G^* = (V, E)$. Then, there exists a weak-isomorphism $h : G \to \overline{G}$ such that for all $x, y \in V$

$$\begin{array}{rcl}\mu_{A^P}(x) & = & \overline{\mu_{A^P}}(h(x)) = \mu_{A^P}(g(x)), \quad \mu_{A^N}(x) = \overline{\mu_{A^N}}(g(x)) = \mu_{A^N}(h(x)),\\ \mu_{B^P}(xy) & \leq & \overline{\mu_{B^P}}(h(x)h(y)), \quad \mu_{B^N}(xy) \geq \overline{\mu_{B^N}}(h(x)h(y)).\end{array}$$

Using the definition of complement in the above inequality, for all $x, y \in V$ we have
$\mu_{B^P}(xy) \leq \overline{\mu_{B^P}}(h(x)h(y)) = \min(\mu_{A^P}(h(x)), \mu_{A^P}(h(y))) - \mu_{B^P}(h(x)h(y))$
$\mu_{B^N}(xy) \geq \overline{\mu_{B^N}}(h(x)h(y)) = \max(\mu_{A^N}(h(x)), \mu_{A^N}(h(y))) - \mu_{B^N}(h(x)h(y))$
$\mu_{B^P}(xy) + \mu_{B^P}(h(x)h(y)) \leq \min(\mu_{A^P}(h(x)), \mu_{A^P}(h(y)))$
$\mu_{B^N}(xy) + \mu_{B^N}(h(x)h(y)) \geq \max(\mu_{A^N}(h(x)), \mu_{A^N}(h(y)))$

$$\Rightarrow \left\{\begin{array}{l}\sum\limits_{x\neq y}\mu_{B^P}(xy) + \sum\limits_{x\neq y}\mu_{B^P}(h(x)h(y)) \leq \sum\limits_{x\neq y}\min(\mu_{A^P}(h(x)), \mu_{A^P}(h(y)))\\ \sum\limits_{x\neq y}\mu_{B^N}(xy) + \sum\limits_{x\neq y}\mu_{B^N}(h(x)h(y)) \geq \sum\limits_{x\neq y}\max(\mu_{A^N}(h(x)), \mu_{A^N}(h(y)))\end{array}\right.$$

$$\Rightarrow \begin{cases} 2\sum_{x\neq y}\mu_{B^P}(xy) \leq \sum_{x\neq y}\min(\mu_{A^P}(x),\mu_{A^P}(y)) \\ 2\sum_{x\neq y}\mu_{B^N}(xy) \geq \sum_{x\neq y}\max(\mu_{A^N}(x),\mu_{A^N}(y)). \end{cases}$$

Hence,

$$\sum_{x\neq y}\mu_{B^P}(xy) \leq \frac{1}{2}\sum_{x\neq y}\min(\mu_{A^P}(x),\mu_{A^P}(y)),$$

$$\sum_{x\neq y}\mu_{B^N}(xy) \geq \frac{1}{2}\sum_{x\neq y}\max(\mu_{A^N}(x),\mu_{A^N}(y)).$$ □

4. Isomorphism properties of μ-complement of highly irregular bipolar fuzzy graphs

Definition 4.1. Let $G = (A, B)$ be a bipolar fuzzy graph. The μ-complement of G is defined as $G^\mu = (A, B^\mu)$ where $B^\mu = (\mu^\mu_{B^P}, \mu^\mu_{B^N})$ and

$$\mu^\mu_{B^P}(xy) = \begin{cases} \mu_{A^P}(x) \wedge \mu_{A^P}(y) - \mu_{B^P}(xy) & \text{if } \mu_{B^P}(xy) > 0 \\ 0 & \text{if } \mu_{B^P}(xy) = 0 \end{cases}$$

$$\mu^\mu_{B^N}(xy) = \begin{cases} \mu_{A^N}(x) \vee \mu_{A^N}(y) - \mu_{B^N}(xy) & \text{if } \mu_{B^N}(xy) < 0 \\ 0 & \text{if } \mu_{B^N}(xy) = 0 \end{cases}$$

Theorem 4.2. *The μ-complement of a highly irregular bipolar fuzzy graph need not be highly irregular.*

Proof. To every vertex, the adjacent vertices with distinct degrees or the non-adjacent vertices with distinct degrees may happen to be adjacent vertices with same degrees. This contradicts the definition of highly irregular bipolar fuzzy graph. □

Theorem 4.3. *Let G_1 and G_2 be two highly irregular bipolar fuzzy graphs. If G_1 and G_2 are isomorphic, then μ-complement of G_1 and G_2 are also isomorphic and vice versa, but the complements need not be highly irregular.*

Proof. The proof is similar to Theorem 3.22. □

Remark 4.4. Let G_1 and G_2 be two highly irregular bipolar fuzzy graphs. If G_1 is weak isomorphic with G_2, then neither μ-complement of G_1 is weak isomorphic with μ-complement of G_2 nor μ-complement of G_2 is weak isomorphic with μ-complement of G_1.

The following example illustrate the above theorem.

Example 4.5. Consider two highly irregular bipolar fuzzy graphs G_1 and G_2 defined as follows.

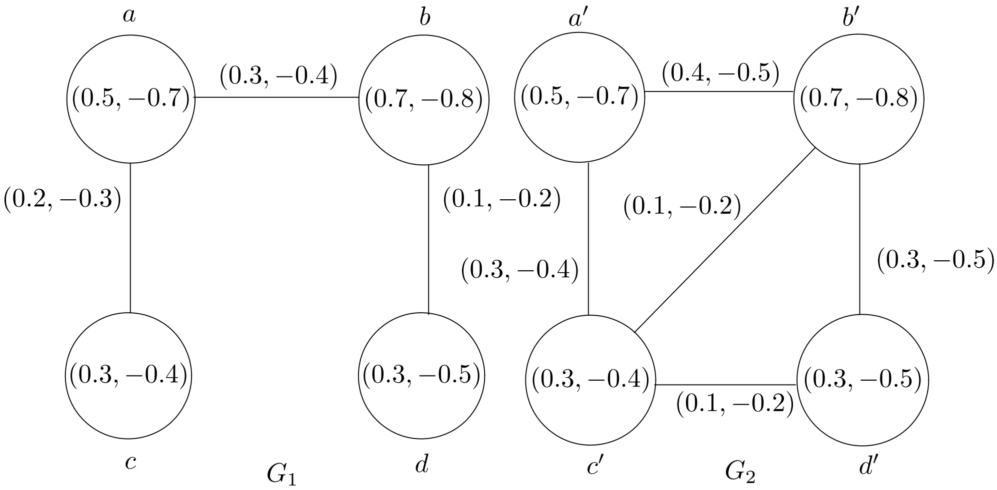
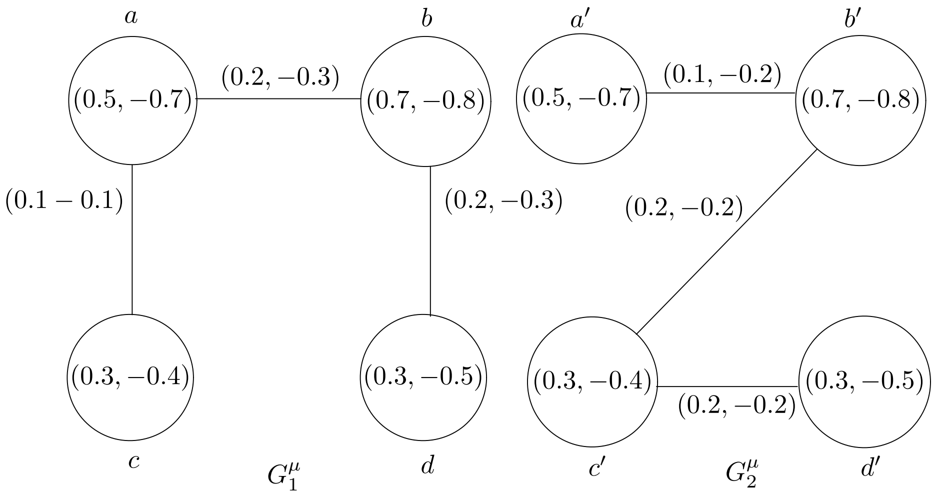


Fig. 11: Highly irregular bipolar fuzzy graphs G_1 and G_2



It is easy to see that G_1 and G_2 are highly irregular bipolar fuzzy graphs and G_1 is weak isomorphic with G_2, but μ-complement of G_1 is not weak isomorphic with μ-complement of G_2.

Theorem 4.6. *If there is a co-weak isomorphism between G_1 and G_2, then μ-complement of G_1 and G_2 need not be co-weak isomorphic, but there can be a homomorphism between μ-complement of G_1 and G_2.*

Proof. It follows from Theorem 3.26. □

Definition 4.7. A bipolar fuzzy graph $G = (A, B)$ is said to be a self μ-complementary bipolar fuzzy graph if $G \cong G^{\mu}$.

Theorem 4.8. *Let G be a highly irregular and self μ-complementary bipolar fuzzy graph, then*
$\sum_{u\neq v}\mu_{B^P}(uv) = \frac{1}{2} = \sum_{u\neq v}\mu_{A^P}(u)\wedge\mu_{A^P}(v)$ *and* $\sum_{u\neq v}\mu_{B^N}(uv) = \frac{1}{2} = \sum_{u\neq v}\mu_{A^N}(u)\vee\mu_{A^N}(v)$.

Proof. Let $G=(A,B)$ be a self μ-complementary highly irregular bipolar fuzzy graph. Since, $G\cong G^\mu$, there exists a bijective map $h:V\to V$ such that
$\mu_{A^P}(u)=\mu^\mu_{A^P}(h(u))=\mu_{A^P}(h(u))$,

$$\mu_{A^N}(u)=\mu^\mu_{A^N}(h(u))=\mu_{A^N}(h(u)) \text{ for all } u,v\in V \tag{4.1}$$

and

$$\mu_{B^P}(uv)=\mu^\mu_{B^P}(h(u)h(v)),\quad \mu_{B^N}(uv)=\mu^\mu_{B^N}(h(u)h(v)) \text{ for all } uv\in E. \tag{4.2}$$

By the definition of μ-complement of a bipolar fuzzy graph we have
$\mu^\mu_{B^P}(uv)=\begin{cases}\mu_{A^P}(u)\wedge\mu_{A^P}(v)-\mu_{B^P}(uv) & \text{if } \mu_{B^P}(uv)>0,\\ 0 & \text{if } \mu_{B^P}(uv)=0.\end{cases}$
Therefore, $\mu^\mu_{B^P}(h(u)h(v))=\mu^\mu_{A^P}(h(u))\wedge\mu^\mu_{A^P}(h(v))-\mu_{B^P}(h(u)h(v))$.
From 4.1 and 4.2 we have
$\mu_{B^P}(uv)=\mu_{A^P}(h(u))\wedge\mu_{A^P}(h(v))-\mu_{B^P}(h(u)h(v))$
$\mu_{B^P}(uv)+\mu_{B^P}(h(u)h(v))=\mu_{A^P}(h(u))\wedge\mu_{A^P}(h(v))$
$2\mu_{B^P}(uv)=\mu_{A^P}(u)\wedge\mu_{A^P}(v)$.
Taking summation,
$2\sum_{u\neq v}\mu_{B^P}(uv)=\sum_{u\neq v}\mu_{A^P}(u)\wedge\mu_{A^P}(v)$. So, $\sum_{u\neq v}\mu_{B^P}(uv)=\frac{1}{2}\sum_{u\neq v}\mu_{A^P}(u)\wedge\mu_{A^P}(v)$.
Similarly, we can show that

$$\sum_{u\neq v}\mu_{B^N}(uv)=\frac{1}{2}\sum_{u\neq v}\mu_{A^N}(u)\vee\mu_{A^N}(v).$$

□

Definition 4.9. A bipolar fuzzy graph $G=(A,B)$ is said to be a self weak μ-complementary bipolar fuzzy graph if G is weak isomorphic with G^μ.

Example 4.10. Here, G and G^μ are highly irregular bipolar fuzzy graphs and also G is weak isomorphic with G^μ. Thus, G is a self weak μ-complementary bipolar fuzzy graph.

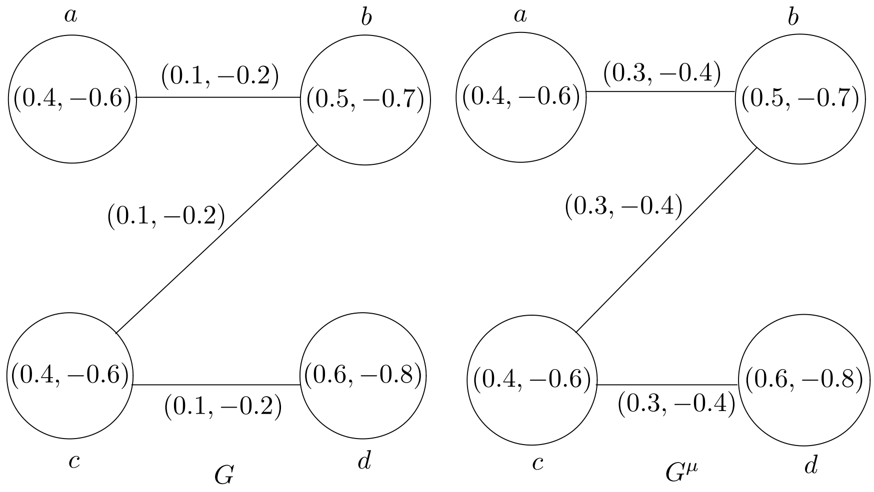


Fig. 12: Highly irregular bipolar fuzzy graphs G and G^μ

5. Conclusions

In this paper, weak isomorphism, co-weak isomorphism and isomorphism of neighbourly irregular bipolar fuzzy graphs and highly irregular bipolar fuzzy graphs are defined. Some results on order, size and degrees of the nodes between isomorphic neighbourly irregular and isomorphic highly irregular bipolar fuzzy graphs are discussed. Finally, isomorphism properties of μ-complement, self μ-complement and self weak μ-complement of highly irregular bipolar fuzzy graphs are stablished.

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Hossein Rashmanlou (rashmanlou@gmail.com)
Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea

R. A. Borzooei (borzooei@sbu.ac.ir)
Department of Mathematics, Shahid Beheshti University, Tehran, Iran