Fuzzy $e$-continuity and fuzzy $e$-open sets

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Abstract. In this paper the concept of fuzzy $e$-open set is introduced and its properties are studied in fuzzy topological spaces. Moreover, we introduce the fuzzy $e$-continuous mapping and other mapping and establish their various characteristic properties. Further fuzzy $e$-separation axioms have been introduced and investigated with the help of fuzzy $e$-open sets.

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1. Introduction

The concept of fuzzy has invaded almost all branches of mathematics with the introduction of fuzzy sets by Zadeh [12] of 1965. The theory of fuzzy topological spaces was introduced and developed by Chang[5]. In 2008, Erdal Ekici[7], has introduced the concept of $e$-open sets in general topology. In this paper, we extend the notion of $e$-open sets to fuzzy topological space in the name fuzzy $e$-open sets and study some properties based on this new concept. We further study the relation between fuzzy $e$-open sets with other types of fuzzy open sets. We also introduce the concepts of fuzzy $e$-continuous mappings and study their nature with separation axioms.

2. Preliminaries

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \gamma)$ (or simply $X$, $Y$ and $Z$) represent non-empty fuzzy topological spaces. Let $A$ be a fuzzy subset of a space $X$. The fuzzy closure of $A$, fuzzy interior of $A$, fuzzy $\delta$-closure of $A$ and the fuzzy $\delta$-interior of $A$ are denoted by $cl(A)$, $int(A)$, $cl_\delta(A)$ and $int_\delta(A)$ respectively. A fuzzy subset $A$ of space $X$ is called fuzzy regular open [2] (resp. fuzzy regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The fuzzy $\delta$-interior of fuzzy subset $A$ of $X$ is the union of all fuzzy regular open sets contained in $A$. A fuzzy subset $A$ is called fuzzy $\delta$-open[11]
if $A = \text{int}_\delta(A)$. The complement of fuzzy $\delta$-open set is called fuzzy $\delta$-closed (i.e., $A = \text{cl}_\delta(A)$).

A fuzzy subset $A$ of a space $X$ is called fuzzy semi open \[2\] (resp. fuzzy $\alpha$-open set \[10\], fuzzy $\beta$-open set \[3\], fuzzy pre-open set \[1\], fuzzy $\gamma$-open \[9\], fuzzy $\delta$-preopen \[1\], fuzzy $\delta$-semi open)\[8\] if $A \leq \text{cl} \text{int} A$ (resp. $A \leq \text{int} \text{cl} (A)$), $A \leq \text{cl} (\text{int} A)$, $A \leq \text{int} (\text{cl} A)$, $A \leq \text{cl} (\text{int} (\text{cl} A))$, $A \leq \text{cl} (\text{int}_\delta A)$). The complement of a fuzzy $\delta$-semiopen set (resp. fuzzy $\delta$-preopen set) is called fuzzy $\delta$-semiclosed (resp. fuzzy $\delta$-preclosed). The union of all fuzzy $\delta$-semi open (resp. fuzzy $\delta$-preopen) sets contained in a fuzzy set $A$ in a fuzzy topological space $X$ is called the fuzzy $\delta$-semi interior \[8\] (resp. fuzzy $\delta$-pre interior \[1\]) of $A$ and it is denoted by $\text{sint}_\delta(A)$ (resp. $\text{pint}_\delta(A)$). The intersection of all fuzzy $\delta$-semi closed (resp. fuzzy $\delta$-preclosed) sets containing a fuzzy set $A$ in a fuzzy topological space $X$ is called the fuzzy $\delta$-semiclosure \[8\] (resp. fuzzy $\delta$-preclosure \[1\]) of $A$ and it is denoted by $\text{scl}_\delta(A)$ (resp. $\text{pcl}_\delta(A)$).

A function $f : X \rightarrow Y$ is called fuzzy $\delta$-pre continuous \[1\] (resp. fuzzy $\delta$-semi continuous \[6\]) if $f^{-1}(\lambda)$ is fuzzy $\delta$-pre open (resp. fuzzy $\delta$-semi open) in $X$ for every fuzzy open set $\lambda$ of $Y$.

### 3. Fuzzy $e$-open set

**Definition 3.1.** A fuzzy subset $\mu$ of a space $X$ is called fuzzy $e$-open (briefly, $fe$-open) if

$$\mu \leq \text{cl}(\text{int}_\delta \mu) \vee \text{int}(\text{cl}_\delta \mu),$$

fuzzy $e$-closed (briefly, $fe$-closed) if

$$\mu \geq \text{cl}(\text{int}_\delta \mu) \wedge \text{int}(\text{cl}_\delta \mu).$$

From the definitions we obtain the following diagram

![Diagram of fuzzy topological properties](attachment:diagram.png)

None of these implications are reversible as shown in the following example.
Example 3.2. Let $X = \{a, b, c\}$ and $v_1, v_2, v_3$ be fuzzy sets of $X$ defined as

\[v_1(a) = 0.2, \ v_2(a) = 0.1, \ v_3(a) = 0.2\]
\[v_1(b) = 0.3, \ v_2(b) = 0.1, \ v_3(b) = 0.4\]
\[v_1(c) = 0.4, \ v_2(c) = 0.4, \ v_3(c) = 0.4\]

Let $\tau = \{0, v_1, v_2, 1\}$, then the fuzzy set $v_3$ is fuzzy $e$-open set. But it is not fuzzy $\delta$-preopen.

Example 3.3. Let $X = \{a, b, c\}$ and $\nu_1, \nu_2, \nu_3, \nu_4$ be fuzzy sets of $X$ defined as

\[\nu_1(a) = 0.3, \ \nu_2(a) = 0.4, \ \nu_3(a) = 0.4, \ \nu_4(a) = 0.3\]
\[\nu_1(b) = 0.5, \ \nu_2(b) = 0.2, \ \nu_3(b) = 0.5, \ \nu_4(b) = 0.5\]
\[\nu_1(c) = 0.5, \ \nu_2(c) = 0.6, \ \nu_3(c) = 0.6, \ \nu_4(c) = 0.4\]

Let $\tau = \{0, \nu_1, \nu_2, \nu_3, \nu_1 \land \nu_2, 1\}$. Then the fuzzy set $\nu_4$ is fuzzy $e$-open but not fuzzy $\delta$-semi open and also not a fuzzy $\beta$-open, fuzzy $\gamma$-open and fuzzy semi open.

Example 3.4. Let $X = \{a, b, c\}$ and $u_1, u_2, u_3, u_4$ be fuzzy sets of $X$ defined as

\[u_1(a) = 0.3, \ u_2(a) = 0.6, \ u_3(a) = 0.6, \ u_4(a) = 0.3\]
\[u_1(b) = 0.4, \ u_2(b) = 0.5, \ u_3(b) = 0.5, \ u_4(b) = 0.4\]
\[u_1(c) = 0.5, \ u_2(c) = 0.5, \ u_3(c) = 0.4, \ u_4(c) = 0.4\]

Let $\tau = \{0, u_1, u_2, u_3, u_4, 1\}$ and let $\lambda$ be fuzzy set defined as $\lambda(a) = 0.7, \ \lambda(b) = 0.6, \ \lambda(c) = 0.4$. Then $\lambda$ is not fuzzy $e$-open set but it is fuzzy $\beta$-open, fuzzy $\gamma$-open and fuzzy semi open.

Lemma 3.5. [18] Let $\mu$ be a fuzzy subset of $X$, then

(i) $pcl_{\delta}(\mu) = \mu \lor cl(int_{\delta}(\mu))$ and $pint_{\delta}(\mu) = \mu \land cl(int_{\delta}(\mu))$

(ii) $sc_{\delta}(\mu) = \mu \lor int(cl_{\delta}(\mu))$ and $sint_{\delta}(\mu) = \mu \land cl(int_{\delta}(\mu))$

Theorem 3.6. For any fuzzy subset $\mu$ of a space $X$, $\mu$ is fuzzy $e$-open if and only if $\mu = pint_{\delta}(\mu) \lor sint_{\delta}(\mu)$.

Proof. Let $\mu$ be fuzzy $e$-open. Then $\mu \leq cl(int_{\delta}(\mu)) \lor int(cl_{\delta}(\mu))$. By lemma [3.5], we have

\[pint_{\delta}(\mu) \lor sint_{\delta}(\mu) = (\mu \land int(cl_{\delta}(\mu))) \lor (\mu \land cl(int_{\delta}(\mu))) = \mu \land (int(cl_{\delta}(\mu)) \lor cl(int_{\delta}(\mu))) = \mu\]

Conversely, if $\mu = pint_{\delta}(\mu) \lor sint_{\delta}(\mu)$ then, by lemma [3.5], $\mu = pint_{\delta}(\mu) \lor sint_{\delta}(\mu) = \mu \land int(cl_{\delta}(\mu)) \lor \mu \land cl(int_{\delta}(\mu)) = \mu \land (int(cl_{\delta}(\mu)) \lor cl(int_{\delta}(\mu))) \leq int(cl_{\delta}(\mu)) \lor cl(int_{\delta}(\mu))$ and hence $\mu$ is fuzzy $e$-open.

Theorem 3.7. In a fuzzy topological space $X$,

(i) Any union of fuzzy $e$-open sets is a fuzzy $e$-open set, and

(ii) Any intersection of fuzzy $e$-closed sets is a fuzzy $e$-closed set.

Proof. (i) Let $\lambda_\alpha$ be a collection of fuzzy $e$-open sets. Then for each $\alpha$, $\lambda_\alpha \leq (cl(int_{\delta}(\lambda_\alpha))) \lor (int(cl_{\delta}(\lambda_\alpha))) \leq (cl(int_{\delta}(\lor \lambda_\alpha))) \lor (int(cl_{\delta}(\lor \lambda_\alpha)))$. Thus $\lor \lambda_\alpha$ is a fuzzy $e$-open set.

(ii) Since $\mu_\alpha = 1 - \lambda_\alpha$ is fuzzy closed set, from (i) we have $\mu_\alpha = 1 - \lambda_\alpha \geq 1 - [(cl(int_{\delta}(\lor \lambda_\alpha))) \lor (int(cl_{\delta}(\lor \lambda_\alpha)))]$. From this we have $\mu_\alpha \geq [1 - (cl(int_{\delta}(\lor \lambda_\alpha))) \land [1 - (int(cl_{\delta}(\lor \lambda_\alpha)))].$ This implies $\mu_\alpha \geq [(int(cl_{\delta}(1-(\lor \lambda_\alpha))))\land[cl(int_{\delta}(1-(\lor \lambda_\alpha)))].$
As 1 - (∨λα) = ∧(1 - λα) we get μα ≥ [(int(clδ(∧(μα))))] ∧ [(cl(intδ(∧(μα))))]. Thus ∧μα is a fuzzy e-closed set.

Definition 3.8. Let μ be any fuzzy set. Then
(i) fe-cl(μ) = ∧{λ : λ ≥ μ, λ is a fuzzy e-closed set of X}
(ii) fe-cl(μ) = ∨{λ : λ ≤ μ, λ is a fuzzy e-open set of X}

Theorem 3.9. In a fuzzy topological space X, λ be a fuzzy e-closed (resp. fuzzy e-open) if and only if λ = fe-cl(λ) (resp. λ = fe-int(λ)).

Proof. Suppose λ = fe-cl(λ) = ∧{μ : μ is a fuzzy e-closed set and μ ≥ λ}. This means λ ∈ ∧{μ : μ is a fuzzy e-closed set and μ ≥ λ} and hence λ is fuzzy e-closed set.

Conversely, suppose λ be a fuzzy e-closed in X. Then we have λ ∈ {μ : μ is a fuzzy e-closed set and μ ≥ λ}. Hence, λ ≤ μ implies λ = ∧{μ : μ is a fuzzy e-closed set and μ ≥ λ}= fe-cl(λ).

Similarly for λ = fe-int(λ).

Theorem 3.10. In a fuzzy topological space X the following holds for fuzzy e-closure sets.
(i) fe-cl(0) = 0.
(ii) fe-cl(λ) is a fuzzy e-closed set in X.
(iii) fe-cl(λ) ≤ fe-cl(μ) if λ ≤ μ.
(iv) fe-cl(fe-cl(λ)) = fe-cl(λ).

Similar results hold for fuzzy e-interiors.

Theorem 3.11. In a fuzzy topological space X, we have
(i) fe-cl(λ ∨ μ) ≥ fe-cl(λ) ∨ fe-cl(μ)
(ii) fe-cl(λ ∧ μ) ≤ fe-cl(λ) ∧ fe-cl(μ).

Proof. (i) λ ≤ λ ∨ μ or μ ≤ λ ∨ μ this implies fe-clλ ≤ fe-cl(λ ∨ μ) or fe-clμ ≤ fe-cl(λ ∨ μ). Therefore fe-cl(λ ∨ μ) ≥ fe-cl(λ) ∨ fe-cl(μ).

(ii) Similar proof of (i).

Theorem 3.12. In a fuzzy topological space X, we have
(i) fe-int(λ ∨ μ) ≥ fe-int(λ) ∨ fe-int(μ) and
(ii) fe-int(λ ∧ μ) ≤ fe-int(λ) ∧ fe-int(μ).

Theorem 3.13. Let u be fuzzy e-open set, we have
(i) If intδ(u) = 0, then u is fuzzy δ-preopen.
(ii) If clδ(u) = 0, then u is fuzzy δ-semiopen.

Theorem 3.14. Let μ be a fuzzy subset of a space X, then, fe-cl(μ) = fpclδ(μ) ∧ fscfδ(μ).

Proof. It is obvious that, fe-cl(μ) ≤ fpclδ(μ) ∧ fscfδ(μ).

Conversely, from definition we have fe-cl(μ) ≥ cl(intδ(e-cl(μ))) ∧ int(clδ(e-cl(μ))) ≥ cl(intδ(μ)) ∧ int(clδ(μ)). Since fe-cl(μ) is fuzzy e-closed, by lemma [3,5] we have fpclδ(μ) ∧ fscfδ(μ) = (μ ∨ cl(intδ(μ))) ∧ (μ ∨ int(clδ(μ))) = μ ∨ (cl(intδ(μ)) ∧ int(clδ(μ))) = μ ≤ fe-cl(μ).
Theorem 3.15. Let μ be a fuzzy subset of a space X, then $fe \text{-int}(\mu) = fpint_{\delta}(\mu) \wedge fsint_{\delta}(\mu)$.

Proof is similar to the above theorem.

Theorem 3.16. Let λ be any fuzzy set in X, then

(i) $fe \text{-cl}(1 - \lambda) = 1 - fe \text{-int}(\lambda)$

(ii) $fe \text{-int}(1 - \lambda) = 1 - fe \text{-cl}(\lambda)$.

Proof. (i) Let $v$ be fuzzy e-open set. Then for a fuzzy e-open set $v \leq \lambda$, $v \geq 1 - \lambda$. Then $fe \text{-int}(\lambda) = \lor\{1 - v, v \text{ is a fuzzy e-closed set and } v \geq 1 - \lambda\} = 1 - \lor\{v : v \text{ is a fuzzy e-closed set and } v \geq 1 - \lambda\} = 1 - fe \text{-cl}(1 - \lambda)$. Thus $fe \text{-cl}(1 - \lambda) = 1 - fe \text{-int}(\lambda)$.

(ii) Let $\mu$ be fuzzy e-open set. Then for a fuzzy e-closed set $\mu \geq \lambda$, $\mu \leq 1 - \lambda$. Then $fe \text{-cl}(\lambda) = \land\{1 - \mu, \mu \text{ is a fuzzy e-open set and } \mu \leq 1 - \lambda\} = 1 - \land\{\mu : \mu \text{ is a fuzzy e-open set and } \mu \leq 1 - \lambda\} = 1 - fe \text{-int}(1 - \lambda)$. Thus $fe \text{-int}(1 - \lambda) = 1 - fe \text{-cl}(\lambda)$.

4. FUZZY E-CONTINUITY AND SEPARATION AXIOMS

Definition 4.1. A mapping $f : X \to Y$ is said to be a fuzzy e-continuous if $f^{-1}(\lambda)$ is fuzzy e-open in X for every fuzzy open set $\lambda$ in Y.

Definition 4.2. A mapping $f : X \to Y$ is said to be a fuzzy e-irresolute if $f^{-1}(\lambda)$ is fuzzy e-open in X for every fuzzy e-open set $\lambda$ in Y.

Theorem 4.3. For a mapping $f : (X, T) \to (Y, S)$, the following statements are equivalent

(i) $f$ is a fuzzy e-continuous.

(ii) For every fuzzy singleton $x_p \in X$ and every fuzzy open set $v$ in Y such that $f(x_p) \leq v$, there exist fuzzy e-open set $u \leq X$ such that $x_p \leq u$ and $f(u) \leq v$.

(iii) $f^{-1}(\lambda) = \text{int}(cl_{\delta}f^{-1}(\lambda)) \lor \text{cl}(\text{int}_{\delta}f^{-1}(\lambda))$ for each fuzzy open set $\lambda$ in Y.

(iv) The inverse image of each fuzzy closed set in Y is fuzzy e-closed.

(v) $\text{cl}(\text{int}_{\delta}f^{-1}(v)) \lor \text{int}(cl_{\delta}f^{-1}(v)) \leq f^{-1}(f(v))$ for each fuzzy set $v$ in Y.

(vi) $f(\text{cl}_{\delta}f^{-1}(u)) \lor \text{int}(cl_{\delta}f^{-1}(u)) \leq cl(f(u))$ for every fuzzy set $u \leq X$.

Proof. (i) ⇒ (ii) : Let the singleton set $x_p$ in $X$ and every fuzzy open set $v$ in Y such that $f(x_p) \leq v$. Since $f$ is fuzzy e-continuous. Then $x_p \in f^{-1}(f(x_p)) \leq f^{-1}(v)$. Let $u = f^{-1}(v)$ which is a fuzzy e-open set in X. So, we have $x_p \leq u$. Now $f(u) = f(f^{-1}(v)) \leq v$.

(ii) ⇒ (iii) : Let $\lambda$ be any fuzzy open set in Y. Let $x_p$ be any fuzzy point in X such that $f(x_p) \leq \lambda$. Then $x_p \in f^{-1}(\lambda)$. By (ii), there exists a fuzzy e-open set $u \leq X$ such that $x_p \leq u$ and $f(u) \leq \lambda$. Therefore, $x_p \in u \leq f^{-1}(f(u)) \leq f^{-1}(\lambda) \leq \text{int}(cl_{\delta}f^{-1}(\lambda)) \lor cl(\text{int}_{\delta}f^{-1}(\lambda))$.

(iii) ⇒ (iv) : Let $\lambda$ be any fuzzy closed set in Y. Then $1 - \lambda$ be a fuzzy open set in Y. By (iii), $f^{-1}(1 - \lambda) \leq \text{int}(cl_{\delta}f^{-1}(1 - \lambda)) \lor cl(\text{int}_{\delta}f^{-1}(1 - \lambda))$. This implies $1 - f^{-1}(\lambda) \leq \text{int}(cl_{\delta}(1 - f^{-1}(\lambda)) \lor cl(\text{int}_{\delta}(1 - f^{-1}(\lambda))) \leq \text{int}(1 - \text{int}_{\delta}f^{-1}(\lambda)) \lor cl(1 - cl_{\delta}f^{-1}(\lambda)) = 1 - cl(\text{int}_{\delta}f^{-1}(\lambda)) \lor 1 - \text{int}(cl_{\delta}f^{-1}(\lambda))$ and hence $1 - f^{-1}(\lambda) =
Let $X, Y$ and $Z$ be fuzzy topological spaces. If $x, y \in X$ pair of distinct points then $f$ is an $\mathcal{E}$-continuous function and this implies $f^{-1}(\lambda)$ is fuzzy $\mathcal{E}$-closed in $X$.

**(iv) $\Rightarrow$ (v):** Let $\nu \leq Y$. Then $f^{-1}(cl(\nu))$ is fuzzy $\mathcal{E}$-closed in $X$. (i.e.$\nu$ is fuzzy $\mathcal{E}$-closed in $Y$ and $cl(\nu)$ is fuzzy $\mathcal{E}$-closed in $Y$).

$$(v) \Rightarrow (vi):$$ Let $u \leq X$. Put $\nu = f(u)$ in $(v)$. Then, $\nu = f^{-1}(cl(\nu)) \leq cl(\nu)$ in $X$. This implies that $\nu = f^{-1}(cl(\nu)) \leq \nu$ in $X$.

$$(vi) \Rightarrow (i):$$ Let $v \leq Y$ be fuzzy open set. Put $u = I_Y - v$ and $u = f^{-1}(v)$ then $f^{-1}(v) \leq f^{-1}(v) \leq cl(v) = v$. That is, $f^{-1}(v)$ is fuzzy $\mathcal{E}$-closed in $X$, so $f$ is fuzzy $\mathcal{E}$-continuous.

we obtain the following diagram hold:

\[
\begin{array}{ccc}
\text{fuzzy $\delta$-pre continuous} & \Longrightarrow & \text{fuzzy $\delta$-semi continuous} \\
\text{fuzzy $\mathcal{E}$-continuous} & \Longleftrightarrow & \text{fuzzy $\delta$-semi continuous}
\end{array}
\]

These implications are not reversible as shown in the following example.

**Example 4.4.** Let $X = \{a, b, c\}$ and $v_1, v_2, v_3$ and $v_4$ be fuzzy sets of $X$ defined as

$v_1(a) = 0.4$, $v_2(a) = 0.6$, $v_3(a) = 0.6$, $v_4(a) = 0.4$

$v_1(b) = 0.6$, $v_2(b) = 0.4$, $v_3(b) = 0.4$, $v_4(b) = 0.5$

$v_1(c) = 0.5$, $v_2(c) = 0.4$, $v_3(c) = 0.5$, $v_4(c) = 0.5$

Let $\tau_1 = \{0, v_1, v_2, v_3 \vee v_2, v_3 \vee v_2, 1\}$, $\tau_2 = \{0, v_3, 1\}$, and $\tau_3 = \{0, v_4, 1\}$ and the mapping $f : (X, \tau_1) \rightarrow (X, \tau_2)$ and $g : (X, \tau_1) \rightarrow (X, \tau_3)$ defined as $f(a) = a, f(b) = b, f(c) = c$. It is clear that $f$ is fuzzy $\mathcal{E}$-continuous, but it is not fuzzy $\delta$-pre continuous and $g$ is fuzzy $\mathcal{E}$-continuous, but it is not fuzzy $\delta$-semi continuous.

**Theorem 4.5.** Let $X, Y$ and $Z$ be fuzzy topological spaces.

(i) If $f : X \rightarrow Y$ fuzzy $\mathcal{E}$-continuous and $g : Y \rightarrow Z$ is fuzzy continuous. Then $g \circ f : X \rightarrow Z$ is fuzzy $\mathcal{E}$-continuous.

(ii) If $f : X \rightarrow Y$ fuzzy $\mathcal{E}$- irresolute and $g : Y \rightarrow Z$ is fuzzy $\mathcal{E}$-continuous. Then $g \circ f : X \rightarrow Z$ is fuzzy $\mathcal{E}$- continuous.

**Proof.** Obvious.

**Definition 4.6.** A fuzzy topological space $(X, \tau)$ is said to be fuzzy $\mathcal{E}$-$T_1$ if for each pair of distinct points $x$ and $y$ of $X$, there exists fuzzy $\mathcal{E}$-closed sets $U_1$ and $U_2$ such that $x \in U_1$ and $y \notin U_2$, $x \notin U_2$ and $y \notin U_1$.

**Theorem 4.7.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is fuzzy $\mathcal{E}$-continuous injective function and $Y$ is fuzzy $\mathcal{E}$-$T_1$ then $X$ is fuzzy $\mathcal{E}$-$T_1$.
Suppose that \( Y \) is fuzzy \( T_1 \). For any two distinct points \( x \) and \( y \) of \( X \), there exists fuzzy open sets \( F_1 \) and \( F_2 \) in \( Y \) such that \( f(x) \in F_1 \), \( f(y) \in F_2 \), \( f(x) \notin F_2 \) and \( f(y) \notin F_1 \). Since \( f \) is injective fuzzy \( e \)-continuous function, we have \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are fuzzy \( e \)-open sets in \( X \). Hence by definition \( X \) is fuzzy \( e \)-\( T_1 \).

**Definition 4.8.** A fuzzy topological space \((X, \tau)\) is said to be fuzzy \( e \)-\( T_2 \) (i.e., fuzzy \( e \)-Hausdorff) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists disjoint fuzzy \( e \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

**Theorem 4.9.** If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy \( e \)-continuous injective function and \( Y \) is fuzzy \( T_2 \) then \( X \) is fuzzy \( e \)-\( T_2 \).

**Proof.** Suppose that \( Y \) is fuzzy \( T_2 \) space. For any two distinct points \( x \) and \( y \) of \( X \), there exists fuzzy open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \), \( f(y) \in V \), \( f(x) \notin V \) and \( f(y) \notin U \). Since \( f \) is injective fuzzy \( e \)-continuous function, we have \( f^{-1}(U) \) and \( f^{-1}(V) \) are fuzzy \( e \)-open sets in \( X \). Hence by definition, \( X \) is fuzzy \( e \)-\( T_2 \).

**Definition 4.10.** A fuzzy topological space \((X, \tau)\) is said to be fuzzy \( e \)-normal if for every two disjoint fuzzy closed sets \( A \) and \( B \) of \( X \), there exist two disjoint fuzzy \( e \)-open sets \( U \) and \( V \) such that \( A \leq U \) and \( B \leq V \) and \( U \cap V = 0 \).

**Theorem 4.11.** If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy \( e \)-continuous closed injective function and \( Y \) is fuzzy normal then \( X \) is fuzzy \( e \)-normal.

**Proof.** Suppose that \( Y \) fuzzy normal. Let \( A \) and \( B \) be closed fuzzy sets in \( X \) such that \( A \cap B = 0 \). Since \( f \) is fuzzy closed injection \( f(A) \) and \( f(B) \) are fuzzy closed in \( Y \) and \( f(A) \cap f(B) = 0 \). Since \( Y \) is normal, there exists fuzzy open sets \( U \) and \( V \) in \( Y \) such that \( f(A) \leq U \), \( f(B) \leq V \) and \( U \cap V = 0 \). Therefore we obtain, \( A \leq f^{-1}(U) \) and \( B \leq f^{-1}(V) \) and \( f^{-1}(U \cap V) = 0 \). Since \( f \) is fuzzy \( e \)- continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are fuzzy \( e \)-open sets. Hence by definition \( X \) is fuzzy \( e \)-normal.

**Definition 4.12.** A space \( X \) is said to be fuzzy \( e \)-regular if for each closed set \( F \) of \( X \) and each \( x \in X - F \), there exist disjoint fuzzy \( e \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( F \leq V \).

**Theorem 4.13.** If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy \( e \)-continuous closed injective function and \( Y \) is fuzzy regular then \( X \) is fuzzy \( e \)-regular.

**Proof.** Let \( F \) be fuzzy closed set in \( Y \) with \( y \notin F \). Take \( y = f(x) \). Since \( Y \) is fuzzy regular, there exists disjoint fuzzy open sets \( U \) and \( V \) such that \( x \in U \) and \( y = f(x) \in f(U) \) and \( F \leq f(V) \) such that \( f(U) \) and \( f(V) \) are disjoint fuzzy open sets. Therefore we obtain that, \( f^{-1}(F) \leq V \). Since \( f \) is fuzzy \( e \)-continuous, \( f^{-1}(F) \) is fuzzy \( e \)-closed set in \( X \) and \( x \notin f^{-1}(F) \). Hence by definition \( X \) is fuzzy \( e \)-regular.

**Definition 4.14.** A fuzzy set \( v \) in a fuzzy topological spaces \((X, \tau)\) is said to be fuzzy \( e \)-connected if and only if \( v \) cannot be expressed as the union of two fuzzy \( e \)-open sets.

**Theorem 4.15.** Let \( f : X \to Y \) be a fuzzy \( e \)-continuous surjective mapping. If \( v \) is a fuzzy \( e \)-connected subset in \( X \) then, \( f(v) \) is fuzzy connected in \( Y \).

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Proof. Suppose that $f(d)$ is not fuzzy connected in $Y$. Then, there exist fuzzy open sets $u$ and $v$ in $Y$ such that $f(d) = u \lor v$. Since $f$ is fuzzy $e$-continuous surjective mapping, $f^{-1}(u)$ and $f^{-1}(v)$ are fuzzy $e$-open set in $X$ and $d = f^{-1}(u \lor v) = f^{-1}(u) \lor f^{-1}(v)$. It is clear that $f^{-1}(u)$ and $f^{-1}(v)$ are fuzzy $e$-open set in $X$. Therefore, $d$ is not fuzzy $e$-connected in $X$, which is a contradiction. Hence, $Y$ is fuzzy connected. □

References