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Soft fuzzy soft homotopy and its topological foldings of a soft fuzzy soft manifold

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Abstract. In this paper, the notions of soft fuzzy soft set, soft fuzzy soft homotopy and soft fuzzy soft fundamental group are introduced. Properties governing these definitions are discussed. Moreover, soft fuzzy soft manifold and soft fuzzy soft topological foldings on it are introduced. In this connection, several properties are established with diagrams wherever necessary.

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1. Introduction

In order to deal with many complicated problems in the field of engineering, social science, economics, medical science, etc., involving uncertainties, classical methods are found to be inadequate in recent times. The concept of fuzzy set was introduced by L.Zadeh [11], which also inadequacy of the parameterization tool of the theory. So, D. Molodtsov [8] proposed the soft set which is free from parameterization inadequacy syndrome of fuzzy set theory, probability theory, etc. The notion of fuzzy soft set was introduced and discussed by P. K. Maji [6]. The concept of soft fuzzy open set was introduced by T. Yogalakshmi, E. Roja, M. K. Uma [10]. The American Mathematician J. W. Alexander[1] was the first to show that knot theory is extremely important in the study of 3-dimensional topology. Further topological folding in manifolds was discussed by El-Ghoul et al.[5]

In this paper, the concept of soft fuzzy soft sets is introduced and studied. The concept of soft fuzzy soft topological spaces is discussed as in [2]. The notions of soft fuzzy soft homotopy, soft fuzzy soft path connected spaces, soft fuzzy soft homotopy equivalent spaces are introduced and several properties are established.

The concept of soft fuzzy soft manifold and its topological foldings is introduced and some interesting properties are studied.

2. Preliminaries

Definition 2.1 ([11])**.** A **fuzzy set** λ is a function from a non-empty set X to a unit interval $I = [0, 1]$. The family of fuzzy sets is denoted by I^X.

Definition 2.2 ([3])**.** A **fuzzy topology** is a family T of fuzzy sets in X which satisfies the following conditions:

(1) $\phi, X \in T$.
(2) If $A, B \in T$ then $A \cap B \in T$.
(3) If $A_i \in T$ for each $i \in I$, then $\cup_i A_i \in T$.

Then the pair (X, T) is called a **fuzzy topological space**.

Definition 2.3 ([8])**.** Let X be a non-empty set, E be the set of all parameters for X and $A \subseteq E$. A pair (F, A) is called **a soft set** over X if F is a mapping defined by $F : A \to 2^X$, where 2^X is the power set of X.

Definition 2.4 ([2])**.** Let λ be a fuzzy subset of X. A collection τ of fuzzy subsets of λ satisfying

(1) $0, \lambda \in \tau$.
(2) $\mu_i \in \tau\ \forall i \in J \Rightarrow \vee\{\mu_i : i \in J\} \in \tau$.
(3) $\mu, \lambda \in \tau \Rightarrow \mu \wedge \lambda \in \tau$.

is called a **fuzzy topology on** λ.

Definition 2.5 ([9])**.** Let X be a topological space with topology T. If Y is a subset of X, the collection

$$T_Y = \{Y \cap U : U \in T\}$$

is a topology on Y, called the **subspace topology**. Then the ordered pair (Y, T_Y) is called a subspace of X.

Definition 2.6 ([9])**.** Let A be the subset of X. A **characteristic function of** A, $\chi_A : X \to \{0, 1\}$ is defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7 ([7])**.** Let $f, g : X \to Y$ be any two continuous functions. We say that **f and g are homotopic**, and denote that by $f \simeq g$, if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for each $x \in X$. The function F is called a **homotopy** between f and g.

Definition 2.8 ([7])**.** Given two paths $f, g : I \to X$ beginning at x_0 and ending at x_1, it is said that f and g are **path homotopic** (notation $f \dot{\simeq} g$) if there is a continuous function $F : I \times I \to X$ such that

(1) $F(t, 0) = f(t)$ and $F(t1) = g(t)$ for all $t \in I$;
(2) $F(0, s) = x_0$ and $F(1, s) = x_1$ for all $t \in I$.

Definition 2.9 ([9]). **Euclidean space** $\mathbb{R}$, is the set of all real numbers together with the topology determined by the Euclidean metric, $d(x,y) = |x-y|$, for all $x, y \in \mathbb{R}$.

Definition 2.10 ([4]). Let (X,T) be a topological spaces. The collection $\tilde{T} = \{D : D \text{ is a fuzzy set in } X \text{ and } D_0 \in T\}$ is a fuzzy topology on X, called the **fuzzy topology on X introduced by T**. $(X,\tilde{T})$ is called the fuzzy topological space introduced by (X,T).

Definition 2.11 ([4]). Let (X,τ) and (Y,σ) be the fuzzy topological spaces and $(I,\widetilde{\varepsilon}_I)$ be the fuzzy topological space introduced by the Euclidean space (I,ε_I). Let $f,g : (X,\tau) \to (Y,\sigma)$ be two fuzzy continuous mappings. If there exists a fuzzy continuous mapping $F : (X,\tau) \times (I,\widetilde{\varepsilon}_I)$ such that $F(x_{\lambda,0}) = f(x_\lambda)$ and $F(x_{\lambda,1}) = g(x_\lambda)$ for every fuzzy point x_λ in (X,τ), then we say that f is **fuzzy homotopic** to g. The mapping F is called a **fuzzy homotopy** between f and g and write $f \simeq g$.

Definition 2.12 ([4]). A fuzzy subset (A,μ) of a fuzzy manifold (M,μ) is called **a fuzzy retraction** if there exist a fuzzy continuous function $\tilde{r} : (M,\mu) \to (A,\mu)$ such that $\tilde{r}(a,\mu(a)) = (a,\mu(a))$, for all $a \in A$. $\mu \in [0,1]$.

Definition 2.13 ([9]). A **manifold** is a Hausdorff space X with a countable basis such that each point x of X has a neighbourhood that is homeomorphic with an open subset of $\mathbb{R}^n$.

Definition 2.14 ([5]). Let (X,T) be a topological space. A map $f : (X,T) \to (X,T)$ is said to be **folding** of a topological space into itself if $f(X) \subset X$ and either for all $G \in T$, $f(G) = U \subset G$, $U \in T$ or for all $G \in T$, $f(G) = G$.

3. Soft fuzzy soft sets

Definition 3.1. Let X be a non-empty set and I be the unit interval. Let P be the set of all parameters and 2^P be the power set of P. A **soft fuzzy soft set** λ_P of X is a function from X to $I \times 2^P$ such that $\lambda_P(x) = (\lambda(x), A)$ where $\lambda : X \to I$ and $A \in 2^P$. The family of all soft fuzzy soft sets over X with parameter P is denoted by $\mathcal{SFS}(X,P)$.

Definition 3.2. Let U be the subset of X and P be the set of all parameters. A **soft fuzzy soft characteristic function of U**, $\chi_U : X \to \{(1,P),(0,\phi)\}$ is defined as

$$\chi_U(x) = \begin{cases} (1,P), & \text{if } x \in U, \\ (0,\phi), & \text{otherwise.} \end{cases}$$

Definition 3.3. Let X be a non-empty set and P be the set of all parameters. Let $x \in X$, $p \in P$ and $\delta_P : X \to I \times 2^P$. Define

$$x_{\delta_P}(y) = \begin{cases} (\delta(x),\{p\}) \ \ (0 < \delta \leq 1), & \text{if } x = y, \\ (0,\phi), & \text{otherwise.} \end{cases}$$

Then the soft fuzzy soft set x_{δ_P} is called the **soft fuzzy soft point** (*inshort*. $\mathcal{SFSP}$).

Definition 3.4. Let X be a non-empty set and P be the set of all parameters. The soft fuzzy soft set λ_P is called the **universal soft fuzzy soft set** if $\lambda_P(x) = (1, P)$, $\forall\, x \in X$ and it is denoted by $(1_X, P)^{\sim}$. The soft fuzzy soft set λ_P is called the **null soft fuzzy soft set** if $\lambda_P(x) = (0, \phi)$, $\forall\, x \in X$ and it is denoted by $(0_X, \phi)^{\sim}$.

Definition 3.5. Let X be a non-empty set and P be the set of all parameters. Let λ_P be the soft fuzzy soft set. Then the **complement** of λ_P, denoted by λ_P^c is defined by $\lambda_P^c(x) = (1 - \lambda(x), P/A)$, for all $x \in X$.

Definition 3.6. Let X be a non-empty set and P be the set of all parameters. Let λ_P and μ_P be any two soft fuzzy soft sets of X such that $\lambda_P(x) = (\lambda(x), A)$ and $\mu_P(x) = (\mu(x), B)\ \forall\, x \in X$. Then

(1) $\lambda_P(x) \sqsubseteq \mu_P(x) = (\lambda(x) \leq \mu(x), A \subseteq B)$.
(2) $\lambda_P(x) \sqsupseteq \mu_P(x) = (\lambda(x) \geq \mu(x), A \supseteq B)$.
(3) $\lambda_P(x) \sqcap \mu_P(x) = (\min\{\lambda(x), \mu(x)\}, A \cap B)$.
(4) $\lambda_P(x) \sqcup \mu_P(x) = (\max\{\lambda(x), \mu(x)\}, A \cup B)$.

Definition 3.7. Let X be a non-empty set and P be the set of all parameters. Let λ_P and μ_P be any two soft fuzzy soft sets of X. Then

(1) $\lambda_P \Subset \mu_P \Leftrightarrow \lambda_P(x) \sqsubseteq \mu_P(x), \forall\, x \in X$.
(2) $\lambda_P \Supset \mu_P \Leftrightarrow \lambda_P(x) \sqsupseteq \mu_P(x), \forall\, x \in X$.
(3) $\lambda_P = \mu_P \Leftrightarrow \lambda_P(x) = \mu_P(x), \forall\, x \in X$.
(4) $\lambda_P \Cup \mu_P \Leftrightarrow \lambda_P(x) \sqcup \mu_P(x), \forall\, x \in X$.
(5) $\lambda_P \Cap \mu_P \Leftrightarrow \lambda_P(x) \sqcap \mu_P(x), \forall\, x \in X$.

Definition 3.8. Let X and Y be any two non-empty sets. Let P and Q be the set of all parameters over X and Y respectively. Let $\mathcal{SFS}(X, P)$, $\mathcal{SFS}(Y, Q)$ be the family of all soft fuzzy soft sets of X, Y respectively and $\psi : P \to Q$ be the function. Let $A \subseteq P$ and $B \subseteq Q$. If $f : \mathcal{SFS}(X, P) \to \mathcal{SFS}(Y, Q)$ and λ_P is a soft fuzzy soft set of X with $\lambda_P(x) = (\lambda(x), A)$, then the **image** under f, $f(\lambda_P)$ is defined by $f(\lambda_P)(y) = (\vee_{x \in f^{-1}(y)} \lambda(x), \psi(N))$ for each $y \in Y$ and N is the second ordinate of $\sqcup_{x \in f^{-1}(y)} \lambda_P(x)$. If μ_Q is the soft fuzzy soft set of Y with $\mu_Q(y) = (\mu(y), B)$, then the **inverse image** under f, $f^{-1}(\mu_Q)$ is defined by $f^{-1}(\mu_Q)(x) = (\mu \circ f(x), \psi^{-1}(B))$ for each $x \in X$.

Definition 3.9. Let X be a non-empty set and P be the set of all parameters. Let λ_P be any soft fuzzy soft set of X and J be an indexed set. A **soft fuzzy soft topology** on X is a family τ_{λ_P} of soft fuzzy soft subsets of λ_P satisfying the following axioms :

(1) $(0_X, \phi)^{\sim}, \lambda_P \in \tau_{\lambda_P}$.
(2) For any family of soft fuzzy soft sets $\{(\lambda_P)_j\}_{j \in J} \in \tau_{\lambda_P} \Rightarrow \Cup_{j \in J} (\lambda_P)_j \in \tau_{\lambda_P}$.
(3) For any finite number of soft fuzzy soft sets $\{(\lambda_P)_j\}_{j=1}^{n} \in \tau_{\lambda_P} \Rightarrow \Cap_{j=1}^{n} (\lambda_P)_j \in \tau_{\lambda_P}$.

Then the pair (X, τ_{λ_P}) is called a **soft fuzzy soft topological space**. The members of τ_{λ_P} is said to be a **soft fuzzy soft open set**. The complement of a soft fuzzy soft open set is a **soft fuzzy soft closed set**.

Definition 3.10. Let (X, τ_{λ_P}) and (Y, σ_{μ_Q}) be any two soft fuzzy soft topological spaces. A function f is said to be a **soft fuzzy soft continuous function** if the inverse image of every soft fuzzy soft open set is soft fuzzy soft open.

4. On soft fuzzy soft contractible spaces

Definition 4.1. Let (X, T) be a topological space and Q be the set of all parameters over X. Let U be the subset of X and χ_U be the soft fuzzy soft characteristic function of U. Then the soft fuzzy soft topology introduced by T is $\omega(T)_Q = \{\chi_U : U \in T\}$ and the pair $(X, \omega(T)_Q)$ is said to be a **soft fuzzy soft topological space introduced by** (X, T).

Note 4.2. Let I be the unit interval and Q be the set of all parameters over I. Let ξ be an Euclidean topology on I and $(I, \omega(\xi)_Q)$ be a soft fuzzy soft topological space introduced by the Euclidean space (I, ξ).

Definition 4.3. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. If Y is the subset of X and χ_Y is the characteristic function of Y, then the collection $\tau_{\lambda_P|Y} = \{\mu_P|Y = \mu_P \Cap \chi_Y : \mu_P \in \tau_{\lambda_P}\}$ is a soft fuzzy soft topology on Y, called the **soft fuzzy soft subspace topology** and the pair $(Y, \tau_{\lambda_P|Y})$ is called a **soft fuzzy soft topological subspace** of (X, τ_{λ_P}).

Definition 4.4. Let (X, τ_{λ_P}) and (Y, σ_{μ_R}) be any two soft fuzzy soft topological spaces and $(I, \omega(\xi)_Q)$ be a soft fuzzy soft topology introduced by the Euclidean space (I, ξ). Let $f, g : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_R})$ be any two soft fuzzy soft continuous functions. If there exists a soft fuzzy soft continuous function $H : (X, \tau_{\lambda_P}) \times (I, \omega(\xi)_Q) \to (Y, \sigma_{\mu_R})$ such that $H(x_{\delta_P}, 0) = f(x_{\delta_P})$ and $H(x_{\delta_P}, 1) = g(x_{\delta_P})$, for each soft fuzzy soft point x_{δ_P} of X, then f is said to be a **soft fuzzy soft homotopic** to g. Moreover, the function H is said to be a **soft fuzzy soft homotopy** between f and g, denoted as $f \simeq g$.

Proposition 4.5. *Let (X, τ_{λ_P}) and (Y, σ_{μ_R}) be any two soft fuzzy soft topological spaces. Let U and V be the subsets of X. Let $\lambda_P = \lambda_P|U \Cup \lambda_P|V$, where $\lambda_P|U$ and $\lambda_P|V$ are the soft fuzzy soft open sets in (X, τ_{λ_P}). Let $f : (U, \tau_{\lambda_P|U}) \to (Y, \sigma_{\mu_R})$ and $h : (V, \tau_{\lambda_P|V}) \to (Y, \sigma_{\mu_R})$ be any two soft fuzzy soft continuous functions. If $f|(U \cap V) = h|(U \cap V)$, then $g : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_R})$ is defined by*

$$g(x) = \begin{cases} f(x), & x \in U \\ h(x), & x \in V \end{cases}$$

is a soft fuzzy soft continuous function.

Proof. Let μ_R be a soft fuzzy soft open set in (Y, σ_{μ_R}). Let $\lambda_P|U$ and $\lambda_P|V$ be the soft fuzzy soft open sets in (X, τ_{λ_P}). Now

$$\begin{aligned} g^{-1}(\mu_R) &= g^{-1}(\mu_R) \Cap \lambda_P \\ &= g^{-1}(\mu_R) \Cap (\lambda_P|U \Cup \lambda_P|V) \\ &= (g^{-1}(\mu_R) \Cap \lambda_P|U) \Cup (g^{-1}(\mu_R) \Cap \lambda_P|V) \\ &= f^{-1}(\mu_R) \Cup h^{-1}(\mu_R) \end{aligned}$$

Since f and h are the soft fuzzy soft continuous functions, $f^{-1}(\mu_R)$ and $h^{-1}(\mu_R)$ are the soft fuzzy soft open sets in $(U, \tau_{\lambda_{P|U}})$ and $(V, \tau_{\lambda_{P|V}})$ respectively. Thus $g^{-1}(\mu_R)$ is a soft fuzzy soft open set. Hence g is a soft fuzzy soft continuous function. □

Proposition 4.6. *The relation $\simeq$ is an equivalence relation.*

Proof. Proof is clear. □

Note 4.7. The equivalence class of "f" under the equivalence relation " $\simeq$ ", is denoted by $[f]$.

Proposition 4.8. *Let (X, τ_{λ_P}), (Y, σ_{μ_Q}) and (Z, ρ_{γ_R}) be the soft fuzzy soft topological spaces. Suppose that f_1 and f_2 are the soft fuzzy soft homotopic functions from (X, τ_{λ_P}) to (Y, σ_{μ_Q}) and that g_1 and g_2 are the soft fuzzy soft homotopic functions from (Y, σ_{μ_Q}) to (Z, ρ_{γ_R}). Then $g_1 \circ f_1 \simeq g_2 \circ f_2$.*

Proof. Proof is clear from the following steps : **(i)** $g_1 \circ f_1 \simeq g_1 \circ f_2$; **(ii)** $g_1 \circ f_2 \simeq g_2 \circ f_2$; **(iii)** transitivity of **(i)** and **(ii)**. □

Proposition 4.9. *Let (X, τ_{λ_P}), (Y, σ_{μ_Q}) and (Z, ρ_{γ_R}) be the soft fuzzy soft topological spaces. Let $f, g : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ be the soft fuzzy soft continuous functions such that $f \simeq g$. If $h : (Y, \sigma_{\mu_Q}) \to (Z, \rho_{\gamma_R})$ is a soft fuzzy soft continuous function, then $h \circ f$, $h \circ g : (X, \tau_{\lambda_P}) \to (Z, \rho_{\gamma_R})$ are soft fuzzy soft continuous and $h \circ f \simeq h \circ g$.*

Proof. Let $(I, \omega(\xi)_Q)$ be a soft fuzzy soft topological space introduced by the Euclidean space (I, ξ). Since h, f, g are the soft fuzzy soft continuous functions, $h \circ f$, $h \circ g$ are the soft fuzzy soft continuous functions. Furthermore, $f \simeq g$ implies that there is a soft fuzzy soft continuous function $G : (X, \tau_{\lambda_P}) \times (I, \omega(\xi)_Q) \to (Y, \sigma_{\mu_Q})$ such that $G(x_{\delta_P}, 0) = f(x_{\delta_P})$, $G(x_{\delta_P}, 1) = g(x_{\delta_P})$, for each soft fuzzy soft point x_{δ_P} of X. Now, $H : (X, \tau_{\lambda_P}) \times (I, \omega(\xi)_Q) \to (Z, \rho_{\gamma_R})$ is given by $H(x_{\delta_P}, t) = h(G(x_{\delta_P}, t))$. Since h, g are the soft fuzzy soft continuous functions, $H = h \circ g$ is a soft fuzzy soft continuous function. Moreover H satisfies the following conditions :

$$H(x_{\delta_P}, 0) = h(G(x_{\delta_P}, 0)) = h(f(x_{\delta_P})) = (h \circ f)(x_{\delta_P})$$
$$H(x_{\delta_P}, 1) = h(G(x_{\delta_P}, 1)) = h(g(x_{\delta_P})) = (h \circ g)(x_{\delta_P})$$

Hence, $h \circ f \simeq h \circ g$. □

Definition 4.10. Two soft fuzzy soft topological spaces (X, τ_{λ_P}) and (Y, σ_{μ_Q}) are said to be a **soft fuzzy soft homotopic equivalent spaces**, if there exist soft fuzzy soft continuous functions $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ and $g : (Y, \sigma_{\mu_Q}) \to (X, \tau_{\lambda_P})$ such that $f \circ g \simeq \mathfrak{id}_Y$ and $g \circ f \simeq \mathfrak{id}_X$, where $\mathfrak{id}_X$ and $\mathfrak{id}_Y$ are the identity functions of X and Y respectively.

Definition 4.11. Let (X, τ_{λ_P}) and (Y, σ_{μ_R}) be any two soft fuzzy soft topological spaces. If the bijective function $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ and its inverse function are soft fuzzy soft continuous functions, then the function f is said to be a **soft fuzzy soft homeomorphism**. Moreover (X, τ_{λ_P}) and (Y, σ_{μ_Q}) are said to be soft fuzzy soft homeomorphic spaces.

Proposition 4.12. *Every soft fuzzy soft homeomorphic spaces are the soft fuzzy soft homotopic equivalent spaces.*

Proof. Proof is obvious. □

Definition 4.13. A soft fuzzy soft topological space (X, τ_{λ_P}) is said to be a **soft fuzzy soft contractible space**, if the identity function $\mathfrak{id}_X : (X, \tau_{\lambda_P}) \to (X, \tau_{\lambda_P})$ is soft fuzzy soft homotopic to a constant function.

Proposition 4.14. *A soft fuzzy soft topological space (X, τ_{λ_P}) is a soft fuzzy soft contractible space if and only if (X, τ_{λ_P}) is a soft fuzzy soft homotopic equivalent to a soft fuzzy soft single-point space.*

Proof. Suppose that the soft fuzzy soft topological space (X, τ_{λ_P}) is a soft fuzzy soft contractible space. Then there exists a constant function $\mathfrak{c} : \mathcal{SFS}(X, P) \to \mathcal{SFS}(X, P)$ defined by $\mathfrak{c}(x_{\delta_P}) = x'_{\delta_P}$ for each soft fuzzy soft point x_{δ_P} of X such that $\mathfrak{id}_X \simeq \mathfrak{c}$. Let $(Y, \sigma_{\mu_Q}) = \{x'_{\alpha_P}\}$ be a soft fuzzy soft single-point space with parameter $Q \subseteq P$ and $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ be a soft fuzzy soft continuous function. Then an inclusion function $\iota : (Y, \sigma_{\mu_Q}) \to (X, \tau_{\lambda_P})$ defined by $\iota(x'_{\alpha_P}) = x'_{\delta_P}$ is a soft fuzzy soft continuous function and $\iota \circ f = \mathfrak{c}$, $f \circ \iota = \mathfrak{id}_Y$. Since $\mathfrak{id}_X \simeq \mathfrak{c}$ and $\simeq$ is an equivalence relation, $\mathfrak{id}_X \simeq \iota \circ f$ and $f \circ \iota \simeq \mathfrak{id}_Y$. Hence (X, τ_{λ_P}) is a soft fuzzy soft homotopic equivalent to a soft fuzzy soft single-point space (Y, σ_{μ_Q}).

Conversely, suppose that $(Y, \sigma_{\mu_Q}) = \{x'_{\alpha_P}\}$ is a soft fuzzy soft single-point space with parameter $Q \subseteq P$ and (X, τ_{λ_P}) is a soft fuzzy soft homotopic equivalent to a soft fuzzy soft single-point space (Y, σ_{μ_Q}). Then there exist soft fuzzy soft continuous functions $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ and $g : (Y, \sigma_{\mu_Q}) \to (X, \tau_{\lambda_P})$ such that $f \circ g \simeq \mathfrak{id}_Y$ and $g \circ f \simeq \mathfrak{id}_X$, where $\mathfrak{id}_X$ and $\mathfrak{id}_Y$ are the identity functions of X and Y respectively. Since (Y, σ_{μ_Q}) is a soft fuzzy soft single-point space, $f(x_{\delta_P}) = x'_{\alpha_P}$, for each soft fuzzy soft point x_{δ_P} of X. Let $g(x'_{\alpha_P}) = x''_{\beta_P}$. Now, it is clear that $g \circ f = \mathfrak{c}$ and $\mathfrak{c} \simeq \mathfrak{id}_X$, where $\mathfrak{c}$ is a constant function. Hence (X, τ_{λ_P}) is a soft fuzzy soft contractible space. □

Proposition 4.15. *The soft fuzzy soft two-point space is not a soft fuzzy soft contractible space.*

Proof. Proof is clear. □

5. On soft fuzzy soft path connected spaces

Definition 5.1. Let $(I, \omega(\xi)_Q)$ be a soft fuzzy soft topological space introduced by the Euclidean space (I, ξ) and (X, τ_{λ_P}) be a soft fuzzy soft topological space. Let x_{δ_P} and x'_{γ_P} be any two soft fuzzy soft points of X. A **soft fuzzy soft path** η in (X, τ_{λ_P}) from x_{δ_P} to x'_{γ_P} is a soft fuzzy soft continuous function $\eta : (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ such that $\eta(0) = x_{\delta_P}$ and $\eta(1) = x'_{\gamma_P}$. Then the soft fuzzy soft points $\eta(0)$ and $\eta(1)$ are called the **origin** and **end points** of η.

Definition 5.2. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. Let x_{δ_P} and x'_{γ_P} be any two soft fuzzy soft points of X. A soft fuzzy soft topological space (X, τ_{λ_P}) is said to be a **soft fuzzy soft path connected space** if there exists a soft fuzzy soft path in (X, τ_{λ_P}) with origin x_{δ_P} and end point x'_{γ_P}.

Definition 5.3. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space and $U \subseteq X$. A soft fuzzy soft set $\lambda_P|U$ is called a **soft fuzzy soft retract** of X if there exists a soft fuzzy soft continuous function $\tilde{r} : (X, \tau_{\lambda_P}) \to (U, \tau_{\lambda_P}|U)$ such that $\tilde{r}(x'_{\delta_P}) = x'_{\delta_P}$ for each soft fuzzy soft point x'_{δ_P} of U. Moreover the function $\tilde{r}$ is said to be a **soft fuzzy soft retraction**.

Definition 5.4. Let $(I, \omega(\xi)_Q)$ be a soft fuzzy soft topological space introduced by the Euclidean space (I, ξ). Let (X, τ_{λ_P}) be a soft fuzzy soft topological space and $U \subseteq X$. A soft fuzzy soft set $\lambda_P|U$ is called a **soft fuzzy soft deformation retract** of X if there exists a soft fuzzy soft retraction $\tilde{r} : (X, \tau_{\lambda_P}) \to (U, \tau_{\lambda_P|U})$ and a soft fuzzy soft homotopy $G : (X, \tau_{\lambda_P}) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ such that $G(x_{\delta_P}, 0) = x_{\delta_P}$, $G(x_{\delta_P}, 1) = \tilde{r}(x_{\delta_P})$, for each soft fuzzy soft point x_{δ_P} of X and $G(u_{\delta_P}, t) = u_{\delta_P}$, for each soft fuzzy soft point u_{δ_P} of U, $t \in I$. Furthermore, the function G is said to be a **soft fuzzy soft deformation retraction**.

Definition 5.5. Let (X, τ_{λ_P}) be the soft fuzzy soft topological space. Let η and θ be the soft fuzzy soft paths in (X, τ_{λ_P}) from x_{δ_P} to x'_{μ_P} and x'_{μ_P} to x''_{γ_P}. The **product of** η **and** θ is the soft fuzzy soft path $\eta\theta$ in (X, τ_{λ_P}) from x_{δ_P} to x''_{γ_P} is defined by

$$\eta\theta(t) = \begin{cases} \eta((2t)_{\alpha_P}), & 0 \le t \le \frac{1}{2} \\ \theta((2t-1)_{\alpha_P}), & \frac{1}{2} \le t \le 1 \end{cases}$$

for all $t \in I$.

Definition 5.6. Let η be the soft fuzzy soft path in (X, τ_{λ_P}) from x_{δ_P} to x'_{μ_P}. The **inverse of** η is the soft fuzzy soft path in (X, τ_{λ_P}) from x'_{μ_P} to x_{δ_P} defined by $\eta^{-1}(t) = \eta(1-t)$ for all $t \in I$.

Definition 5.7. Let $(I, \omega(\xi)_Q)$ and $(I, \omega(\zeta)_R)$ be any two soft fuzzy soft topological spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. Two soft fuzzy soft paths η and θ in (X, τ_{λ_P}) from x_{δ_P} to x'_{μ_P} are said to be a **soft fuzzy soft path homotopy** (in short, $\eta \cong \theta$) if there exists a soft fuzzy soft continuous function $G : (I, \omega(\zeta)_R) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ such that

$$\begin{array}{lclcl} G(0, t_{\alpha_Q}) = x_{\delta_P} & \text{and} & G(1, t_{\alpha_Q}) = x'_{\mu_P} & & \text{for all } t_{\alpha_Q} \in \mathcal{SFS}(I, Q); \\ G(t_{\beta_R}, 0) = \eta(t_{\beta_R}) & \text{and} & G(t_{\beta_R}, 1) = \theta(t_{\beta_R}) & & \text{for all } t_{\beta_R} \in \mathcal{SFS}(I, R). \end{array}$$

Proposition 5.8. *Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. Let $\eta_0, \eta_1, \theta_0, \theta_1$ be the soft fuzzy soft paths in (X, τ_{λ_P}) respectively. If $\eta_0 \cong \eta_1$, $\theta_0 \cong \theta_1$ and $\eta_0\theta_0$ is defined, then $\eta_1\theta_1$ is defined and $\eta_1\theta_1 \cong \eta_0\theta_0$.*

Proof. Let $(I, \omega(\xi)_Q)$ and $(I, \omega(\zeta)_R)$ be the soft fuzzy soft topological spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. It is clear that $\eta_1\theta_1$ is well-defined. Let $G, H : (I, \omega(\zeta)_R) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ be the soft fuzzy soft path homotopies from η_0 to η_1 and from θ_0 to θ_1 respectively. Then there exists a soft fuzzy soft path homotopy $E : (I, \omega(\zeta)_R) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ from $\eta_0\theta_0$ to $\eta_1\theta_1$ is given by

$$E(t, t'_{\delta_Q}) = \begin{cases} G(2t, t'_{\delta_Q}), & 0 \le t \le \frac{1}{2} \\ H(2t-1, t'_{\delta_Q}), & \frac{1}{2} \le t \le 1 \end{cases}$$

for each $t'_{\delta_Q} \in \mathcal{SFS}(I,Q)$, $t \in I$. By Proposition 4.5., E is a soft fuzzy soft continuous function. Hence, $\eta_1\theta_1 \cong \eta_0\theta_0$. □

Proposition 5.9. *Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. Let η_0 and η_1 be any two soft fuzzy soft paths in (X, τ_{λ_P}). If $\eta_0 \cong \eta_1$, then $\eta_0^{-1} \cong \eta_1^{-1}$.*

Proof. Let $(I, \omega(\xi)_Q)$ and $(I, \omega(\zeta)_R)$ be the soft fuzzy soft topological spaces introduced by the Euclidean spaces (I, ξ) and (I, ζ) respectively. Let $G : (I, \omega(\zeta)_R) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ be a soft fuzzy soft path homotopy from η_0 to η_1. A soft fuzzy soft path homotopy $H : (I, \omega(\zeta)_R) \times (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ from η_0^{-1} to η_1^{-1} is given by $H(t, t'_{\delta_Q}) = G(1-t, t'_{\delta_Q})$ for each $t'_{\delta_Q} \in \mathcal{SFS}(I,Q)$. Hence, $\eta_0^{-1} \cong \eta_1^{-1}$. □

Notation 5.10. $\langle\langle\eta\rangle\rangle$ denotes the set of all soft fuzzy soft paths homotopic to η; that is, equivalence class of η.

Definition 5.11. The **product** and **inverse** of the equivalence classes of the soft fuzzy soft paths η and θ are defined by $\langle\langle\eta\rangle\rangle \circ \langle\langle\theta\rangle\rangle = \langle\langle\eta\theta\rangle\rangle$ and $\langle\langle\eta\rangle\rangle^{-1} = \langle\langle\eta^{-1}\rangle\rangle$.

Definition 5.12. Let $\mathfrak{e} : (I, \omega(\xi)_Q) \to (X, \tau_{\lambda_P})$ be a soft fuzzy soft path in (X, τ_{λ_P}), defined by $\mathfrak{e}(t_{\gamma_Q}) = x_{\delta_P}$ for each soft fuzzy soft points t_{γ_Q} and x_{δ_P} of I and X respectively. The set of all soft fuzzy soft homotopic equivalence classes of paths with origin and end x_{δ_P} is called a **soft fuzzy soft fundamental group of X at** x_{δ_P} if the following conditions are satisfied.

(i) If $\langle\langle\eta\rangle\rangle$ has origin and end x_{δ_P}, then $\langle\langle\mathfrak{e}\rangle\rangle \circ \langle\langle\eta\rangle\rangle = \langle\langle\eta\rangle\rangle \circ \langle\langle\mathfrak{e}\rangle\rangle = \langle\langle\eta\rangle\rangle$.
(ii) If $\langle\langle\eta\rangle\rangle$ has origin and end x_{δ_P}, then $\langle\langle\eta\rangle\rangle \circ \langle\langle\eta^{-1}\rangle\rangle = \langle\langle\eta^{-1}\rangle\rangle \circ \langle\langle\eta\rangle\rangle = \langle\langle\mathfrak{e}\rangle\rangle$.
(iii) If $(\eta\theta)\kappa$ is defined, then $(\langle\langle\eta\rangle\rangle \circ \langle\langle\theta\rangle\rangle) \circ \langle\langle\kappa\rangle\rangle = \langle\langle\eta\rangle\rangle \circ (\langle\langle\theta\rangle\rangle \circ \langle\langle\kappa\rangle\rangle)$.

Notation 5.13. Soft fuzzy soft fundamental group of X at x_{δ_P} is denoted by $\pi_1(X, x_{\delta_P})$.

Definition 5.14. Let $\pi_1(X, x_{\delta_P})$ and $\pi_1(Y, y_{\mu_Q})$ be any two soft fuzzy soft fundamental groups. A function $f : \pi_1(X, x_{\delta_P}) \to \pi_1(Y, y_{\mu_Q})$ is said to be a **soft fuzzy soft homomorphism** if $f(\langle\langle\theta\rangle\rangle \circ \langle\langle\eta\rangle\rangle) = f(\langle\langle\theta\rangle\rangle) \circ f(\langle\langle\eta\rangle\rangle)$ for all $\langle\langle\theta\rangle\rangle, \langle\langle\eta\rangle\rangle \in \pi_1(X, x_{\delta_P})$. Moreover the soft fuzzy soft homomorphism is said to be a **soft fuzzy soft isomorphism** if it is bijective.

Proposition 5.15. *Let (X, τ_{λ_P}) be a soft fuzzy soft path connected space. Let x_{δ_P}, x'_{μ_P} be any two soft fuzzy soft points of X. Then there exists a soft fuzzy soft group isomorphism of $\pi_1(X, x_{\delta_P})$ onto $\pi_1(X, x'_{\mu_P})$.*

Proof. Proof is obvious. □

Note 5.16. If $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ is a soft fuzzy soft continuous function and if η, θ are the soft fuzzy soft paths in (X, τ_{λ_P}) with $\eta(1) = \theta(0)$, then $f(\eta\theta) = (f\eta).(f\theta)$.

Definition 5.17. Let (X, τ_{λ_P}) and (Y, σ_{μ_Q}) be any two soft fuzzy soft path connected spaces and $f : (X, \tau_{\lambda_P}) \to (Y, \sigma_{\mu_Q})$ be a soft fuzzy soft continuous function. Then the function, f_*: $\pi_1(X, x'_{\mu_P}) \to \pi_1(Y, f(x'_{\mu_P}))$ is called the **soft fuzzy soft homomorphism induced by f** if $f_*(\langle\langle\eta\rangle\rangle) = \langle\langle f\eta\rangle\rangle$, for all $\langle\langle\eta\rangle\rangle \in \pi_1(X, x'_{\mu_P})$.

Proposition 5.18. *Let (X,τ_{λ_P}), (Y,σ_{μ_Q}) and (Z,ρ_{γ_R}) be any three soft fuzzy soft path connected spaces. Let $f:(X,\tau_{\lambda_P})\to(Y,\sigma_{\mu_Q})$ be a soft fuzzy soft continuous function and x'_{μ_P} be a soft fuzzy soft point of X. Then f induces a soft fuzzy soft homomorphism f_*: $\pi_1(X,x'_{\mu_P})\to\pi_1(Y,f(x'_{\mu_P}))$.*

Proof. For each $\langle\langle\theta\rangle\rangle$, $\langle\langle\eta\rangle\rangle\in\pi_1(X,x'_{\mu_P})$,

$$\begin{aligned}
f_*(\langle\langle\theta\rangle\rangle\circ\langle\langle\eta\rangle\rangle) &= f_*(\langle\langle\theta\eta\rangle\rangle)\\
&= \langle\langle f(\theta\eta)\rangle\rangle\\
&= \langle\langle (f\theta).(f\eta)\rangle\rangle\\
&= \langle\langle (f\theta)\rangle\rangle\circ\langle\langle (f\eta)\rangle\rangle\\
&= f_*(\langle\langle\theta\rangle\rangle)\circ f_*(\langle\langle\eta\rangle\rangle)
\end{aligned}$$

Thus f_* is a soft fuzzy soft homomorphism. □

Proposition 5.19. *Let (X,τ_{λ_P}), (Y,σ_{μ_Q}), and (Z,ρ_{γ_R}) be any three soft fuzzy soft path connected spaces. Let x'_{δ_P} be a soft fuzzy soft point of X. If $g:(X,\tau_{\lambda_P})\to(Y,\sigma_{\mu_Q})$ and $f:(Y,\sigma_{\mu_Q})\to(Z,\rho_{\gamma_R})$ are the soft fuzzy soft continuous functions, then $(f\circ g)_*=f_*\circ g_*$.*

Proof. For each $\langle\langle\eta\rangle\rangle\in\pi_1(X,x'_{\delta_P})$,

$$\begin{aligned}
(f\circ g)_*(\langle\langle\eta\rangle\rangle) &= \langle\langle (f\circ g)\eta\rangle\rangle\\
&= \langle\langle f(g(\eta))\rangle\rangle\\
&= f_*(\langle\langle g(\eta)\rangle\rangle)\\
&= f_*(g_*(\langle\langle\eta\rangle\rangle))\\
&= (f_*\circ g_*)(\langle\langle\eta\rangle\rangle)
\end{aligned}$$

Hence, $(f\circ g)_*=f_*\circ g_*$. □

Proposition 5.20. *Let h be a soft fuzzy soft homeomorphism between the soft fuzzy soft path connected spaces (X,τ_{λ_P}) and (Y,σ_{μ_Q}). Then $h_*:\pi_1(X,x'_{\mu_P})\to\pi_1(Y,h(x'_{\mu_P}))$ is a soft fuzzy soft isomorphism.*

Proof. Let (X,τ_{λ_P}) and (Y,σ_{μ_Q}) be any two soft fuzzy soft topological spaces. Let $\mathfrak{id}_X:(X,\tau_{\lambda_P})\to(X,\tau_{\lambda_P})$ and $\mathfrak{id}_{\pi_1(X,x'_{\mu_P})}:\pi_1(X,x'_{\mu_P})\to\pi_1(X,x'_{\mu_P})$ be any two identity functions. For each $\langle\langle\eta\rangle\rangle\in\pi_1(X,x'_{\mu_P})$, $(\mathfrak{id}_X)_*(\langle\langle\eta\rangle\rangle)=\langle\langle\mathfrak{id}_X\eta\rangle\rangle=\langle\langle\eta\rangle\rangle=\mathfrak{id}_{\pi_1(X,x'_{\mu_P})}(\langle\langle\eta\rangle\rangle)$. Thus, $(\mathfrak{id}_X)_*=\mathfrak{id}_{\pi_1(X,x'_{\mu_P})}$. Since there is a soft fuzzy soft homeomorphism h from (X,τ_{λ_P}) to (Y,σ_{μ_Q}), h is a bijective function. Now, $(h^{-1})_*\circ h_*=(h^{-1}\circ h)_*=(\mathfrak{id}_X)_*=\mathfrak{id}_{\pi_1(X,x'_{\mu_P})}$ and similarly, $h_*\circ(h^{-1})_*=\mathfrak{id}_{\pi_1(Y,h(x'_{\mu_P}))}$. Since $(h_*)^{-1}=(h^{-1})_*$, we have h_* is bijective. Therefore by Proposition 5.18., h induces a soft fuzzy soft homomorphism h_* with an inverse $(h_*)^{-1}$. Hence h_* is a soft fuzzy soft isomorphism from $\pi_1(X,x'_{\mu_P})$ to $\pi_1(Y,h(x'_{\mu_P}))$. □

Proposition 5.21. *Let $U\subseteq X$ and x_{δ_P} is a soft fuzzy soft point of U. If $\lambda_P|U$ is the soft fuzzy soft deformation retract of X, then $\pi_1(U,x_{\delta_P})$ is soft fuzzy soft isomorphic to $\pi_1(X,x_{\delta_P})$.*

Proof. Proof is simple. □

6. Soft fuzzy soft manifold and its topological foldings

Definition 6.1. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space and J be an indexed set. A collection $\mathcal{B}$ of soft fuzzy soft open sets of τ_{λ_P} is said to be a **soft fuzzy soft basis** if for every soft fuzzy soft open set μ_P in τ_{λ_P}, there exists a subcollection $\{(\gamma_P)_j\}_{j\in J}$ of $\mathcal{B}$ such that $\mu_P = \Cup_{j\in J}\{(\gamma_P)_j\}$. If the collection $\mathcal{B}$ containing countable number of soft fuzzy soft open sets, then (X, τ_{λ_P}) is said to be a **soft fuzzy soft second countable base**.

Definition 6.2. A soft fuzzy soft topological space (X, τ_{λ_P}) is said to be a **soft fuzzy soft Hausdorff space** if for each pair of distinct soft fuzzy soft points x_{δ_P}, x'_{μ_P} of X, there exist soft fuzzy soft open sets $\alpha_P \Supset x_{\delta_P}$, $\beta_P \Supset x'_{\mu_P}$ such that $\alpha_P \Cap \beta_P = (0, \phi)^{\sim}$.

Definition 6.3. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space and $(\mathbb{R}^n, \omega(\xi)_Q)$ be a soft fuzzy soft topological space introduced by the Euclidean space $(\mathbb{R}, \xi)$. A **soft fuzzy soft manifold** having dimension n is a soft fuzzy soft topological space (X, τ_{λ_P}) having the following properties:

(1) (X, τ_{λ_P}) is a soft fuzzy soft second countable space.
(2) (X, τ_{λ_P}) is a soft fuzzy soft Hausdorff space.
(3) The function $f : (X, \tau_{\lambda_P}) \to (\mathbb{R}^n, \omega(\xi)_Q)$ and the inverse function f^{-1} are the soft fuzzy soft continuous functions.
(4) The function f is bijective, defined on the set $\{x \in X : \gamma_P(x) \sqsupset (0, \phi)\}$, which maps soft fuzzy soft open set γ_P in (X, τ_{λ_P}) onto a soft fuzzy soft open set $f(\gamma_P)$ in $(\mathbb{R}^n, \omega(\xi)_Q)$.

Definition 6.4. Let (X, τ_{λ_P}) be a soft fuzzy soft topological space. A function $\psi : (X, \tau_{\lambda_P}) \to (X, \tau_{\lambda_P})$ is said to be a **soft fuzzy soft topological folding** if $\psi(X) \subset X$ and $\forall \mu_P \in \tau_{\lambda_P}$, either $\psi(\mu_P) \in \tau_{\lambda_P}$ or $\psi(\mu_P) = \mu_P$.

Proposition 6.5. *Let τ_{λ_P} and σ_{μ_Q} be the soft fuzzy soft topologies on the soft fuzzy soft manifolds $\mathbb{M}$ and $\mathbb{N}$ respectively. Let $\mathbb{A} \subseteq \mathbb{M}$, $\mathbb{B} \subseteq \mathbb{N}$ and $\lambda_P|\mathbb{A}$, $\mu_Q|\mathbb{B}$ be the soft fuzzy soft deformation retract of $\mathbb{M}$, $\mathbb{N}$ respectively. Then $(\mathbb{M}, \tau_{\lambda_P})$ and $(\mathbb{N}, \sigma_{\mu_Q})$ are the soft fuzzy soft homotopic equivalent spaces if and only if $(\mathbb{A}, \tau_{\lambda_P|\mathbb{A}})$ and $(\mathbb{B}, \sigma_{\mu_Q|\mathbb{B}})$ are the soft fuzzy soft homotopy equivalent spaces.*

Proof. Let $(\mathbb{M}, \tau_{\lambda_P})$ and $(\mathbb{N}, \sigma_{\mu_Q})$ be any two soft fuzzy soft topological spaces. Let $(I, \omega(\xi)_R)$ be the soft fuzzy soft topological space introduced by the Euclidean space (I, ξ). Since $(\mathbb{M}, \tau_{\lambda_P})$ and $(\mathbb{N}, \sigma_{\mu_Q})$ are the soft fuzzy soft homotopic equivalent spaces, there exist soft fuzzy soft continuous functions $f : (\mathbb{M}, \tau_{\lambda_P}) \to (\mathbb{N}, \sigma_{\mu_Q})$ and $g : (\mathbb{N}, \sigma_{\mu_Q}) \to (\mathbb{M}, \tau_{\lambda_P})$ such that $f \circ g \simeq \mathfrak{id}_{\mathbb{N}}$ and $g \circ f \simeq \mathfrak{id}_{\mathbb{M}}$, where $\mathfrak{id}_{\mathbb{M}}$, $\mathfrak{id}_{\mathbb{N}}$ are the identity functions on $\mathbb{M}, \mathbb{N}$ respectively. Since $\lambda_P|\mathbb{A}$ is a soft fuzzy soft deformation retract of $\mathbb{M}$, there exist a soft fuzzy soft retraction $\tilde{r_1} : (\mathbb{M}, \tau_{\lambda_P}) \to (\mathbb{A}, \tau_{\lambda_P|\mathbb{A}})$ and a soft fuzzy soft homotopy $G : (\mathbb{M}, \tau_{\lambda_P}) \times (I, \omega(\xi)_R) \to (\mathbb{M}, \tau_{\lambda_P})$ such that $G(x_{\delta_P}, 0) = x_{\delta_P}$, $G(x_{\delta_P}, 1) = \tilde{r}_1(x_{\delta_P})$, for each soft fuzzy soft point x_{δ_P} of $\mathbb{M}$ and $G(a_{\delta_P}, t) = a_{\delta_P}$, for each soft fuzzy soft point a_{δ_P} of $\mathbb{A}$, $t \in I$. Since $\mu_Q|\mathbb{B}$ is a soft fuzzy soft deformation retract of $\mathbb{N}$, there exist soft fuzzy soft retraction $\tilde{r}_2 : (\mathbb{N}, \sigma_{\mu_Q}) \to (\mathbb{B}, \sigma_{\mu_Q|\mathbb{B}})$ and soft fuzzy soft homotopy $H : (\mathbb{N}, \sigma_{\mu_Q}) \times (I, \omega(\xi)_R) \to (\mathbb{N}, \sigma_{\mu_Q})$ such that $H(y_{\gamma_Q}, 0) = y_{\gamma_Q}$, $H(Y_{\gamma_Q}, 1) = \tilde{r}_2(y_{\gamma_Q})$, for each soft fuzzy soft point y_{γ_Q} of $\mathbb{N}$ and $H(b_{\gamma_Q}, t) = b_{\gamma_Q}$, for each soft fuzzy soft point b_{γ_Q} of $\mathbb{B}$, $t \in I$. Now if $f' = f|\mathbb{A}$ and $g' = g|\mathbb{B}$, then for each soft fuzzy soft point b_{γ_Q} of $\mathbb{B}$, $((\tilde{r_2}f') \circ (\tilde{r_1}g'))(b_{\gamma_Q}) = (\tilde{r_2}f')(a_{\delta_P}) = b_{\gamma_Q} = \mathfrak{id}_{\mathbb{B}}(b_{\gamma_Q})$ and for each soft fuzzy soft point a_{δ_P} of $\mathbb{A}$, $((\tilde{r_1}g') \circ (\tilde{r_2}f'))(a_{\delta_P}) = \mathfrak{id}_{\mathbb{A}}(a_{\delta_P})$, where $\mathfrak{id}_{\mathbb{A}}$ and $\mathfrak{id}_{\mathbb{B}}$ are the identity functions of $\mathbb{A}$ and $\mathbb{B}$ respectively. Therefore there exist soft fuzzy soft continuous functions $(\tilde{r_2} \circ f') : (\mathbb{A}, \tau_{\lambda_P|\mathbb{A}}) \to (\mathbb{B}, \sigma_{\mu_Q|\mathbb{B}})$ and $(\tilde{r_1} \circ g') : (\mathbb{B}, \sigma_{\mu_Q|\mathbb{B}}) \to (\mathbb{A}, \tau_{\lambda_P|\mathbb{A}})$ with $(\tilde{r_2}f') \circ (\tilde{r_1}g') \simeq \mathfrak{id}_{\mathbb{B}}$ and $(\tilde{r_1}g') \circ (\tilde{r_2}f') \simeq \mathfrak{id}_{\mathbb{A}}$. Hence $(\mathbb{A}, \tau_{\lambda_P|\mathbb{A}})$ and $(\mathbb{B}, \sigma_{\mu_Q|\mathbb{B}})$ are the soft fuzzy soft homotopy equivalent spaces.

By Proposition 5.21., the converse part is clear. □

Definition 6.6. A **soft fuzzy soft knot** is a simple closed curve in $\mathbb{R}^3$ that can be broken into a finite number of line segments. That is, a soft fuzzy soft knot is a subset of $\mathbb{R}^3$ which is soft fuzzy soft homeomorphic to the circle.

Example 6.7. Figure 1 and Figure 2 are the examples of a soft fuzzy soft knot.

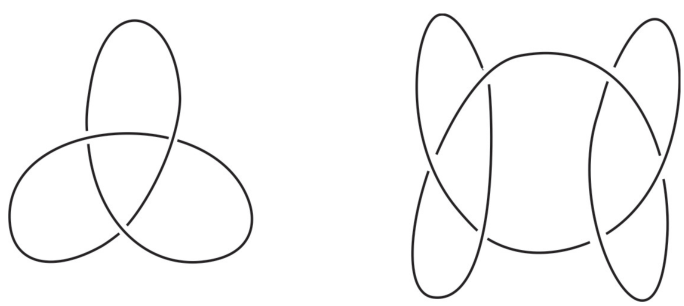

Figure 1. soft fuzzy soft cloverleaf knot

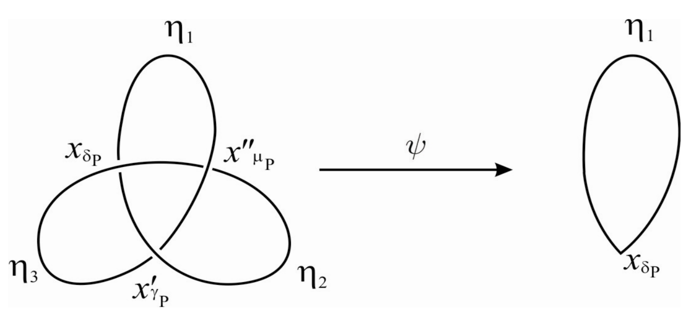

Figure 2. soft fuzzy soft square knot

Definition 6.8. A **soft fuzzy soft singular knot** is a soft fuzzy soft knot with self-intersections.

Proposition 6.9. *The soft fuzzy soft topological foldings of a soft fuzzy soft knot is not necessary a soft fuzzy soft knot.*

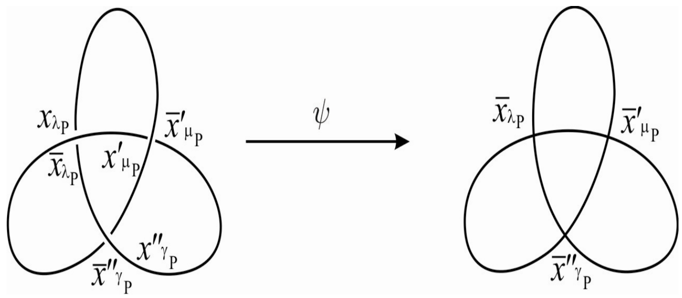

FIGURE 3. $\psi(x_{\lambda_P}) = \bar{x}_{\lambda_P}; \psi(x'_{\mu_P}) = \bar{x'}_{\mu_P}; \psi(x''_{\gamma_P}) = \bar{x''}_{\gamma_P}$

Proof. Proof is clear by defining ψ as $\psi(x_{\delta_P}) = \psi(x'_{\gamma_P}) = \psi(x''_{\mu_P}) = x_{\delta_P}$ and $\psi(\eta_1) = \psi(\eta_2) = \psi(\eta_3) = \eta_1$ in the Figure 3. □

Definition 6.10. A **soft fuzzy soft cross folding** is the soft fuzzy soft topological folding which folds a soft fuzzy soft point of upper path crossing on a soft fuzzy soft point of lower path crossing.

Proposition 6.11. *Every soft fuzzy soft cross folding of a soft fuzzy soft knot into itself gives a soft fuzzy soft singular knot.*

Proof. If K is a soft fuzzy soft knot with soft fuzzy soft topology τ_{λ_P} and $\psi : (K, \tau_{\lambda_P}) \to (K, \tau_{\lambda_P})$ is a soft fuzzy soft cross folding, then $\psi((K, \tau_{\lambda_P}))$ is not a simple closed curve in $\mathbb{R}^3$. This implies that, $\psi((K, \tau_{\lambda_P}))$ is a soft fuzzy soft singular knot by the Figure 4, Figure 5, Figure 6.

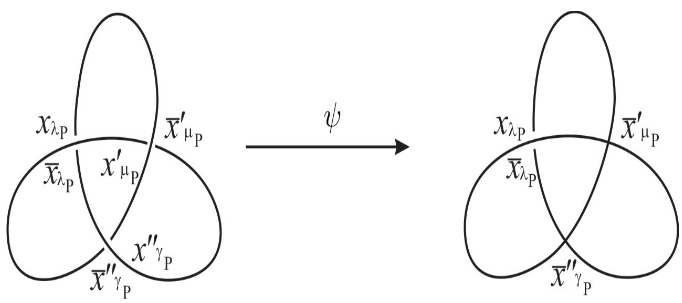

FIGURE 4. $\psi(x'_{\mu_P}) = \bar{x'}_{\mu_P}; \psi(x''_{\gamma_P}) = \bar{x''}_{\gamma_P}$

□

Proposition 6.12. *Every soft fuzzy soft topological foldings of a soft fuzzy soft singular knot need not be a soft fuzzy soft singular knot.*

Proof. Proof follows from Figure 7 and Figure 8.

□

Proposition 6.13. *The limit of soft fuzzy soft topological foldings of a soft fuzzy soft knot into itself is a soft fuzzy soft point.*

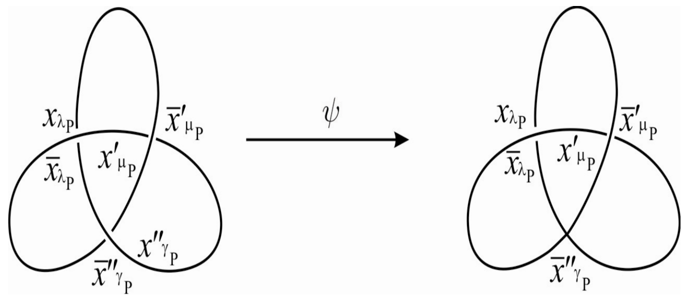

FIGURE 5. $\psi(x''_{\gamma_P}) = \bar{x''}_{\gamma_P}$

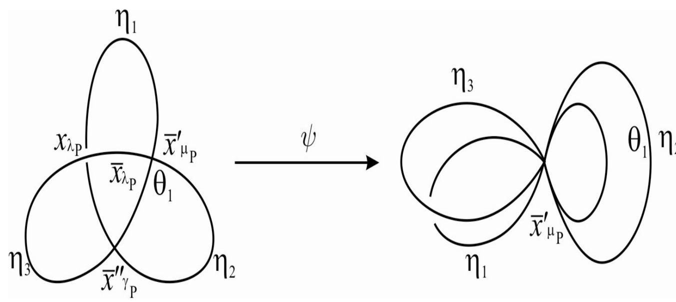

FIGURE 6. $\psi(\bar{x''}_{\gamma_P}) = \bar{x'}_{\mu_P}$

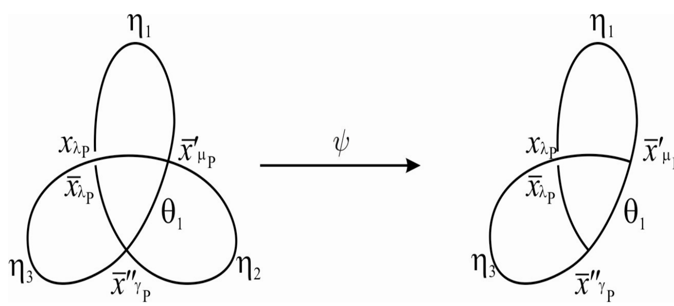

FIGURE 7. $\psi(\eta_2) = \theta_1$

Proof. Applying soft fuzzy soft topological foldings $\psi_i, (i = 1, 2, ...)$ in a successive steps on the soft fuzzy soft knot K getting an equivalent soft fuzzy soft knots successively until reaching a soft fuzzy soft point x_{δ_P}. That is, if $\psi_1 : K \to K$, $\psi_2 : \psi_1(K) \to \psi_1(K)$,.... $\psi_i : \psi_{i-1}(\psi_{i-2}....(K)...) \to \psi_{i-1}(\psi_{i-2}....(K)...)$, then $\lim\limits_{i\to\infty} \psi_i(\psi_{i-1}(\psi_{i-2}\ (....(K)...))) = x_{\delta_P}$ is a soft fuzzy soft point. This is represented in the Figure 8. □

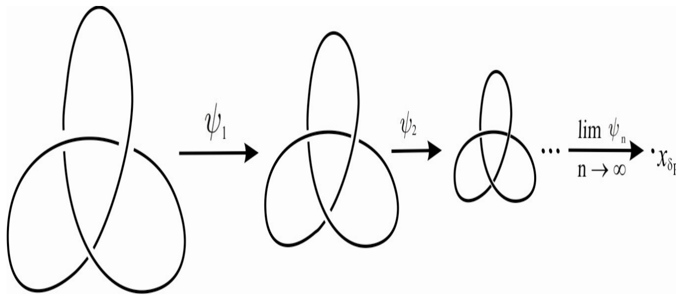

Figure 8.

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