Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 1, (July 2014), pp. 1–5
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy cosmall submodules

Taybeh Amouzegar, Yahya Talebi

Received 14 August 2013; Revised 29 October 2013; Accepted 26 December 2013

Abstract. Let M be a module, μ and ν fuzzy submodules of M with $\mu \subseteq \nu$. Then μ is called a fuzzy cosmall submodule of ν in M if $\nu/\mu \ll_f M/\mu (= \chi_M/\mu^*)$. In this paper we investigate fuzzy cosmall submodules.

2010 AMS Classification: 16D10, 08A72

Keywords: Fuzzy small submodule, Fuzzy cosmall submodule.

Corresponding Author: Yahya Talebi (talebi@umz.ac.ir)

1. Introduction

After the introduction of fuzzy sets by Zadeh [9], a number of applications of this fundamental concept have come up. Naegoita and Ralescu [4] applied this concept to modules and defined fuzzy submodules of a module. Consequently, fuzzy finitely generated submodules, fuzzy quotient modules [5], radical of fuzzy submodules, and primary fuzzy submodules [2, 7] were investigated. In this paper we investigate fuzzy small submodules, fuzzy hollow modules and fuzzy cosmall submodules of a module. Throughout this paper R will denote an arbitrary associative ring with identity and M will be unital right R-module.

2. Preliminaries

In this section we briefly introduce some definitions and results of fuzzy sets and fuzzy submodules, which we need to develop our paper.

By a fuzzy set of a module M we mean any mapping μ from M to $[0, 1]$. The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by $\mu^* = \{x \in M \mid \mu(x) > 0\}$. The subset μ_* of M is defined as $\mu_* = \{x \in M \mid \mu(x) = 1\}$.

We denote the set of all fuzzy submodules of M by $F(M)$.

Definition 2.1 ([3])**.** Let M be an R-module. A fuzzy subset μ of M is said to be a *fuzzy submodule*, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu(0) = 1$;

(ii) $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$;
(iii) $\mu(rx) \geq \mu(x)$.

Definition 2.2 ([3]). Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$(\nu/\mu)([x]) = sup\{\nu(z) \mid z \in [x]\}, \ \forall x \in \nu^*$$

where $[x]$ denotes the coset $x + \mu^*$.

Lemma 2.3 ([3]). *μ_* is a submodule of M if and only if μ is a fuzzy submodule of M.*

Let $N \leq M$, then characteristic function of N (χ_N) is defined as

$$\chi_N(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4 ([3]). *Let $\mu \in F(M)$. Then $\mu_* = M$ if and only if $\mu = \chi_M$. Also if $\sigma \in F(M)$ and $\mu \subseteq \sigma$, then $\mu_* \subseteq \sigma_*$.*

Lemma 2.5 ([1]). *Let $\mu, \sigma \in F(M)$, then $(\mu \cap \sigma)_* = \mu_* \cap \sigma_*$, $(\mu \cup \sigma)_* = \mu_* \cup \sigma_*$. Further if μ and σ have finite images then $(\mu+\sigma)_* = \mu_* + \sigma_*$, where the sum of two fuzzy submodules is defined as $(\mu+\sigma)(x) = sup\{min\{\mu(a), \sigma(b)\} \mid a, b \in M, \ x = a + b\}$.*

Definition 2.6 ([3]). Let $\mu, \sigma \in F(M)$. The sum $\mu + \sigma$ is called the *direct sum* of μ and σ if $\mu \cap \sigma = \chi_0$ and it is denoted by $\mu \oplus \sigma$.

Definition 2.7 ([1]). A fuzzy submodule $\mu(\neq \chi_0)$ of M is said to be *fuzzy indecomposable* if there is not $(\chi_0, \chi_M \neq)\sigma, \nu \in F(M)$ such that $\mu = \nu \oplus \sigma$.

Definition 2.8 ([1]). Let M be an R-module and $\mu \in F(M)$. Then μ is said to be a *fuzzy small submodule* if for any $\nu \in F(M)$, $\mu + \nu = \chi_M$ implies that $\nu = \chi_M$. It is indicated by the notation $\mu \ll_f M$ or $\mu \ll_f \chi_M$.

It is obvious that χ_0 is always a fuzzy small submodule of M.

Lemma 2.9 ([1]). *Let $\mu \in F(M)$. Then $\mu \ll_f M$ if and only if $\mu_* \ll M$.*

3. Fuzzy hollow modules and fuzzy cosmall submodules

Let μ and σ be any two fuzzy submodule of M such that $\mu \subseteq \sigma$, then μ is called a *fuzzy submodule* of σ. μ is called a *fuzzy small submodule* in σ, denoted by $\mu \ll_f \sigma$, if $\mu \ll_f \sigma^*$.

Definition 3.1. Let M be an R-module. χ_M is said to be a *fuzzy hollow module*, if every proper fuzzy submodule of χ_M is fuzzy small submodule in χ_M.

Proposition 3.2. *Let M be a module. Then:*
(i) *If χ_M is a fuzzy hollow module, then every factor of χ_M is fuzzy hollow.*
(ii) *If χ_M is fuzzy hollow, then χ_M is fuzzy indecomposable.*
(iii) *If $\sigma \ll_f \chi_M$ and χ_M/σ is fuzzy hollow, then χ_M is fuzzy hollow.*

Proof. (i) By [1, Proposition 3.8].

(ii) It is clear.

(iii) Let $\mu \in F(M)$ be a proper fuzzy submodule of χ_M. Assume that $\mu+\nu=\chi_M$ for $\nu \in F(M)$. Then $\mu+\nu+\sigma=\chi_M$. Hence $(\mu+\sigma)/\sigma+(\nu+\sigma)/\sigma=\chi_M/\sigma$. Since χ_M/σ is fuzzy hollow, then $(\nu+\sigma)/\sigma=\chi_M/\sigma$. Thus $\nu+\sigma=\chi_M$. As $\sigma \ll_f \chi_M$, $\nu=\chi_M$. Therefore χ_M is fuzzy hollow. □

Let M be a module and $N \leq L \leq M$. Recall that N is called a *cosmall submodule* of L in M if $L/N \ll M/N$. The notation $N \overset{cs}{\underset{M}{\hookrightarrow}} L$ indicates that N is a cosmall submodule of L in M [8]. In [6], authors generalized this concept in fuzzy settings. In this section, we get some interesting results.

Definition 3.3 ([6])**.** Let M be a module and $\mu,\nu \in F(M)$ with $\mu \subseteq \nu$. Then μ is called a *fuzzy cosmall submodule* of ν in M if $\nu/\mu \ll_f M/\mu(=\chi_M/\mu^*)$. The notation $\mu \overset{fcs}{\underset{M}{\hookrightarrow}} \nu$ indicates that μ is a fuzzy cosmall submodule of ν in M.

Example 3.4. Consider $M=\mathbb{Z}_8=\{0,1,2,3,4,5,6,7\}$ under addition modulo 8. Then M is a module over the ring $\mathbb{Z}$. Let $S=\{0,2,4,6\}$. Define $\mu : M \to [0,1]$ as follows:

$$\mu(x)=\begin{cases} 1 & \text{if } x \in S \\ \alpha & \text{otherwise} \end{cases}$$

where $0 \leq \alpha < 1$. Then χ_0 is a fuzzy cosmall submodule of μ in M.

Proposition 3.5. *Let $\mu,\sigma \in F(M)$. Then $\mu \overset{fcs}{\underset{M}{\hookrightarrow}} \sigma$ if and only if $\mu_* \overset{cs}{\underset{M}{\hookrightarrow}} \sigma_*$.*

Proof. Let $\mu,\sigma \in F(M)$. $\mu \overset{fcs}{\underset{M}{\hookrightarrow}} \sigma$ if and only if $\sigma/\mu \ll_f M/\mu$. By Lemma 2.9 $(\sigma/\mu)_* \ll (M/\mu)_*$. By [1, Proposition 3.6], $\sigma_*/\mu_* \ll M/\mu_*$. Hence $\mu_* \overset{cs}{\underset{M}{\hookrightarrow}} \sigma_*$. □

Proposition 3.6. *Let M be a module. If $\mu \subseteq \sigma \subseteq \chi_M$ and $\sigma=\mu+\nu$, where $\nu \ll_f \mu$, Then μ is a fuzzy cosmall submodule of σ in M.*

Proof. Let $\chi_M=\sigma+\gamma$ for some $\gamma \in F(M)$. Then $\chi_M=\mu+\nu+\gamma=\mu+\gamma$, since $\nu \ll_f M$. So, by [6, Theorem 4.24], μ is a fuzzy cosmall submodule of σ in M. □

Definition 3.7 ([3])**.** Let X and Y be any two nonempty sets, and $f : X \to Y$ be a mapping. Let μ be a fuzzy subset of X and σ a fuzzy subset of Y, then the *image* $f(\mu)$ and the *inverse image* $f^{-1}(\sigma)$ are defined as follows: for all $y \in Y$

$$f(\mu)(y)=\begin{cases} sup\{\mu(x) \mid x \in X, f(x)=y\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and $f^{-1}(\sigma)(x)=\sigma(f(x))$ for all $x \in X$.

Lemma 3.8. *Let $f : M \to N$ be an epimorphism. If $\mu \in F(M)$ and $\sigma \in F(N)$, then:*

(i) $f(\mu)_*=f(\mu_*)$;

(ii) $f^{-1}(\sigma)_*=f^{-1}(\sigma_*)$.

Proof. (i) Let $x \in f(\mu)_*$, then $(f(\mu))(x) = 1$. Thus $sup\{\mu(y) \mid f(y) = x\} = 1$. Hence $\mu(y) = 1$ for some $y \in f^{-1}(x)$. Then $y \in \mu_*$, where $f(y) = x$, that is, $x \in f(\mu_*)$. So $f(\mu)_* \subseteq f(\mu_*)$. Conversely, let $x \in f(\mu_*)$, then $x = f(y)$, for some $y \in \mu_*$. So $y \in f^{-1}(x)$, where $\mu(y) = 1$. Hence $sup\{\mu(y) \mid y \in f^{-1}(x)\} \geq \mu(y) = 1$. Thus $(f(\mu))(x) = 1$, and so $x \in f(\mu)_*$. Therefore the equality follows.

(ii) $x \in f^{-1}(\sigma)_*$ iff $f^{-1}(\sigma)(x) = 1$ iff $\sigma(f(x)) = 1$ iff $f(x) \in \sigma_*$ iff $x \in f^{-1}(\sigma_*)$. □

Theorem 3.9. *Let $f : M \to N$ be an epimorphism. If $\mu \subseteq \nu \subseteq \chi_M$ and $\mu \overset{fcs}{\underset{M}{\hookrightarrow}} \nu$, then $f(\mu) \overset{fcs}{\underset{N}{\hookrightarrow}} f(\nu)$.*

Proof. Let $\sigma \subseteq \chi_N$ and $f(\nu) + \sigma = \chi_N$. Then, by Proposition 3.6 and Lemma 3.8, $f(\nu_*) + \sigma_* = N$. Since f is an epimorphism, there exists $K \leq M$ such that $f(K) = \sigma_*$. It is clear that $K = (\chi_K)_*$ and so $f(K) = f((\chi_K)_*) = \sigma_*$. Hence $f(\nu_* + (\chi_K)_*) = f(\nu_*) + f((\chi_K)_*) = N = f(M)$. So $\nu_* + (\chi_K)_* = M$. By Lemma 2.5 and Lemma 2.4, $\nu + \chi_K = \chi_M$. By hypothesis, $\mu + \chi_K = \chi_M$. Then $\mu_* + (\chi_K)_* = M$. Thus $f(\mu_*) + f((\chi_K)_*) = N$, so $f(\mu_*) + \sigma_* = N$. Hence $f(\mu) + \sigma = \chi_N$. By [6, Theorem 4.24], $f(\mu) \overset{fcs}{\underset{N}{\hookrightarrow}} f(\nu)$. □

Theorem 3.10. *Let $f : M \to N$ be an epimorphism. If $\nu \subseteq \sigma \subseteq \chi_N$, then $\nu \overset{fcs}{\underset{N}{\hookrightarrow}} \sigma$ if and only if $f^{-1}(\nu) \overset{fcs}{\underset{M}{\hookrightarrow}} f^{-1}(\sigma)$.*

Proof. Let $\mu \in F(M)$ and $f^{-1}(\sigma) + \mu = \chi_M$. By Proposition 3.6, $(f^{-1}(\sigma))_* + \mu_* = M$. From Lemma 3.8, $f^{-1}(\sigma_*) + \mu_* = M$. Thus $\sigma_* + f(\mu_*) = N$. Using Proposition 3.5, $\nu_* \overset{cs}{\underset{N}{\hookrightarrow}} \sigma_*$, and so $\nu_* + f(\mu_*) = N$. Hence $f^{-1}(\nu_*) + \mu_* = M$. Then $f^{-1}(\nu) + \mu = N$. Therefore $f^{-1}(\nu) \overset{fcs}{\underset{M}{\hookrightarrow}} f^{-1}(\sigma)$, by [6, Theorem 4.24]. Conversely, let $\sigma + \mu = \chi_N$ for some $\mu \in F(M)$. Then $\sigma_* + \mu_* = N$, so $f^{-1}(\sigma_*) + f^{-1}(\mu_*) = \mu$. Thus $f^{-1}(\sigma) + f^{-1}(\mu) = \chi_M$. Hence $f^{-1}(\nu_*) + f^{-1}(\mu_*) = M$, then $\nu_* + \mu_* = N$. So $\nu + \mu = \chi_N$. Therefore $\nu \overset{fcs}{\underset{N}{\hookrightarrow}} \sigma$. □

Acknowledgements. This research partially is supported by the "Research Center in Algebriac Hyperstructure and Fuzzy Mathematics" University of Mazandaran, Babolsar Iran.

References

[1] D. K. Basnet, N. K. Sarma and L. B. Singh, Fuzzy superfluous submodule, In Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and Its Applications (2010) 330–335.

[2] R. Kumar, S. K. Bhambri and P. Kumar, Fuzzy submodules: some analogues and deviations, Fuzzy Sets and Systems 70 (1995) 125–130.

[3] J. N. Mordeson and D. S. Malik, Fuzzy Commutative Algebra, World Scientific, River Edge, NJ, USA. (1998).

[4] C. V. Naegoita and D. A. Ralescu, Application of Fuzzy Sets in System Analysis, Birkhauser, Basel, Switzerland (1975).

[5] F. Z. Pan, Fuzzy finitely generated modules, Fuzzy Sets and Systems 21 (1987) 105–113.
[6] S. Rahman and H. K. Saikia, Fuzzy small submodule and Jacobcon L-radical, Int. J. Math. Math. Sci. 1 (2011) 1–12.
[7] F. I. Sidky, On radicals of fuzzy submodules and primary fuzzy submodules, Fuzzy Sets and Systems 119 (2001) 419–425.
[8] R. Wisbaure, Foundations of Module and Ring Theory, Gordon and Breach: Philadelphia (1991).
[9] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1964) 338–353.

Taybeh Amouzegar (t.amoozegar@yahoo.com)
Department of Mathematics, Quchan University of Advanced Technology, Quchan, Iran

Yahya Talebi (talebi@umz.ac.ir)
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran