Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 5, (November 2014), pp. 769–783
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Soft Hausdorff spaces and their some properties

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Received 14 January 2014; Revised 21 April 2014; Accepted 17 May 2014

Abstract. Molodtsov (1999) [24] defined the notion of a soft set for coping with uncertainties. Shabir and Naz (2011) [30] introduced the topological structures of soft sets and investigated many properties. In the present paper, we contribute to the progress of the soft topological structures. We introduce soft Hausdorff spaces using the definition of soft point in [10, 19] and investigate some of their properties. We give some new concepts such as cluster soft point, soft net and support them with examples. Then, by using these concepts, we obtain some properties with respect to soft Hausdorff spaces which are important for further research on soft topology.

2010 AMS Classification: 06D72, 54A40

Keywords: Soft set, Soft topology, Cluster soft point, Soft net, Soft filter, Soft Hausdorff space.

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1. Introduction

There are diverse uncertainties for most of complex problems in economics, engineering, environmental science and social science. Several set theories can be regarded as mathematical tools for dealing with these uncertainties, for example the theory of fuzzy sets [32], the theory of intuitionistic fuzzy sets [3, 4], the theory of vague sets [12], the theory of interval mathematics [4, 14] and the theory of rough sets [26]. However, these theories have their inherent difficulties because of the inadequacy of the parameterization tool of the theories as cited by Molodtsov [24].

In 1999, Molodtsov [24] initiated the concept of soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. This new theory does not require the specification of a parameter. We can utilize any parametrization with the aid of words, sentences, real numbers and so on. This implies that the problem of setting the membership function does not arise. Hence, soft set theory has compelling applications in several diverse fields, most of these applications was shown by Molodtsov [24].

Maji et al. [22] gave the first practical application of soft sets in decision making problems. Chen et al. [8] presented a novel concept of parameterization reduction in soft sets. Kong et al. [18] introduced the notion of normal parameter reduction and presented a algorithm for normal parameter reduction. Then, Ma et al. [20] proposed a simpler and more easily comprehensible algorithm. Pei and Miao [27] showed that soft sets are a class of special information systems. Maji et al. [21] studied on soft set theory in detail. Ali et al. [2] presented some new algebraic operations on soft sets. Aktas and Cagman [1] introduced the soft group and also compared soft sets to fuzzy set and rough set. Jun [15] defined and studied the concept of soft BCK/BCI-algebras. Also, Feng et al. [11] worked on soft semirings, soft ideals and idealistic soft semirings. Babitha and Sunil [6] introduced the soft set relation and discussed many related concepts such as equivalent soft set relation, partition and composition. Kharal and Ahmad [17] defined the notion of a mapping on soft classes and worked some properties of images and inverse images of soft sets. Das and Samanta [9] introduced the notions of soft real sets and soft real numbers and studied their properties. Shabir and Naz [30] initiated the study of soft topological spaces. Also, they defined fundamental notions such as soft open sets, soft closed sets, soft interior, soft closure and soft separation axioms and established their several properties. Nazmul and Samanta [25] defined new concepts such as soft element, soft interior operator and soft closure operator and worked their properties. Aygunoglu and Aygun [5] introduced soft product topology, soft compactness and generalized Tychonoff theorem to the soft topological spaces. Zorlutuna et al. [33] also studied on soft topological spaces and obtained the relation between soft topology and fuzzy topology. Das and Samanta [10] introduced soft metric spaces and investigated some their fundamental properties. Also, they redefined the notion of soft point. Studies on the soft topological spaces have been accelerated [13, 28, 29, 31].

In this work, we first remind the fundamental concepts of soft set theory. Also, we recall notion of soft point which is defined in [10, 19] and some of its basic properties. We then recall some concepts of soft topological spaces and introduce some new notions such as cluster soft point and soft net. Moreover, we introduce convergence of soft net and soft filter. Finally, we define soft Hausdorff spaces by using notion of soft point in [10, 19]. We study some basic properties of soft Hausdorff spaces. Also, we show that a soft net and a soft filter converge to at most one soft point in soft Hausdorff space. These study not only can form the theoretical basis for further applications of soft topology but also lead to the advance of information systems.

2. Preliminaries

In this section, we recall some basic notions regarding soft sets, most of which exist in [7, 17], and we present the concept of soft point introduced in [10, 19]. Throughout this work, let X be an initial universe, $P(X)$ be the power set of X, E be the set of all parameters for X and $A \subseteq E$.

Aygunoglu and Aygun [5] modified the definition of soft set defined in [7, 23, 24] as follows.

Definition 2.1. A soft set F_A on the universe X with the set E of parameters is defined by the set of ordered pairs

$$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(X)\}$$

where $F_A : E \to P(X)$ such that $F_A(e) \neq \varnothing$ if $e \in A \subseteq E$ and $F_A(e) = \varnothing$ if $e \notin A$.

Throughout this paper, the family of all soft sets over X is denoted by $S(X, E)$. [5]

Definition 2.2 ([7]). The soft set $F_\varnothing \in S(X, E)$ is called null soft set if $F_\varnothing(e) = \varnothing$ for every $e \in E$.

The null soft set is denoted by Φ. [21]

Definition 2.3 ([7]). Let $F_A \in S(X, E)$. If $F_A(e) = X$ for all $e \in A$, then F_A is called A-absolute soft set. If $A = E$, then the A-absolute soft set is called absolute soft set.

The A-absolute soft set and absolute soft set are denoted by $\widetilde{A}$ and $\widetilde{E}$, respectively. [21]

Definition 2.4 ([7]). Let $F_A, G_B \in S(X, E)$. F_A is a soft subset of G_B if $F_A(e) \subseteq G_B(e)$ for each $e \in E$. It is denoted by $F_A \sqsubseteq G_B$.

Definition 2.5 ([7]). Let $F_A, G_B \in S(X, E)$. F_A and G_B are soft equal if $F_A \sqsubseteq G_B$ and $G_B \sqsubseteq F_A$. It is denoted by $F_A = G_B$.

Definition 2.6 ([7]). Let $F_A \in S(X, E)$. The complement of F_A is denoted by F_A^c, where $F_A^c : E \to P(X)$ is a mapping defined by $F_A^c(e) = X - F_A(e)$ for all $e \in E$.

Definition 2.7 ([7]). Let $F_A, G_B \in S(X, E)$. The union of F_A and G_B is a soft set H_C, which is defined by $H_C(e) = F_A(e) \cup G_B(e)$ for all $e \in E$.

H_C is denoted by $F_A \sqcup G_B$.

Definition 2.8 ([7]). Let $F_A, G_B \in S(X, E)$. Then, the intersection of F_A and G_B is a soft set H_C, which is defined by $H_C(e) = F_A(e) \cap G_B(e)$ for all $e \in E$.

H_C is denoted by $F_A \sqcap G_B$.

Definition 2.9 ([33]). Let J be an arbitrary index set and let $\{(F_A)_i\}_{i \in J}$ be a family of soft sets over X. Then,

(i) The union of these soft sets is the soft set H_C defined by $H_C(e) = \bigcup_{i \in J}(F_A)_i(e)$ for every $e \in E$ and this soft set is denoted by $\bigsqcup_{i \in J}(F_A)_i$.

(ii) The intersection of these soft sets is the soft set H_C defined by $H_C(e) = \bigcap_{i \in J}(F_A)_i(e)$ for every $e \in E$ and this soft set is denoted by $\sqcap_{i \in J}(F_A)_i$.

Theorem 2.10 ([5, 7, 33]). *Let J be an index set and F_A, G_B, $(F_A)_i$, $(G_B)_i \in S(X, E)$, for all $i \in J$. Then, the following statements are satisfied.*

(1) $F_A \sqcap \left(\bigsqcup_{i \in J}(G_B)_i\right) = \bigsqcup_{i \in J}(F_A \sqcap (G_B)_i)$.
(2) $F_A \sqcup \left(\sqcap_{i \in J}(G_B)_i\right) = \sqcap_{i \in J}(F_A \sqcup (G_B)_i)$.
(3) $(F_A^c)^c = F_A$.
(4) $\left(\sqcap_{i \in J}(F_A)_i\right)^c = \bigsqcup_{i \in J}(F_A)_i^c, \quad \left(\bigsqcup_{i \in J}(F_A)_i\right)^c = \sqcap_{i \in J}(F_A)_i^c$.
(5) *If $F_A \sqsubseteq G_B$, then $G_B^c \sqsubseteq F_A^c$.*
(6) $F_A \sqcup F_A^c = \widetilde{E}$, $F_A \sqcap F_A^c = \Phi$.

Definition 2.11 ([10, 19]). A soft set P_A over X is said to be a soft point, which is called soft element in [25] , if there is $e \in A$ such that $P_A(e) = \{x\}$ for some $x \in X$ and $P_A(e') = \varnothing$ for all $e' \in E \backslash \{e\}$. This soft point is denoted by P_e^x

The soft point P_e^x is said to belongs to the soft set F_A, denoted by $P_e^x \widetilde{\in} F_A$, if $x \in F_A(e)$. [10, 25]

From now on, the family of all soft points over X will be denoted by $SP(X, E)$.

Definition 2.12 ([10]). Two soft points $P_{e_1}^{x_1}, P_{e_2}^{x_2}$ are said to be equal if $e_1 = e_2$ and $x_1 = x_2$. Thus, $P_{e_1}^{x_1} \neq P_{e_2}^{x_2} \Leftrightarrow x_1 \neq x_2$ or $e_1 \neq e_2$.

Proposition 2.13 ([10, 25]). *Let $F_A, G_B \in S(X, E)$ and let $P_e^x \in SP(X, E)$. Then we have:*

(1) $P_e^x \widetilde{\in} F_A$ *iff* $P_e^x \widetilde{\notin} F_A^c$.
(2) $P_e^x \widetilde{\in} F_A \sqcup G_B$ *iff* $P_e^x \widetilde{\in} F_A$ *or* $P_e^x \widetilde{\in} G_B$.
(3) $P_e^x \widetilde{\in} F_A \sqcap G_B$ *iff* $P_e^x \widetilde{\in} F_A$ *and* $P_e^x \widetilde{\in} G_B$.
(4) $F_A \sqsubseteq G_B$ *iff* $P_e^x \widetilde{\in} F_A$ *imply* $P_e^x \widetilde{\in} G_B$.

Definition 2.14 ([17]). Let $S(X, E)$ and $S(Y, K)$ be the families of all soft sets over X and Y, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two mappings. Then, the mapping φ_ψ is called a soft mapping from X to Y, denoted by $\varphi_\psi : S(X, E) \to S(Y, K)$, for which:

(1) Let $F_A \in S(X, E)$. Then $\varphi_\psi(F_A)$ is the soft set over Y defined as follows:

$$\varphi_\psi(F_A)(k) = \begin{cases} \bigcup_{e \in \psi^{-1}(k) \cap A} \varphi(F_A(e)), & \text{if } \psi^{-1}(k) \cap A \neq \varnothing; \\ \varnothing, & \text{otherwise.} \end{cases}$$

for all $k \in K$.

$\varphi_\psi(F_A)$ is called a soft image of a soft set F_A.

(2) Let $G_B \in S(Y, K)$. Then $\varphi_\psi^{-1}(G_B)$ is the soft set over X defined as follows:

$$\varphi_\psi^{-1}(G_B)(e) = \begin{cases} \varphi^{-1}(G_B(\psi(e))), & \text{if } \psi(e) \in B; \\ \varnothing, & \text{otherwise.} \end{cases}$$

for all $e \in E$.

$\varphi_\psi^{-1}(G_B)$ is called a soft inverse image of a soft set G_B.

The soft mapping φ_ψ is called injective, if φ and ψ are injective. The soft mapping φ_ψ is called surjective, if φ and ψ are surjective. [5, 33]

Theorem 2.15 ([17]). *Let $(F_A)_i := (F_i)_{A_i} \in S(X, E)$ and $(G_B)_i := (G_i)_{B_i} \in S(Y, K)$ for all $i \in J$ where J is an index set. Then, for a soft mapping $\varphi_\psi : S(X, E) \to S(Y, K)$, the following conditions are satisfied.*

(1) *If* $(F_A)_1 \sqsubseteq (F_A)_2$, *then* $\varphi_\psi((F_A)_1) \sqsubseteq \varphi_\psi((F_A)_2)$.
(2) *If* $(G_B)_1 \sqsubseteq (G_B)_2$, *then* $\varphi_\psi^{-1}((G_B)_1) \sqsubseteq \varphi_\psi^{-1}((G_B)_2)$.
(3) $\varphi_\psi\big(\bigsqcup_{i \in J}(F_A)_i\big) = \bigsqcup_{i \in J} \varphi_\psi((F_A)_i)$, $\quad \varphi_\psi\big(\sqcap_{i \in J} (F_A)_i\big) \sqsubseteq \sqcap_{i \in J} \varphi_\psi((F_A)_i)$.
(4) $\varphi_\psi^{-1}\big(\bigsqcup_{i \in J}(G_B)_i\big) = \bigsqcup_{i \in J} \varphi_\psi^{-1}((G_B)_i)$, $\quad \varphi_\psi^{-1}\big(\sqcap_{i \in J}(G_B)_i\big) = \sqcap_{i \in J} \varphi_\psi^{-1}((G_B)_i)$.
(5) $\varphi_\psi^{-1}(\widetilde{K}) = \widetilde{E}$, $\varphi_\psi^{-1}(\Phi) = \Phi$.
(6) $\varphi_\psi(\Phi) = \Phi$.

Theorem 2.16 ([5, 33])**.** *Let $F_A, (F_A)_i := (F_i)_{A_i} \in S(X,E)$ for all $i \in J$ where J is an index set and let $G_B \in S(Y,K)$. Then, for a soft mapping $\varphi_\psi : S(X,E) \to S(Y,K)$, the following conditions are satisfied.*

(1) $F_A \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(F_A))$, *the equality holds if φ_ψ is injective.*
(2) $\varphi_\psi(\varphi_\psi^{-1}(G_B)) \sqsubseteq G_B$, *the equality holds if φ_ψ is surjective.*
(3) $\varphi_\psi\big(\sqcap_{i\in J}(F_A)_i\big) = \sqcap_{i\in J}\varphi_\psi((F_A)_i)$ *if φ_ψ is injective.*
(4) $\varphi_\psi(\widetilde{E}) = \widetilde{K}$ *if φ_ψ is surjective.*

Definition 2.17 ([6])**.** Let $F_A \in S(X,E)$ and $G_B \in S(Y,K)$. The cartesian product $F_A \times G_B$ is defined by $H_{A\times B}$, where $H_{A\times B} : E\times K \to P(X\times Y)$ and $H_{A\times B}(e,k) = F_A(e)\times G_B(k)$ for all $(e,k)\in E\times K$.

Definition 2.18 ([5])**.** Let $F_A \in S(X,E), G_B \in S(Y,K)$ and let $p_1 : X\times Y \to X$, $q_1 : E\times K \to E$ and $p_2 : X\times Y \to Y$, $q_2 : E\times K \to K$ be the projection mappings in classical meaning. The soft mappings $(p_q)_1 := (p_1)_{q_1}$ and $(p_q)_2 := (p_2)_{q_2}$ are called soft projection mappings from $X\times Y$ to X and $X\times Y$ to Y, respectively, where $(p_q)_1(F_A\times G_B) = F_A$ and $(p_q)_2(F_A\times G_B) = G_B$.

3. Soft topological spaces, soft nets and convergence

In this section, we recall some fundamental properties of soft topological spaces and give new definitions such as cluster soft point and soft net. In the next section, these properties and definitions will be used.

Definition 3.1 ([30])**.** Let τ be a collection of soft sets over X, then τ is said to be a soft topology on X if

(st_1) $\Phi, \widetilde{E}$ belong to τ.
(st_2) the union of any number of soft sets in τ belongs to τ.
(st_3) the intersection of any two soft sets in τ belongs to τ.

The pair (X,τ) is called a soft topological space. The members of τ are called soft open sets in X. A soft set F_A over X is called a soft closed in X if $F_A^c \in \tau$.

Example 3.2. Let X be a non-empty set and E be a non-empty set of parameters. Let $P_e^x \in SP(X,E)$ be a fixed soft point. Then,

$$\tau = \{F_E : P_e^x \widetilde{\notin} F_E \ or\ if\ P_e^x \widetilde{\in} F_E,\ then \bigcup_{e\in E} X\backslash F_E(e)\ is\ finite\} \cup \{\Phi\}$$

is a soft topology on X and hence (X,τ) is a soft topological space over X.

Definition 3.3 ([5, 25])**.** Let (X,τ) be a soft topological space. A subcollection $\mathcal{B}$ of τ is called a base for τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.

Proposition 3.4 ([25])**.** *A family $\mathcal{B}$ of soft sets over X forms a base of a soft topology over X iff the following conditions are satisfied.*

(i) $\Phi \in \mathcal{B}$;
(ii) $\widetilde{E}$ *is union of the members of* $\mathcal{B}$;
(iii) If $F_A, G_B \in \mathcal{B}$ then $F_A \sqcap G_B$ is union of some members of $\mathcal{B}$, i.e., for $F_A, G_B \in \mathcal{B}$ and $P_e^x \widetilde{\in} F_A \sqcap G_B$ there exists a $H_C \in \mathcal{B}$ such that $P_e^x \widetilde{\in} H_C \sqsubseteq F_A \sqcap G_B$.

Remark 3.5. Let $\mathcal{D} = \{(F_A)_i : i \in J\}$. Then, $\bigsqcup_{i\in\varnothing}(F_A)_i = \Phi$. Assume that $\bigsqcup_{i\in\varnothing}(F_A)_i \neq \Phi$. Then, for $e \in A_{i_0}$ there exists $i_0 \in \varnothing$ such that $(F_A)_{i_0}(e) \neq \varnothing$, and we have a contradiction. Moreover, let $\mathcal{C} = \{(F_A)_i^c : i \in J\}$. Then, $\bigsqcup_{i\in\varnothing}(F_A)_i^c = \Phi$. If we pass to the complement, then we have $\sqcap_{i\in\varnothing}(F_A)_i = \widetilde{E}$.

From now on, we will omit the condition (i) in Proposition 3.4.

Corollary 3.6. *A family of soft sets $\mathcal{B}$ satisfying the conditions (ii) and (iii) of Proposition 3.4 generate a unique soft topology $\tau = \left\{\bigsqcup_{F_A\in\psi} F_A : \psi \subseteq \mathcal{B}\right\}$ over X.*

Example 3.7. Let $\mathbb{R}$ be the real numbers, $E = \{e\}$ and $(F_\lambda)_E = \{\,(e,\,\{\lambda\}\,)\,\}$, where $\lambda \in \mathbb{R}$. Consider the family $\mathcal{B} = \{(F_\lambda)_E : \lambda \in \mathbb{R}\}$; since $\mathcal{B}$ satisfy (ii) and (iii), it is base and generate the soft topology $\tau = \left\{\bigsqcup_{\lambda\in J}(F_\lambda)_E : J \subseteq \mathbb{R}\right\}$ over $\mathbb{R}$.

Definition 3.8 ([5])**.** Let (X, τ_1) and (Y, τ_2) be two soft topological spaces. A soft mapping $\varphi_\psi : (X, \tau_1) \to (Y, \tau_2)$ is called soft continuous if $\varphi_\psi^{-1}(G_B) \in \tau_1$ for every $G_B \in \tau_2$.

A soft mapping $\varphi_\psi : (X, \tau_1) \to (Y, \tau_2)$ is called soft open if $\varphi_\psi(F_A) \in \tau_2$ for every $F_A \in \tau_1$.

Definition 3.9 ([31])**.** Let (X, τ_1) and (Y, τ_2) be two soft topological spaces. A soft mapping $\varphi_\psi : (X, \tau_1) \to (Y, \tau_2)$ is called homeomorphism if φ_ψ is bijective, soft continuous and soft open.

Definition 3.10 ([5])**.** Let (X, τ) be a soft topological space. A subcollection $\mathcal{S}$ of τ is called a subbase for τ if the family of all finite intersections of members of $\mathcal{S}$ forms a base for τ.

Theorem 3.11 ([5])**.** *Let $\mathcal{S}$ be a family of soft sets over X such that $\Phi, \widetilde{E} \in \mathcal{S}$. Then $\mathcal{S}$ is a subbase for the soft topology τ, whose members are of the form $\bigsqcup_{i\in J}\big(\sqcap_{\lambda\in\Lambda_i} (F_A)_{i,\lambda}\big)$ where J is arbitrary index set and for each $i \in J$, Λ_i is a finite index set, $(F_A)_{i,\lambda} \in \mathcal{S}$ for $i \in J$ and $\lambda \in \Lambda_i$.*

Definition 3.12 ([5])**.** Let $\{(\varphi_\psi)_i : S(X, E) \to (Y_i, \tau_i)\}_{i\in J}$ be a family of soft mappings and $\{(Y_i, \tau_i)\}_{i\in J}$ be a family of soft topological spaces. Then, the soft topology τ generated from the subbase $\mathcal{S} = \{(\varphi_\psi)_i^{-1}(F_A) : F_A \in \tau_i, i \in J\}$ is called the soft topology (or initial soft topology) induced by the family of soft mappings $\{(\varphi_\psi)_i\}_{i\in J}$.

Theorem 3.13 ([5])**.** *The initial soft topology τ on X induced by the family $\{(\varphi_\psi)_i : S(X, E) \to (Y_i, \tau_i)\}_{i\in J}$ is the coarsest soft topology making $(\varphi_\psi)_i : (X, \tau) \to (Y_i, \tau_i)$ soft continuous, for all $i \in J$.*

Definition 3.14 ([5])**.** Let $\{(X_i, \tau_i)\}_{i\in J}$ be a family of soft topological spaces. Then, the initial soft topology on $X\big(= \prod_{i\in J} X_i\big)$ generated by the family $\{(p_q)_i\}_{i\in J}$ is called product soft topology on X (Here, $(p_q)_i$ is the soft projection mapping from X to $X_i, i \in J$).

The product soft topology is denoted by $\prod_{i\in J} \tau_i$.

Definition 3.15 ([33])**.** Let (X, τ) be a soft topological space and $F_A \in S(X, E)$. The soft interior of F_A is the soft set

$$(F_A)^o = \bigsqcup\{G_B : G_B \text{ is a soft open set and } G_B \sqsubseteq F_A\}.$$

By property (st_2) for soft open sets, $(F_A)^o$ is soft open. It is the largest soft open set contained in F_A.

Definition 3.16 ([30]). Let (X, τ) be a soft topological space and $F_A \in S(X, E)$. The soft closure of F_A is the soft set

$$\overline{(F_A)} = \sqcap\{G_B : G_B \text{ is a soft closed set and } F_A \sqsubseteq G_B\}.$$

Clearly $\overline{(F_A)}$ is the smallest soft closed set over X which contains F_A.

Definition 3.17 ([25]). A soft set F_A in a soft topological space (X, τ) is called a soft neighborhood of the soft point P_e^x if there exists a soft open set G_B such that $P_e^x \widetilde{\in} G_B \sqsubseteq F_A$.

The soft neighborhood system of a soft point P_e^x, denoted by $\mathcal{N}_\tau(P_e^x)$, is the family of all its soft neighborhoods.

Definition 3.18 ([25, 33]). A soft set F_A in a soft topological space (X, τ) is called a soft neighborhood of the soft set G_B if there exists a soft open set H_C such that $G_B \sqsubseteq H_C \sqsubseteq F_A$.

Definition 3.19 ([25]). Let (X, τ) be a soft topological space and $\{\mathcal{N}_\tau(P_e^x) : P_e^x \widetilde{\in} \widetilde{E}\}$ be the system of soft neighborhoods. Then,

(sn_1) $\mathcal{N}_\tau(P_e^x) \neq \varnothing$, for all $P_e^x \in SP(X, E)$,
(sn_2) If $F_A \in \mathcal{N}_\tau(P_e^x)$, then $P_e^x \widetilde{\in} F_A$,
(sn_3) If $F_A \in \mathcal{N}_\tau(P_e^x)$ and $F_A \sqsubseteq G_B$, then $G_B \in \mathcal{N}_\tau(P_e^x)$,
(sn_4) If $F_A, G_B \in \mathcal{N}_\tau(P_e^x)$, then $F_A \sqcap G_B \in \mathcal{N}_\tau(P_e^x)$,
(sn_5) If $F_A \in \mathcal{N}_\tau(P_e^x)$, then there is a $G_B \in \mathcal{N}_\tau(P_e^x)$ such that $G_B \sqsubseteq F_A$ and $G_B \in \mathcal{N}_\tau(P_\alpha^y)$, for each $P_\alpha^y \widetilde{\in} G_B$.

Theorem 3.20 ([19]). *Let (X, τ) be a soft topological space. A soft point $P_e^x \widetilde{\in} \overline{(F_A)}$ iff each soft neighborhood of P_e^x intersects F_A.*

Definition 3.21 ([28]). Let (X, τ) be a soft topological space and let $\mathcal{E}(P_e^x)$ be a family of soft neighborhoods of a soft point P_e^x. If, for each soft neighborhood F_A of P_e^x, there exists a $G_B \in \mathcal{E}(P_e^x)$ such that $P_e^x \widetilde{\in} G_B \sqsubseteq F_A$, then we say that $\mathcal{E}(P_e^x)$ is a soft neighborhood base at P_e^x.

Definition 3.22 ([30]). Let (X, τ) be a soft topological space and S be a non-empty subset of X. Then, $\tau_S = \{\widetilde{E_S} \sqcap F_A : F_A \in \tau\}$ is called the soft relative topology on S and (S, τ_S) is called a soft subspace of (X, τ).

Here, $\widetilde{E_S}$ is the soft set over X defined by $\widetilde{E_S}(e) = S$ for all $e \in E$.

Proposition 3.23 ([30]). *Let (S, τ_S) be a soft subspace of (X, τ) and $F_A \in S(X, E)$. Then, F_A is soft open in S if and only if $F_A = \widetilde{E_S} \sqcap G_B$, for some $G_B \in \tau$.*

Definition 3.24. Let (X, τ) be a soft topological space, $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ be a sequence of soft points in (X, τ) and $P_e^x \in SP(X, E)$. The sequence $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ is said to converge to P_e^x, and we write $P_{e_n}^{x_n} \to P_e^x$, if for every $F_A \in \mathcal{N}_\tau(P_e^x)$, there exists an $n_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \widetilde{\in} F_A$ for all $n \geq n_0$.

Example 3.25. Let $(\mathbb{R}, \tau)$ be a soft topological space which is defined in Example 3.7 and $P_e^{x_n} \to P_e^x$ in (X, τ). Then, for every $F_A \in \mathcal{N}_\tau(P_e^x)$, there exists an $n_0 \in \mathbb{N}$ such that $P_e^{x_n} \widetilde{\in} F_A$ for all $n \geq n_0$. Since $P_e^x \in \tau$, there exists an $n' \in \mathbb{N}$ such that $P_e^{x_n} \widetilde{\in} P_e^x$ for all $n \geq n'$. Hence, $x_n = x$, for all $n \geq n'$.

Definition 3.26. Let (X, τ) be a soft topological space, $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ be a sequence of soft points in (X, τ) and $P_e^x \in SP(X, E)$. P_e^x is called a cluster soft point of the sequence $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ if for every $F_A \in \mathcal{N}_\tau(P_e^x)$ and for every $n_0 \in \mathbb{N}$, there is some $n \geq n_0$ such that $P_{e_n}^{x_n} \widetilde{\in} F_A$.

Remark 3.27. If a sequence $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ in a soft topological space (X, τ) converges to $P_e^x \in SP(X, E)$, then this soft point is a cluster soft point of this sequence. But the converse is not always true. For example, consider the soft topological space $(\mathbb{R}, \tau)$ as defined in Example 3.7. The sequence $\{P_e^{(-1)^n} : n \in \mathbb{N}\}$ of soft points in $(\mathbb{R}, \tau)$ has two cluster soft points, P_e^1 and P_e^{-1}; but does not converge.

Definition 3.28 ([16])**.** A directed set is a pair $(D, \gtrsim)$ where D is a non-empty set and $\gtrsim$ is a binary relation on D satisfying:

(d_1) For all $m \in D$, $m \gtrsim m$,

(d_2) For all $m, n, p \in D$, $m \gtrsim n$ and $n \gtrsim p$ imply $m \gtrsim p$,

(d_3) For all $m, n \in D$, there is some $p \in D$ such that $p \gtrsim m, p \gtrsim n$.

We also say that the relation $\gtrsim$ directs the set D.

Example 3.29. Let $\mathcal{N}_\tau(P_e^x)$ be a soft neighborhood system of a soft point P_e^x in a soft topological space (X, τ). Then, the set $\mathcal{N}_\tau(P_e^x)$ with the relation $\sqsubseteq^*$ (that is, $F_A \sqsubseteq^* G_B$ if and only if $G_B \sqsubseteq F_A$) forms a directed set.

Definition 3.30. Let X be a set and $(D, \gtrsim)$ be a directed set. The function $T : D \to SP(X, E)$ is called a soft net in X. In other words, a soft net is a pair $(T, \gtrsim)$ such that $T : D \to SP(X, E)$ is a function and $\gtrsim$ directs the domain of T. For $n \in D$, $T(n)$ is denoted by T_n and hence a soft net is denoted by $\{T_n : n \in D\}$.

Example 3.31. Since $\left(\mathcal{N}_\tau(P_e^x), \sqsubseteq^*\right)$ is a directed set, the function $T : \mathcal{N}_\tau(P_e^x) \to SP(X, E)$ is a soft net, denoted by $\{T_{F_A} : F_A \in \mathcal{N}_\tau(P_e^x)\}$.

Definition 3.32. Let (X, τ) be a soft topological space, $\{T_n : n \in D\}$ be a soft net in X and $F_A \in S(X, E)$.

(i) The soft net $\{T_n : n \in D\}$ is called in F_A if $T_n \widetilde{\in} F_A$, for all $n \in D$.

(ii) The soft net $\{T_n : n \in D\}$ is called eventually in F_A if there exists some $m \in D$ such that $T_n \widetilde{\in} F_A$, for all $n \gtrsim m$.

Definition 3.33. A soft net $\{T_n : n \in D\}$ in a soft topological space (X, τ) is said to converge to $P_e^x \in SP(X, E)$, and we write $T_n \to P_e^x$, if it is eventually in every soft neighborhood of P_e^x.

Example 3.34. Let $X = \{a, b\},\ E = \{e_1, e_2\}$ and

$$\tau = \big\{\Phi, \widetilde{E},\ F_E = \{(e_1, X), (e_2, \{b\})\},\ G_E = \{(e_1, \{a\}), (e_2, X)\},$$
$$F_E \sqcap G_E = \{(e_1, \{a\}), (e_2, \{b\})\}\big\}.$$

The set $\mathcal{N}_\tau(P_{e_1}^a)$ with the relation $\sqsubseteq^*$ form a directed set. Then, $\{T_{F_A} : F_A \in \mathcal{N}_\tau(P_{e_1}^a)\}$ is a soft net in X, where $T_{F_E} = P_{e_2}^a,\ T_{G_E} = P_{e_2}^b,\ T_{F_E \sqcap G_E} = P_{e_1}^a,\ T_{\widetilde{E}} = P_{e_1}^b$, and converges to $P_{e_1}^a$.

Definition 3.35 ([29]). A soft filter on X is a non-empty subset $\mathcal{F} \subseteq S(X,E)$ such that:

(sf_1) $\Phi \notin \mathcal{F}$,

(sf_2) If $F_A, G_B \in \mathcal{F}$, then $F_A \sqcap G_B \in \mathcal{F}$,

(sf_3) If $F_A \in \mathcal{F}$ and $F_A \sqsubseteq G_B$, then $G_B \in \mathcal{F}$.

The definition implies that the intersection of a finite number of members of a filter is non-empty and the union of any number of members of a filter belongs to filter. Also, from (sf_3) it follows that $\widetilde{E} \in \mathcal{F}$.

Example 3.36. Let (X,τ) be a soft topological space. Then, the soft neighborhood system $\mathcal{N}_\tau(P_e^x)$ of a soft point P_e^x is a soft filter on X. It is called the soft neighborhood filter at P_e^x.

Definition 3.37 ([29]). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two soft filters on X. Then, $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ (or $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$) if $\mathcal{F}_1 \subset \mathcal{F}_2$.

Definition 3.38. A soft filter $\mathcal{F}$ on a soft topological space (X,τ) is said to converge to $P_e^x \in SP(X,E)$, and we write $\mathcal{F} \to P_e^x$, if $\mathcal{N}_\tau(P_e^x) \subset \mathcal{F}$.

Definition 3.38 implies that $\mathcal{F}$ converges to P_e^x if $\mathcal{F}$ is finer than the soft neighborhood filter at P_e^x.

4. Soft Hausdorff spaces

In this section, we introduce soft Hausdorff space and establish its some properties. Also, we show that a soft net and a soft filter converge to at most one soft point in soft Hausdorff space.

Definition 4.1. A soft topological space (X,τ) is called soft Hausdorff space or soft T_2-space if for any two distinct soft points $P_{e_1}^{x_1}$, $P_{e_2}^{x_2} \in SP(X,E)$ there exist soft open sets F_A and G_B such that $P_{e_1}^{x_1} \widetilde{\in} F_A, P_{e_2}^{x_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$.

Example 4.2. Let (X,τ) be a soft topological space which is defined in Example 3.2. Then, (X,τ) is a soft Hausdorff space.

Definition 4.3 ([28]). A soft topological space (X,τ) is soft first-countable if there exists a countable soft neighborhood base at every soft point over X.

Lemma 4.4 ([28]). *(X,τ) is a soft first-countable space if and only if every soft point over X has a countable soft open neighborhood base $\{(F_A)_n\}_{n\in\mathbb{N}}$ such that $(F_A)_{n+1} \sqsubseteq (F_A)_n$ for each $n \in \mathbb{N}$.*

Theorem 4.5. *Let (X,τ) be a soft first-countable space.*

(X,τ) is a soft Hausdorff space if and only if every sequence $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ of soft points in (X,τ) converges to at most one soft point.

Proof. Let (X,τ) be a soft Hausdorff space and let us assume that a sequence $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ of soft points in (X,τ) converges to two distinct soft points P_e^x and P_α^y. By the soft Hausdorff property, there exist soft open sets F_A and G_B such that $P_e^x \widetilde{\in} F_A, P_\alpha^y \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Since $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ converges to P_e^x and P_α^y there exists an index $p_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \widetilde{\in} F_A$ and $P_{e_n}^{x_n} \widetilde{\in} G_B$ for all $n \geq p_0$, a contradiction. This mean that every sequence of soft points in (X,τ) converges to at most one soft point.

Conversely, let (X,τ) be a soft first-countable space and assume that every sequence of soft points in (X,τ) converges to at most one soft point, but suppose that (X,τ) is not a soft Hausdorff space. Then, there exist two distinct soft points $P_e^x, P_\alpha^y \in SP(X,E)$ such that every soft neighborhood of P_e^x intersects every soft neighborhood of P_α^y. By Lemma 4.4, P_e^x and P_α^y have countable soft open neighborhood bases $\{(F_A)_n\}_{n\in\mathbb{N}}$ and $\{(G_B)_n\}_{n\in\mathbb{N}}$, respectively such that $(F_A)_{n+1} \sqsubseteq (F_A)_n$ and $(G_B)_{n+1} \sqsubseteq (G_B)_n$ for each $n \in \mathbb{N}$. Because $(F_A)_n \sqcap (G_B)_n \neq \Phi$, for every $n \in \mathbb{N}$ we may choose a soft point $P_{e_n}^{x_n} \widetilde{\in} (F_A)_n \sqcap (G_B)_n$. Then, $\{P_{e_n}^{x_n} : n \in \mathbb{N}\}$ is a sequence of soft points in (X,τ) and converges to both P_e^x and P_α^y, which contradicts our assumption. □

If (X,τ) is a soft topological space which is not soft first-countable, then the sufficient condition of Theorem 4.5 does not hold.

Example 4.6. Let $\mathbb{R}$ be the real numbers, $E = \{e\}$ and

$$\tau = \{F_E : \mathbb{R}\backslash F_E(e) \textit{ is countable}\} \cup \{\Phi\}.$$

Then, $(\mathbb{R},\tau)$ is a soft topological space which is not soft first-countable. Now, let $\{P_e^{x_n} : n \in \mathbb{N}\}$ be a sequence of soft points in $(\mathbb{R},\tau)$ and let us consider this sequence converges to $P_e^x \in SP(\mathbb{R},E)$. Then, this sequence does not converge to any soft point expect for P_e^x since it has the form $\{P_e^{x_1}, P_e^{x_2}, ..., P_e^{x_{n_0}}, P_e^x, P_e^x, ...\}$. Hence, every sequence of soft points in $(\mathbb{R},\tau)$ converges to at most one soft point; but $(\mathbb{R},\tau)$ is not soft Hausdorff space.

Lemma 4.7. *Let (X,τ) be a soft topological space, where $E = \{e\}$, and let $\{P_e^{x_n} : n \in \mathbb{N}\}$ be a sequence of soft points in (X,τ). Then, $P_e^x \in SP(X,E)$ is a cluster soft point of the sequence $\{P_e^{x_n} : n \in \mathbb{N}\}$ if and only if $P_e^x \widetilde{\in} \overline{(F_n)_E}$, for all $n \in \mathbb{N}$, where $(F_n)_E = \bigsqcup_{n' \geq n} P_e^{x_{n'}}$.*

Proof. The sufficiency is clear from Theorem 3.20 and the definition of cluster soft point.

To prove necessity, let $P_e^x \in SP(X,E)$ be a cluster soft point of $\{P_e^{x_n} : n \in \mathbb{N}\}$. Then, for every $F_E \in \mathcal{N}_\tau(P_e^x)$ and for every $n_0 \in \mathbb{N}$, there is some $n \geq n_0$ such that $P_e^{x_n} \widetilde{\in} F_E$. Therefore, every soft neighborhood of P_e^x intersects $(F_n)_E$ for all $n \in \mathbb{N}$. By Theorem 3.20, $P_e^x \widetilde{\in} \overline{(F_n)_E}$ for all $n \in \mathbb{N}$. □

Theorem 4.8. *Let (X,τ) be a soft Hausdorff space, where $E = \{e\}$, and let $\{P_e^{x_n} : n \in \mathbb{N}\}$ be a sequence of soft points in (X,τ). If $\{P_e^{x_n} : n \in \mathbb{N}\} \to P_e^x \in SP(X,E)$, then P_e^x is the unique cluster soft point of $\{P_e^{x_n} : n \in \mathbb{N}\}$.*

Proof. Firstly, we show that the soft point P_e^x is a cluster soft point of $\{P_e^{x_n} : n \in \mathbb{N}\}$. By Lemma 4.7, we show that $P_e^x \widetilde{\in} \overline{(F_n)_E}$ for all $n \in \mathbb{N}$, where $(F_n)_E = \bigsqcup_{n' \geq n} P_e^{x_{n'}}$. Since $\{P_e^{x_n} : n \in \mathbb{N}\} \to P_e^x$, for every $F_E \in \mathcal{N}_\tau(P_e^x)$, there exists an $n_0 \in \mathbb{N}$ such that $P_e^{x_n} \widetilde{\in} F_E$ for all $n \geq n_0$. Then, $F_E \sqcap (F_n)_E \neq \Phi$ for all $n \geq n_0$. If $n < n_0$, then $(F_{n_0})_E \sqsubseteq (F_n)_E$ and so $F_E \sqcap (F_n)_E \neq \Phi$. Hence $P_e^x \widetilde{\in} \overline{(F_n)_E}$ for all $n \in \mathbb{N}$.

Now, we show that the soft point P_e^x is unique. Assume that $P_e^y \in SP(X,E)$ $(x \neq y)$ is also a cluster soft point of $\{P_e^{x_n} : n \in \mathbb{N}\}$. Because (X,τ) is a soft Hausdorff space, there exist soft open sets F_E and G_E such that $P_e^x \widetilde{\in} F_E, P_e^y \widetilde{\in} G_E$

and $F_E \sqcap G_E = \Phi$. Since $\{P_e^{x_n} : n \in \mathbb{N}\} \to P_e^x$, there exists an $n_0 \in \mathbb{N}$ such that $P_e^{x_n} \widetilde{\in} F_E$ for all $n \geq n_0$. Therefore, $(F_{n_0})_E \sqsubseteq F_E$. On the other hand, $P_e^y \widetilde{\in} \overline{(F_n)_E}$ for all $n \in \mathbb{N}$ from Lemma 4.7. Then, for $n = n_0$, we have $P_e^y \widetilde{\in} \overline{(F_{n_0})_E}$ and by Theorem 3.20, $G_E \sqcap (F_{n_0})_E \neq \Phi$. This is a contradiction. □

Theorem 4.9. *A soft topological space (X, τ) is a soft Hausdorff space if and only if every soft net in (X, τ) converges to at most one soft point.*

Proof. Let (X, τ) be a soft Hausdorff space and let us assume that a soft net $\{T_n : n \in D\}$ converges to two distinct soft points $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$. By the soft Hausdorff property, there exist soft open sets F_A and G_B such that $P_{e_1}^{x_1} \widetilde{\in} F_A, P_{e_2}^{x_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. As $\{T_n : n \in D\}$ converges to $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$ there exists an index $k \in D$ such that $T_k \widetilde{\in} F_A$ and $T_k \widetilde{\in} G_B$, a contradiction. This mean that every soft net in X converges to at most one soft point.

For the converse, assume that every soft net in (X, τ) converges to at most one soft point, but suppose that (X, τ) is not a soft Hausdorff space. This implies that there exist two distinct soft points $P_{e_1}^{x_1}, P_{e_2}^{x_2} \in SP(X, E)$ such that every soft neighborhood of $P_{e_1}^{x_1}$ intersects every soft neighborhood of $P_{e_2}^{x_2}$. Since $\mathcal{N}_\tau(P_{e_1}^{x_1})$ and $\mathcal{N}_\tau(P_{e_2}^{x_2})$ are directed sets, $\mathcal{N}_\tau(P_{e_1}^{x_1}) \times \mathcal{N}_\tau(P_{e_2}^{x_2})$ with the relation $\sqsubseteq^*$ (that is, $(F_A, G_B) \sqsubseteq^* ((F_A)_1, (G_B)_1)$ if and only if $F_A \sqsupseteq (F_A)_1$ and $G_B \sqsupseteq (G_B)_1$) form a directed set. As $F_A \sqcap G_B \neq \Phi$, for every $(F_A, G_B) \in \mathcal{N}_\tau(P_{e_1}^{x_1}) \times \mathcal{N}_\tau(P_{e_2}^{x_2})$ we may select a soft point $T_{(F_A,G_B)} \widetilde{\in} F_A \sqcap G_B$. Then, $\{T_{(F_A,G_B)} : (F_A, G_B) \in \mathcal{N}_\tau(P_{e_1}^{x_1}) \times \mathcal{N}_\tau(P_{e_2}^{x_2})\}$ is a soft net. Now, we observe that $T_{((F_A)_1,(G_B)_1)} \widetilde{\in} (F_A)_1 \sqcap (G_B)_1 \sqsubseteq F_A \sqcap G_B$, for all $((F_A)_1, (G_B)_1) \in \mathcal{N}_\tau(P_{e_1}^{x_1}) \times \mathcal{N}_\tau(P_{e_2}^{x_2})$ such that $(F_A, G_B) \sqsubseteq^* ((F_A)_1, (G_B)_1)$. Consequently, this soft net converges to $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$, which contradicts our assumption. □

In a soft topological space which is not soft Hausdorff space, a soft net can converge to more than one soft point.

Example 4.10. Let $\mathbb{R}$ be the real numbers, $E = \{e\}$ and $(F_\lambda)_E = \{ \big(e, (2 - \lambda, 2 + \lambda)\big)\}$, where $\lambda \in \mathbb{R}^+$. Then, $\tau = \{(F_\lambda)_E : \lambda \in \mathbb{R}^+\} \cup \{\Phi, \widetilde{E}\}$ is a soft topological space which is not soft Hausdorff space. Since the set $\mathbb{R}\backslash\{0\}$ with the relation $\leq$ form a directed set, $\{T_n : n \in \mathbb{R}\backslash\{0\}\}$ is a soft net in $\mathbb{R}$, where $T_n = P_e^{2-\frac{1}{n}}$ for all $n \in \mathbb{R}\backslash\{0\}$, and converge to more than one soft point.

Theorem 4.11. *A soft topological space (X, τ) is a soft Hausdorff space if and only if every soft filter on (X, τ) converges to at most one soft point.*

Proof. Let (X, τ) be a soft Hausdorff space and $\mathcal{F}$ be a soft filter on (X, τ). Suppose that $\mathcal{F}$ converges to two distinct soft points $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$. By the soft Hausdorff property, there exist soft open sets F_A and G_B such that $P_{e_1}^{x_1} \widetilde{\in} F_A, P_{e_2}^{x_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Since $\mathcal{F}$ converges to $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$, then $F_A, G_B \in \mathcal{F}$. Therefore, $F_A \sqcap G_B = \Phi \in \mathcal{F}$, contradicting the definition of soft filter.

Conversely, assume that every soft filter on (X, τ) converges to at most one soft point, but suppose that (X, τ) is not a soft Hausdorff space. Then, there are two distinct soft points $P_{e_1}^{x_1}, P_{e_2}^{x_2} \in SP(X, E)$ such that every pair of soft neighborhoods F_A of $P_{e_1}^{x_1}$ and G_B of $P_{e_2}^{x_2}$ intersect. Thus, $\mathcal{F} = \{F_A \sqcap G_B : F_A \in \mathcal{N}_\tau(P_{e_1}^{x_1}), G_B \in$

$\mathcal{N}_\tau(P_{e_2}^{x_2})\}$ is a soft filter on X. Since every soft neighborhood of $P_{e_1}^{x_1}$ and every soft neighborhood of $P_{e_2}^{x_2}$ belongs to $\mathcal{F}$, we get $\mathcal{F} \to P_{e_1}^{x_1}$ and $\mathcal{F} \to P_{e_2}^{x_2}$. This is a contradiction. □

Theorem 4.12. *A soft topological space (X, τ) is a soft Hausdorff space if and only if the intersection of all soft closed neighborhoods of a soft point equals the soft point itself.*

Proof. Let (X, τ) be a soft Hausdorff space and $P_{e_1}^{x_1} \in SP(X,E)$. If $P_{e_2}^{x_2} \in SP(X,E)$ $(P_{e_1}^{x_1} \neq P_{e_2}^{x_2})$ then, since (X, τ) is a soft Hausdorff space, there exist soft open sets F_A and G_B such that $P_{e_1}^{x_1} \widetilde{\in} F_A, P_{e_2}^{x_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. If $F_A \sqcap G_B = \Phi$, then we have $P_{e_1}^{x_1} \widetilde{\in} F_A \sqsubseteq G_B^c$. Therefore, G_B^c is a soft closed neighborhood of $P_{e_1}^{x_1}$ not containing $P_{e_2}^{x_2}$. Hence, the intersection of all soft closed neighborhoods of $P_{e_1}^{x_1}$ does not contain any soft point except for $P_{e_1}^{x_1}$.

Conversely, let $P_{e_1}^{x_1}, P_{e_2}^{x_2} \in SP(X,E)$ be any two distinct soft points and let $\{(F_A)_i\}_{i\in J}$ be a family of all soft closed neighborhoods of $P_{e_1}^{x_1}$. Since $\sqcap_{i\in J}(F_A)_i = P_{e_1}^{x_1}$, there exists an $i_0 \in J$ such that $x_2 \notin (F_A)_{i_0}(e_2)$. Then, there exist soft neighborhoods $(F_A)_{i_0}$ of $P_{e_1}^{x_1}$ and $(F_A)_{i_0}^c$ of $P_{e_2}^{x_2}$ such that $(F_A)_{i_0} \sqcap (F_A)_{i_0}^c = \Phi$. Thus, (X, τ) is a soft Hausdorff space. □

Proposition 4.13. *Let $\varphi_\psi : S(X,E) \to S(Y,K)$ be a soft mapping and $P_e^x \in SP(X,E)$. Then $\varphi_\psi(P_e^x) = P_{\psi(e)}^{\varphi(x)} \in SP(Y,K)$.*

Proof. It is clear from the Definition 2.14. □

Proposition 4.14. *Let $\varphi_\psi : S(X,E) \to S(Y,K)$ be a soft mapping and $P_k^y \in SP(Y,K)$. If φ_ψ is bijective, then $\varphi_\psi^{-1}(P_k^y) = P_{\psi^{-1}(k)}^{\varphi^{-1}(y)} \in SP(X,E)$.*

Proof. Using the Definition 2.14, we can easily prove it. □

Theorem 4.15. *Let the soft mapping $\varphi_\psi : (X,\tau_1) \to (Y,\tau_2)$ be injective and soft continuous. If (Y,τ_2) is soft Hausdorff space, then (X,τ_1) is also soft Hausdorff space.*

Proof. Let $P_{e_1}^{x_1}, P_{e_2}^{x_2} \in SP(X,E)$ be any two distinct soft points. By Proposition 4.13, $\varphi_\psi(P_{e_1}^{x_1}) = P_{\psi(e_1)}^{\varphi(x_1)}$, $\varphi_\psi(P_{e_2}^{x_2}) = P_{\psi(e_2)}^{\varphi(x_2)} \in SP(Y,K)$. Since φ_ψ is injective, we have $P_{\psi(e_1)}^{\varphi(x_1)} \neq P_{\psi(e_2)}^{\varphi(x_2)}$. Since (Y,τ_2) is soft Hausdorff space, there exist soft open sets F_A and G_B such that $P_{\psi(e_1)}^{\varphi(x_1)} \widetilde{\in} F_A, P_{\psi(e_2)}^{\varphi(x_2)} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Because φ_ψ is soft continuous, the sets $\varphi_\psi^{-1}(F_A)$ and $\varphi_\psi^{-1}(G_B)$ are disjoint soft open sets in (X,τ_1) containing $P_{e_1}^{x_1}$ and $P_{e_2}^{x_2}$, respectively. This implies that (X,τ_1) is a soft Hausdorff space. □

Theorem 4.16. *Let the soft mapping $\varphi_\psi : (X,\tau_1) \to (Y,\tau_2)$ be bijective and soft open. If (X,τ_1) is soft Hausdorff space, then (Y,τ_2) is also soft Hausdorff space.*

Proof. Let $P_{k_1}^{y_1}, P_{k_2}^{y_2} \in SP(Y,K)$ be any two distinct soft points. By Proposition 4.14, $\varphi_\psi^{-1}(P_{k_1}^{y_1}) = P_{\psi^{-1}(k_1)}^{\varphi^{-1}(y_1)}$, $\varphi_\psi^{-1}(P_{k_2}^{y_2}) = P_{\psi^{-1}(k_2)}^{\varphi^{-1}(y_2)} \in SP(X,E)$. Since φ_ψ is surjective, we have $P_{\psi^{-1}(k_1)}^{\varphi^{-1}(y_1)} \neq P_{\psi^{-1}(k_2)}^{\varphi^{-1}(y_2)}$. Since (X,τ_1) is soft Hausdorff space, there exist soft open sets F_A and G_B such that $P^{\varphi^{-1}(y_1)}_{\psi^{-1}(k_1)} \widetilde{\in} F_A, P^{\varphi^{-1}(y_2)}_{\psi^{-1}(k_2)} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. As φ_ψ is soft open, the sets $\varphi_\psi(F_A)$ and $\varphi_\psi(G_B)$ are soft open sets in (Y, τ_2) containing $P^{y_1}_{k_1}$ and $P^{y_2}_{k_2}$, respectively. Also, $\varphi_\psi(F_A)$ and $\varphi_\psi(G_B)$ are disjoint because φ_ψ is injective. Thus, (Y, τ_2) is a soft Hausdorff space. □

Theorem 4.17. *Let (X, τ) be a soft topological space and S be a non-empty subset of X. If (X, τ) is a soft Hausdorff space, then (S, τ_S) is a soft Hausdorff space.*

Proof. Let $P^{x_1}_{e_1}, P^{x_2}_{e_2} \widetilde{\in} \widetilde{E_S}$ be any two distinct soft points. Then, $P^{x_1}_{e_1}, P^{x_2}_{e_2} \widetilde{\in} \widetilde{E}$ and so there exist soft open sets F_A and G_B in (X, τ) such that $P^{x_1}_{e_1} \widetilde{\in} F_A, P^{x_2}_{e_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Therefore, $\widetilde{E_S} \sqcap F_A$ and $\widetilde{E_S} \sqcap G_B$ are soft open sets in (S, τ_S) containing $P^{x_1}_{e_1}$ and $P^{x_2}_{e_2}$, respectively. Also, $\widetilde{E_S} \sqcap F_A$ and $\widetilde{E_S} \sqcap G_B$ are disjoint because $F_A \sqcap G_B = \Phi$. Hence, (S, τ_S) is a soft Hausdorff space. □

Lemma 4.18 ([31])**.** *Let (X, τ_1) and (Y, τ_2) be two soft topological spaces. Then, X and Y are homeomorphic to a subspace of $X \times Y$.*

Theorem 4.19. *Let (X, τ_1) and (Y, τ_2) be two soft topological spaces. Then, (X, τ_1) and (Y, τ_2) are soft Hausdorff spaces if and only if $(X \times Y, \tau_1 \times \tau_2)$ is soft Hausdorff space.*

Proof. Let (X, τ_1) and (Y, τ_2) be soft Hausdorff spaces and let $P^{(x_1,y_1)}_{(e_1,k_1)}, P^{(x_2,y_2)}_{(e_2,k_2)} \in SP(X \times Y, E \times K)$ be any two distinct soft points. Then either $P^{x_1}_{e_1} \neq P^{x_2}_{e_2}$ or $P^{y_1}_{k_1} \neq P^{y_2}_{k_2}$. Take $P^{x_1}_{e_1} \neq P^{x_2}_{e_2}$. Since (X, τ_1) is soft Hausdorff space, there exist soft open sets F_A and G_B in (X, τ_1) such that $P^{x_1}_{e_1} \widetilde{\in} F_A, P^{x_2}_{e_2} \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Then, $F_A \times \widetilde{K}$ and $G_B \times \widetilde{K}$ are soft open sets in $(X \times Y, \tau_1 \times \tau_2)$ such that $P^{(x_1,y_1)}_{(e_1,k_1)} \widetilde{\in} F_A \times \widetilde{K}, P^{(x_2,y_2)}_{(e_2,k_2)} \widetilde{\in} G_B \times \widetilde{K}$ and $(F_A \times \widetilde{K}) \sqcap (G_B \times \widetilde{K}) = \Phi$. Thus, $(X \times Y, \tau_1 \times \tau_2)$ is soft Hausdorff space.

For the converse, let $X \times Y$ be soft Hausdorff space. It follows directly from the Theorem 4.17 and Lemma 4.18 that X and Y are soft Hausdorff spaces. □

5. Conclusions

In the present work, we mainly introduce soft Hausdorff spaces and establish some of their properties. Also, we have shown that a soft net and a soft filter converge to at most one soft point in soft Hausdorff space. Since there are close relations between soft sets and information systems, we can utilize the results inferred from this study to improve these kinds of relations. We believe that these results will help the researchers to advance and promote the further study on soft topology to carry out a general framework for their applications in practical life.

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