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Some classes of fuzzy sets in a generalized fuzzy topological spaces and certain unifications

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Abstract. The paper deals basically with a kind of generalized fuzzy closed sets and the corresponding fuzzy open-like sets in a generalized fuzzy topological space. The motive behind such an investigation is to present a unified approach towards different earlier studies concerning many types of such generalized open-like fuzzy sets. Two fuzzy separation axioms viz. regularity and normality are also introduced and studied in our setting, which yield many characterizations and properties of similar such separation axioms in different particular settings.

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1. Introduction and preliminary results

It is observed from the literature that a large number of papers is devoted to the study of different generalized types of fuzzy open-like sets in fuzzy topological spaces. Mention may be made, for instance, of gf-open sets [3], Fsg-open sets [2, 12], Fpg-open sets [7], $Fg\alpha$-open sets [15], fbg-open sets [5] etc. It is seen from such investigations that many features and properties of these apparently different looking sets are common or similar. It has thus been the aim of this paper to unify such existing types of generalized fuzzy open-like sets and their properties by using the concept of generalized fuzzy topology introduced by Chetty [10] in 2008.

Let X be a nonempty set and I^X (where I = [0,1]) be the collection of all fuzzy sets[20] defined on X. A subcollection μ of I^X is called a generalized fuzzy topology (GFT, for short)[10] if the constant zero fuzzy set $0_X \in \mu$ and μ is closed under arbitrary unions of its members. A nonempty set X together with a GFT μ on X is denoted by (X, μ), called a generalized fuzzy topological space (GFTS, for short). We shall call the members of μ fuzzy μ-open (briefly μf-open) sets, and the complements of μf-open sets will be called fuzzy μ-closed (μf-closed, for short). For any $A \in I^X$, the μ-closure of A, denoted by $c_\mu(A)$, will mean the fuzzy set given by $c_\mu(A) = \bigwedge\{F : A \leq F,\ F$ is μf-closed $\}$. Similarly, the μ-interior of A is denoted by $i_\mu(A)$ and is given by $i_\mu(A) = \bigvee\{V \in \mu : V \leq A\}$.

We now recall that a fuzzy set A in a fuzzy topological space (henceforth abbreviated as an fts)[9] (X, τ) is called fuzzy semiopen [1] (preopen [18], α-open [6], b-open [5]) if $A \leq \mathrm{cl}(\mathrm{int}(A))$ (resp. $A \leq \mathrm{int}(\mathrm{cl}(A))$, $A \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, $A \leq (\mathrm{int}(\mathrm{cl}(A)) \bigvee \mathrm{cl}(\mathrm{int}(A)))$. The collections of such sets are denoted by $FSO(X)$, $FPO(X)$, $F\alpha O(X)$ and $FBO(X)$ respectively.

Remark 1.1. It is now clear that for an fts (X, τ), if we take $\mu = \tau$, $FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $FBO(X)$ then c_μ becomes cl (fuzzy closure), scl (fuzzy semi-closure [19]), pcl (fuzzy pre-closure [18]), αcl (fuzzy α-closure [14]), bcl (fuzzy b-closure [4]) respectively.

It is straightforward to check that μ-closure and μ-interior, as defined above, obey the following usual properties of closure and interior operators.

Result 1.2. *In a GFTS (X, μ), the following properties hold :*

(a) *For any A, $B \in I^X$; $A \leq B \Rightarrow i_\mu(A) \leq i_\mu(B)$ and $c_\mu(A) \leq c_\mu(B)$.*

(b) *For any $A \in I^X$, $i_\mu(A)$ (resp. $c_\mu(A)$) is a fuzzy μ-open (resp. μ-close) set contained in (resp. containing) A.*

(c) *$A \in I^X$ is fuzzy μ-open (resp. μ-closed) iff $A = i_\mu(A)$ (resp. $A = c_\mu(A)$).*

(d) *For any $A \in I^X$, $i_\mu(i_\mu(A)) = i_\mu(A)$ and $c_\mu(c_\mu(A)) = c_\mu(A)$.*

(e) *For any $A \in I^X$, $1 - c_\mu(A) = i_\mu(1 - A)$.*

Proof. We verify only two of the results, which go in the usual way and the straightforward demonstrations of other parts are omitted.

(d) We obviously have $i_\mu(A) \leq A$ and hence $i_\mu(i_\mu(A)) \leq i_\mu(A)$. Now, $V \in \mu$ such that $V \leq A \Rightarrow V \leq i_\mu(A) \Rightarrow V \leq i_\mu(i_\mu(A)) \Rightarrow i_\mu(A) \leq i_\mu(i_\mu(A))$.

(e) We have, $i_\mu(1-A)$ is the largest μf-open set contained in $1-A \Rightarrow 1-i_\mu(1-A)$ is the smallest μf-closed set containing $A \Rightarrow c_\mu(A) = 1 - i_\mu(1 - A) \Rightarrow 1 - c_\mu(A) = i_\mu(1 - A)$. □

As defined by Pu and Liu [13], a fuzzy set A is called quasi-coincident with a fuzzy set B, written as AqB, if there exists some $x \in X$ such that $A(x) + B(x) > 1$; the negation of such a statement will be denoted as $A\overline{q}B$. Following Pu and Liu [13], we shall call a μf-open set U a μf-open q-neighbourhood of a fuzzy point x_α in a GFTS (X, μ) if $x_\alpha qU$ (i. e., $\alpha + U(x) > 1$). It is shown by Pu and Liu [13] that a fuzzy point x_α is in the closure of a fuzzy set A in an fts (X, τ) iff every fuzzy open q-neighbouhood U of x_α is quasi-coincident with A. A similar demonstration proves the following:

Result 1.3. *For a fuzzy set A and a fuzzy point x_α in a GFTS (X, μ), $x_\alpha \leq c_\mu(A)$ iff every μf-open q-neighbourhood U of x_α is quasi-coincident with A.*

In Section 2 of this paper, we initiate the study of $\mu g f$-closed sets and some of their properties. Section 3 is devoted to the introduction of μf-regular and μf-normal spaces, where we obtain some characterizations of such spaces. In the last section, we introduce and investigate the generalized fuzzy (μ, λ)-continuous functions and also obtain some preservation theorems of μf-closed sets, μf-regular and μf-normal spaces.

Our prime motivation in all these deliberations is first to demonstrate the simple nature of the proofs of many existing results and secondly to obtain unified versions of many more new results under different particular interpretations and choices of μ.

2. Generalized fuzzy μ-closed sets

Definition 2.1. Let (X, μ) be a GFTS. Then $A \in I^X$ is said to be a generalized fuzzy μ-closed set (μgf-closed, for short) if $c_\mu(A) \leq U$ whenever $A \leq U$ and $U \in \mu$. The complement of a μgf-closed set is said to be generalized fuzzy μ-open (μgf-open, for short).

Remark 2.2. (i) Let (X, τ) be an fts. Then the definition of gf-open [3] (resp. Fsg-open [2, 12], Fpg-open [7], $Fg\alpha$-open [15] and fbg-open [5]) set can be obtained by taking $\mu = \tau$ ($= FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $FBO(X)$ respectively).

(ii) Clearly every μf-closed set in a GFTS (X, μ) is a μgf-closed set; that the converse is false is shown by the following example.

Example 2.3. Let $X = \{a, b\}$. We define $A, B : X \to [0, 1]$ by

$$A(a) = 0,\ A(b) = \tfrac{1}{3};$$
$$B(a) = \tfrac{1}{2},\ B(b) = \tfrac{1}{3}.$$

Then (X, μ) is a GFTS where $\mu = \{O_X, 1_X, A\}$. It is easy to see that B is a μgf-closed set but it is not μf-closed.

Theorem 2.4. *Let (X, μ) be a GFTS and $A, B \in I^X$ be such that $A \leq B \leq c_\mu(A)$. If A is μgf-closed then B is also μgf-closed.*

Proof. Let U be a μf-open set such that $B \leq U$. Since $A \leq B$, we have $A \leq U$. But A is μgf-closed , so $c_\mu(A) \leq U$. Again since $B \leq c_\mu(A)$, $c_\mu(B) \leq c_\mu(c_\mu(A)) = c_\mu(A) \leq U \Rightarrow B$ is μgf-closed. □

Corollary 2.5. *μ-closure of every μgf-closed set is μgf-closed.*

Corollary 2.6. *Let (X, μ) be a GFTS and $A, B \in I^X$ be such that $i_\mu(A) \leq B \leq A$. If A is μgf-open then B is also μgf-open.*

Theorem 2.7. *Let (X, μ) be a GFTS. A fuzzy set $A \in I^X$ is μgf-open iff $F \leq i_\mu(A)$ whenever F is μf-closed and $F \leq A$.*

Proof. Let A be a μgf-open set and let F be a μf-closed set such that $F \leq A$. Then $1 - A \leq 1 - F$ and hence by hypothesis, $c_\mu(1 - A) = 1 - i_\mu(A) \leq 1 - F \Rightarrow F \leq i_\mu(A)$. Conversely suppose that A is a fuzzy set such that $F \leq i_\mu(A)$ whenever F is μf-closed and $F \leq A$. We claim that $1 - A$ is μgf-closed. In fact, let $1 - A \leq U$ where $U \in \mu$. Then $1 - U \leq A$ and so by hypothesis, we have $1 - U \leq i_\mu(A) \Rightarrow 1 - i_\mu(A) = c_\mu(1 - A) \leq U \Rightarrow 1 - A$ is a μgf-closed set $\Rightarrow A$ is μgf-open. □

Theorem 2.8. *Let (X, μ) be a GFTS. Then every fuzzy subset in X is μgf-closed iff $\mu = \lambda$ (the collection of all μf-closed sets in X).*

Proof. Suppose that every fuzzy set is a μgf-closed set. Let $A \in \mu$. Then A is μgf-closed and thus $c_\mu(A) \leq A \Rightarrow A \in \lambda$. Again let $F \in \lambda$. Then $1 - F$ is μf-open i.e., $1 - F \in \mu \Rightarrow 1 - F \in \lambda$ (as $\mu \leq \lambda$) $\Rightarrow F \in \mu$.
Conversely suppose that $\mu = \lambda$. Let $A \in I^X$ be such that $A \leq U$, where $U \in \mu$. Then U is μf-closed. Thus $c_\mu(A) \leq c_\mu(U) = U \Rightarrow A$ is a μgf-closed set. □

Example 2.9. The intersection of two μgf-closed sets is not in general a μgf-closed set. In fact, let $X = \{a, b, c\}$. Consider the fuzzy sets A, B, C as follows:

$$A(a) = 1,\ A(b) = 0,\ A(c) = 0;$$
$$B(a) = 1,\ B(b) = 1,\ B(c) = 0;$$
$$C(a) = 1,\ C(b) = 0,\ C(c) = 1.$$

Then (X, μ) is a GFTS, where $\mu = \{0_X, 1_X, A\}$. It is easy to check that B, C are μgf-closed, but their intersection $B \bigwedge C$ is not a μgf-closed set.

Remark 2.10. As seen from the above example, the union of two μgf-open sets is not necessarily μgf-open.

3. Fuzzy μ-regular and μ-normal spaces

Definition 3.1. A GFTS (X, μ) is said to be fuzzy μ-regular (μf-regular, for short) if for each fuzzy point x_α in X and each $U \in \mu$ with $x_\alpha qU$, there exists $V \in \mu$ such that $x_\alpha qV$ and $c_\mu(V) \leq U$.

Theorem 3.2. *For a GFTS (X, μ), the following are equivalent:*
(a) X is μf-regular.
(b) For each fuzzy point x_α and each μf-closed set F with $x_\alpha \not\leq F$, there exists a $U \in \mu$ such that $x_\alpha \not\leq c_\mu(U)$ and $F \leq U$.
(c) For each fuzzy point x_α and each μf-closed set F with $x_\alpha \not\leq F$, there exist U, $V \in \mu$, such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$.
(d) For each $A \in I^X$ and any μf-closed set F with $A \not\leq F$, there exist U, $V \in \mu$ such that AqU, $F \leq V$ and $U\overline{q}V$.
(e) For any $A \in I^X$ and any $U \in \mu$ with AqU, there exists a $V \in \mu$ such that AqV and $c_\mu(V) \leq U$.
(f) For each fuzzy point x_α and each μf-closed set F with $x_\alpha \not\leq F$, there exist a μf-open set U and a μgf-open set V such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$.
(g) For each $A \in I^X$ and any μf-closed set F with $A \not\leq F$, there exist a $U \in \mu$ and a μgf-open set V such that AqU, $F \leq V$, $F \leq V$ and $U\overline{q}V$.

Proof. (a) $\Rightarrow$ (b): Suppose x_α is any fuzzy point in a μf-regular space X and F is a μf-closed set such that $x_\alpha \not\leq F$. Then $x_\alpha q(1-F) = V$ (say) and V is a μf-open set. Then by (a), there exists some $W \in \mu$ such that $x_\alpha qW$ and $c_\mu(W) \leq V$. Put $U = 1 - c_\mu(W)$. Now, $x_\alpha q\ i_\mu(c_\mu(W)) \Rightarrow x_\alpha \not\leq 1 - i_\mu(c_\mu(W)) = c_\mu(1 - c_\mu(W)) = c_\mu(U)$. Thus U is a μf-open set such that $x_\alpha \not\leq c_\mu(U)$ and also $F = 1-V \leq 1-c_\mu(W) = U$.

(b) $\Rightarrow$ (c): Let x_α be any fuzzy point in X and F any μf-closed set such that $x_\alpha \not\leq F$. Then by (b), there exists a μf-open set V such that $x_\alpha \not\leq c_\mu(V)$ and $F \leq V$. Now $x_\alpha \not\leq c_\mu(V)$ and hence by Result 1.3, there exists a μf-open sets U such that $x_\alpha qU$ and $U\overline{q}V$.

$(c) \Rightarrow (d)$: Let $A \in I^X$ be such that $A \nleq F$, where F is μf-closed. Then there exists a fuzzy point $x_\alpha \leq A$ such that $x_\alpha \nleq F$. By (c), there exist μf-open sets U and V such that $x_\alpha q U$, $F \leq V$ and $U \overline{q} V$. Since $x_\alpha \leq A$, we have AqU.

$(d) \Rightarrow (e)$: Let $A \in I^X$ be such that AqU where $U \in \mu$. Then $A \nleq 1 - U$ where $1 - U$ is μf-closed and hence by (d), there exist μf-open sets V and W such that AqV, $1 - U \leq W$ and $V \overline{q} W$. Thus $V \leq 1 - W \Rightarrow c_\mu(V) \leq 1 - W \leq U$ i.e., there exists a $V \in \mu$ such that AqV and $c_\mu(V) \leq U$.

$(e) \Rightarrow (a)$: Obvious.

$(c) \Rightarrow (f)$: Follows from the fact that every μf-open set is $\mu g f$-open.

$(f) \Rightarrow (g)$: Let $A \in I^X$ and F be a μf-closed set such that $A \nleq F$. Then there exists a fuzzy point $x_\alpha \leq A$ such that $x_\alpha \nleq F$. Then by (f), there exist a μf-open set U and a $\mu g f$-open set V such that $x_\alpha q U$, $F \leq V$ and $U \overline{q} V$. Since $x_\alpha \leq A$, we have AqU.

$(g) \Rightarrow (c)$: Let x_α be a fuzzy point in X and F any μf-closed set such that $x_\alpha \nleq F$. Then by (g), there exist a $U \in \mu$ and a $\mu g f$-open set W such that $x_\alpha q U$, $F \leq W$ and $U \overline{q} W$. By Theorem 2.7, we have $F \leq i_\mu(W)$. Put $V = i_\mu(W)$. Then V is a μf-open set such that $F \leq V$ and $U \overline{q} V$. □

Definition 3.3. A GFTS (X, μ) is said to be fuzzy μ-normal (μf-normal, for short) if for any μf-closed set F and any μf-open set U with $F \leq U$, there exists a μf-open set V such that $F \leq V \leq c_\mu(V) \leq U$.

Theorem 3.4. *For a GFTS (X, μ) the following are equivalent:*

(a) *X is μf-normal.*

(b) *For any two μf-closed sets F and G with $F \overline{q} G$, there exist μf-open sets U and V such that $F \leq U$, $G \leq V$ and $U \overline{q} V$.*

(c) *For any two μf-closed sets F and G with $F \overline{q} G$, there exist $\mu g f$-open sets U and V such that $F \leq U$, $G \leq V$ and $U \overline{q} V$.*

(d) *For any μf-closed set F and any μf-open set U with $F \leq U$, there exists a $\mu g f$-open set V such that $F \leq V \leq c_\mu(V) \leq U$.*

(e) *For any μf-closed set F and a $\mu g f$-open set U with $F \leq U$, there exists a μf-open set V such that $F \leq V \leq c_\mu(V) \leq i_\mu(U)$.*

(f) *For any $\mu g f$-closed set F and any μf-open set U with $F \leq U$, there exists a μf-open set V such that $F \leq c_\mu(F) \leq V \leq c_\mu(V) \leq U$.*

Proof. $(a) \Rightarrow (b)$: For any two μf-closed sets F and G with $F \overline{q} G$, we have $F \leq 1 - G$ where $1 - G$ is μf-open. Then there exists a μf-open set U such that $F \leq U \leq c_\mu(U) \leq 1 - G$. Put $V = (1 - c_\mu(U))$. Then U and V are μf-open sets such that $F \leq U$, $G \leq V$ and $U \overline{q} V$.

$(b) \Rightarrow (c)$: Follows from Remark 2.2.

$(c) \Rightarrow (d)$: Let F be a μf-closed set and U a μf-open set such that $F \leq U$. Then $F \overline{q} (1 - U)$ and hence by (c), there exist $\mu g f$-open sets V and W such that $F \leq V$, $1 - U \leq W$ and $V \overline{q} W$. Now $V \overline{q} W \Rightarrow V \leq 1 - W$ so that $F \leq V \leq 1 - W \leq U$. Again, since U is μf-open and $1 - W$ is $\mu g f$-closed, we have $c_\mu(1 - W) \leq U$. Hence $F \leq V \leq c_\mu(V) \leq U$.

$(d) \Rightarrow (e)$: Let F be a μf-closed set and U a $\mu g f$-open set such that $F \leq U$. Since U is $\mu g f$-open and $F \leq U$ where F is μf-closed, by Theorem 2.7 we have $F \leq i_\mu(U)$. Hence by (d), there exists a μf-open set V such that $F \leq V \leq c_\mu(V) \leq i_\mu(U)$.

$(e) \Rightarrow (f)$: Let F be a μgf-closed set and U a μf-open set such that $F \leq U$. Then $c_\mu(F) \leq U$. Again U is μgf-open (by Remark 2.2). Hence by (e), there exists a μf-open set V such that $F \leq c_\mu(F) \leq V \leq c_\mu(V) \leq i_\mu(U) \Rightarrow F \leq c_\mu(F) \leq V \leq c_\mu(V) \leq U$.

$(f) \Rightarrow (a)$: Obvious. □

4. Generalized fuzzy continuous functions

Definition 4.1. Let (X, μ) and (Y, λ) be two GFTS's. A mapping $f : (X, \mu) \to (Y, \lambda)$ is said to be

(i) fuzzy (μ, λ)-continuous if $f^{-1}(F)$ is μf-closed for every λf-closed set F in Y.

(ii) fuzzy (μ, λ)-open ((μ, λ)-closed) if for every μf-open (resp. μf-closed) set F in X, $f(F)$ is λf-open (resp. λf-closed) in Y.

Remark 4.2. Let (X, μ) and (Y, λ) be two GFTS's and $f : (X, \mu) \to (Y, \lambda)$ be a mapping. Then f is fuzzy (μ, λ)-continuous iff $f^{-1}(U)$ is μf-open for every λf-open set U in Y.

Theorem 4.3. *Let (X, μ) and (Y, λ) be two GFTS's. A surjective mapping $f : (X, \mu) \to (Y, \lambda)$ is fuzzy (μ, λ)-closed iff for each $A \in I^Y$ and each μf-open set U in X with $f^{-1}(A) \leq U$, there exists a λf-open set V in Y such that $A \leq V$, $f^{-1}(V) \leq U$.*

Proof. Let $f : (X, \mu) \to (Y, \lambda)$ be fuzzy (μ, λ)-closed and surjective. Also let $A \in I^X$ be such that $f^{-1}(A) \leq U$ where U is a μf-open set in X. Then $1_X - U \leq 1_X - f^{-1}(A)$ $\Rightarrow f(1_X - U) \leq f(1_X - f^{-1}(A)) \leq 1_Y - A$. Take $V = 1_Y - f(1_X - U)$. Then V is a λf-open set such that $A \leq V$ and $f^{-1}(V) = f^{-1}(1_Y - f(1_X - U)) \leq U$.
Conversely, let the condition hold and F be a μf-closed set in X. Take $A = 1_Y - f(F)$. Then $f^{-1}(A) \leq 1_X - F$, where $1_X - F$ is a μf-open set in X. Thus by the given condition, there exists a λf-open set V in Y such that $A \leq V$ and $f^{-1}(V) \leq 1_X - F$. Then $V \leq f(1_X - F) \leq 1_Y - f(F) = A$. Thus $A = 1_Y - f(F) = V \Rightarrow f(F)$ is λf-closed in Y. Hence f is fuzzy (μ, λ)-closed. □

Theorem 4.4. *Let (X, μ) and (Y, λ) be two GFTS's. For a mapping $f : (X, \mu) \to (Y, \lambda)$, the following are equivalent:*

(a) f is fuzzy (μ, λ)-continuous.

(b) $f(c_\mu(A)) \leq c_\lambda(f(A))$ for any $A \in I^X$.

(c) $c_\mu(f^{-1}(B) \leq f^{-1}(c_\lambda(B))$ for any $B \in I^Y$.

Proof. $(a) \Rightarrow (b)$: Since $c_\lambda(f(A))$ is λf-closed in Y and f is fuzzy (μ, λ)-continuous, $f^{-1}(c_\lambda(f(A)))$ is μf-closed in X. Now, since $A \leq f^{-1}(c_\lambda(f(A)))$, so $c_\mu(A) \leq f^{-1}(c_\lambda(f(A))) \Rightarrow f(c_\mu(A)) \leq c_\lambda(f(A))$.

$(b) \Rightarrow (c)$: Let $B \in I^X$. Then by (b), we have $f(c_\mu(f^{-1}(B))) \leq c_\lambda(ff^{-1}(B)) \leq c_\lambda(B) \Rightarrow c_\mu(f^{-1}(B)) \leq f^{-1}(c_\lambda(B))$.

$(c) \Rightarrow (a)$: Let F be any λf-closed in Y. Then by (c), $c_\mu(f^{-1}(F)) \leq f^{-1}(c_\lambda(F)) = f^{-1}(F) \Rightarrow f^{-1}(F)$ is μf-closed in $X \Rightarrow f$ is fuzzy (μ, λ)-continuous. □

Theorem 4.5. *Let (X, μ) and (Y, λ) be two GFTS's and $f : (X, \mu) \to (Y, \lambda)$ be fuzzy (μ, λ)-continuous and fuzzy (μ, λ)-closed mapping. If A is μgf-closed in X, then $f(A)$ is a λgf-closed set in Y.*

Proof. Let $f(A) \leq U$ where U is λf-open in Y. Then $A \leq f^{-1}(U)$, where $f^{-1}(U)$ is μf-open in X. A being a μgf-closed set in X, $c_\mu(A) \leq f^{-1}(U)$. Thus $f(c_\mu(A)) \leq U$ and $f(c_\mu(A))$ is λf-closed in Y. Hence $c_\lambda(f(A)) \leq c_\lambda(f(c_\mu(A))) = f(c_\mu(A)) \leq U$. It follows that $f(A)$ is λgf-closed in Y. □

Remark 4.6. If we drop fuzzy (μ, λ)-closedness of f in the above theorem, the conclusion of the above theorem may not remain valid, as we see below.

Example 4.7. Let $X = \{a, b\}$, $Y = \{c, d\}$. We define the fuzzy sets A, B and C by:

$$A(a) = 0.5,\ A(b) = 0.3;$$
$$B(a) = 0.7,\ B(b) = 0.9;$$
$$C(c) = 0.9,\ C(d) = 0.7.$$

Let $\mu = \{0_X, 1_X, A, B\}$, $\lambda = \{0_Y, 1_Y, C\}$. Then (X, μ) and (Y, λ) are GFTS's.
We define a mapping $f : (X, \mu) \to (Y, \lambda)$ by:

$$f(a) = d,\ f(b) = c.$$

Then it is easy to check that f is fuzzy (μ, λ)-continuous but not fuzzy (μ, λ)-closed. Now consider a fuzzy set $F : X \to [0, 1]$ given by:

$$F(a) = 0.5,\ F(b) = 0.7.$$

Then F is μf-closed in X and hence it is μgf-closed in X. But $f(F)$ is not λgf-closed in Y. In fact, $f(F)(c) = sup_{x \in f^{-1}(c)} F(x) = F(b) = 0.7$ and $f(F)(d) = sup_{x \in f^{-1}(d)} F(x) = F(a) = 0.5$. Thus $f(F) \leq C$, where $C \in \lambda$ but $c_\lambda(f(F)) \nleq C$.

Theorem 4.8. *Let (X, μ) and (Y, λ) be two GFTS's and $f : (X, \mu) \to (Y, \lambda)$ be a fuzzy (μ, λ)-continuous and fuzzy (μ, λ)-open surjection. If X is μf-regular, then Y is λf-regular.*

Proof. Let F be any λf-closed set in Y and y_α a fuzzy point in Y such that $y_\alpha \nleq F$. Then $f^{-1}(y_\alpha) \nleq f^{-1}(F)$. Since f is fuzzy (μ, λ)-continuous, $f^{-1}(F)$ is μf-closed in X. Select some $x \in f^{-1}(y)$ (as f is onto) and consider the fuzzy point x_α in X. Then $x_\alpha \nleq f^{-1}(F)$. Since X is μf-regular, there exist μf-open sets U and V such that $x_\alpha q U$, $f^{-1}(F) \leq V$ and $U \overline{q} V$. Since f is fuzzy (μ, λ)-open and surjective, $f(U)$ and $f(V)$ are λf-open sets in Y such that $y_\alpha q f(U)$, $F \leq f(V)$ and $f(U) \overline{q} f(V)$. Thus Y is λf-regular. □

Theorem 4.9. *Let (X, μ) and (Y, λ) be two GFTS's and $f : (X, \mu) \to (Y, \lambda)$ be fuzzy (μ, λ)-continuous, fuzzy (μ, λ)-open and surjective. If X is μf-normal, then Y is λf-normal.*

Proof. Let F and G be two λf-closed sets in Y such that $F \overline{q} G$. Then $f^{-1}(F) \overline{q} f^{-1}(G)$. Since f is fuzzy (μ, λ)-continuous, $f^{-1}(F)$ and $f^{-1}(G)$ are μf-closed sets in X. Since X is μf-normal, there exist two μf-open sets U and V such that $f^{-1}(F) \leq U$, $f^{-1}(G) \leq V$ and $U \overline{q} V$. Since f is fuzzy (μ, λ)-open and surjective, $f(U)$ and $f(V)$ are λf-open sets in Y such that $F \leq f(U)$, $G \leq f(V)$ and $f(U) \overline{q} f(V)$. Hence Y is λf-normal. □

5. Concluding remark:

As we have already observed, the results of this article unify different existing results in special settings, e.g. Theorem 2.4 of [3] and Theorem 2 of [11] are obtained as particular cases of Theorem 2.4 here. Similarly, Theorem 2.7 of [3] and Theorem 3 of [11] (resp. Theorem 2.9 of [3] and Theorem 4 of [11], Theorem 2.5 of [3] and Theorem 5 of [11]) follow immediately from Corollary 2.6 (resp. Theorem 2.7, Theorem 2.8) of this paper if one take $FO(X)$ and $FSO(X)$ respectively for μ. Moreover, from what has been done here, many more new definitions and results for an fts can be arrived at by respective choices of $FO(X), FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $FBO(X)$ etc. in place of μ. In the present context, we may refer to the papers [7, 8, 16, 17], the results of which can similarly be subjected to suitable process of unification.

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