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Fuzzy soft product topology

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Abstract. In this paper, we introduce the concept of Cartesian product of two fuzzy soft sets. Fuzzy soft product topology over $X \times Y$ is defined and some properties of it are investigated. The notion of fuzzy soft point and fuzzy soft neighborhood are also studied and fuzzy soft Hausdorff spaces are considered.

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1. Introduction

To solve problems dealing with uncertainties, where classical mathematical methods can not be used effectively, several theories like probability theory and fuzzy set theory have been introduced. Although these theories are two powerful mathematical approaches to deal with vagueness, lack of parametrization tools restricts the application of these methods in both theory and practice. Soft set theory, originally introduced by Molodtsov[10], is a new approach to deal with uncertainty. A soft set is in fact a set-valued map which is used to describe the universe of discourse based on some parameters. Due to theory of soft set is more general than the formers (see [10]), it has received much attention from researchers and was developed sharply.

Study on soft set theory after introduction by Molodtsov[10], was started by Maji et al.[7, 8] in 2000. In [7], the authors defined and studied some basic concepts like complement, union, and intersection of soft sets and then, in [8] applied this method to solve a decision-making problem successfully. Aktaş and Çağman[2], Sezgin and Atagun[14] and Irfan Ali et al.[3] improved the work of Maji et al.[7] and proposed some new operations and properties of soft set theory. Topological study on soft sets was begun by Çağman et al.[5]. They defined a topology on a soft set as a collection of soft subsets of it and consider some basic concepts and properties of this new space. Shabir and Naz[15] also worked on soft topology. They proposed a new definition of soft topology which is defined over an initial universe with a fixed parameter set.

However soft set theory can solve a wide range of problems, its inherent difficulty still remained. This theory can not be applied effectively to deal with real-life problems since in these situations we face imprecise environment and inexact information which are modelled by fuzzy set theory. The extension of the classical soft set theory from crisp cases to the world of vague concepts has been proposed by Maji et al.[9]. They combined fuzzy set theory and soft set theory to develop theory of soft set and introduced the new notion "fuzzy soft set" as a fuzzy generalization of soft sets. Then, in [11], they discussed practical applications of fuzzy soft sets in decision making problems. In 2009, Kharal and Ahmad[1] extended some operations in classical set theory to fuzzy soft set theory and in [6], they introduced the concept of fuzzy soft map and considered some properties of image and inverse image of a fuzzy soft set under fuzzy soft maps.

Topological study on fuzzy soft sets was started by Tanay and Kandemir[17]. They applied classical definition of topology to construct a topology over a fuzzy soft set and called this new topological space fuzzy soft topology. Furthermore, they studied some fundamental topological structures such as interior and closure of a fuzzy soft set, fuzzy soft base and fuzzy soft subspace. Besides this, they modified the definition of fuzzy soft complement as a fuzzification of soft complement which had been introduced in [3] by Irfan Ali et al. Simsekler and Yuksel[16] improved the concept of fuzzy soft topology, proposed in [17], and redefined it with a fixed set of parameters. Roy and Samanta[12] remarked the new definition of fuzzy soft topology. They proposed the notion of fuzzy soft topology over an ordinary set by applying fuzzy soft subsets of it, where parameter set is supposed fixed every where over the space. They also studied the concept of base and subbase for this space. Then in[13], they continued study on fuzzy soft topological spaces by considering fuzzy soft point and different neighborhood structures of a fuzzy soft point.

The aim of this study is considering product topology of fuzzy soft sets. So the paper is organized as followings: first in section (2), some basic definitions and properties of fuzzy soft set theory is given. Then in section (3), some operations in fuzzy soft set theory are considered. It is also revealed that while having different parameters' set, De Morgan laws do not hold in general. Fuzzy soft topology and some properties of it are studied in section (4). In section (5), the concept of Cartesian product of two fuzzy soft sets is given and then product topology of fuzzy soft sets is studied. In the last section, it is shown that whether product of two fuzzy soft Hausdorff spaces is Hausdorff.

2. Preliminaries

In this section, we recall some basic definitions of fuzzy soft set theory which are needed in the following sections.

Suppose X is the set of objects under consideration and E is the set of parameters. Let $A \subseteq E$ and 2^X shows the set of all subsets of X, and I^X denotes the set of all fuzzy subsets of X.

Molodtsov[10] introduced the concept of soft set as follow:

Definition 2.1 ([10]). A pair (F, A) is called a soft set over universal set X if F is a mapping given by $F : A \to 2^X$ such that $F(e) \subseteq X$ for all $e \in A \subseteq E$.

Maji et al.[9] combined the concept of soft set and fuzzy set and introduced a new concept called fuzzy soft set as below:

Definition 2.2 ([9]). A pair (f, A) is called a fuzzy soft set, F.S set in brief, over X if f is a mapping given by $f : A \to I^X$ where $\forall e \in A$, $f(e)$ is a fuzzy subset of X with membership function $f_e : X \to [0, 1]$.

So for each parameter $e \in A \subseteq E$, $f_e(x)$ indicates the membership degree of element x of X in $f(e)$. In fact, it shows how much each member of X has the attribute $e \in A$.
From now, we will show the F.S set (f, A) by f_A.

Definition 2.3 ([6]). The collection of all F.S sets over X with regards to parameter set E is called fuzzy soft space, say F.S space, over X and is denoted by $\mathcal{FS}(X, E)$ or X_E.

Definition 2.4. (Rules of Fuzzy Soft Sets) For two F.S sets f_A and g_B where $f_A, g_B \in X_E$ we have

i. ([9, 17]) f_A is a fuzzy soft subset of g_B, denoted by $f_A \tilde{\leq} g_B$, if and only if:
(1) $A \subseteq B$,
(2) For all e in A, $f_e(x) \leq g_e(x)$ $\forall x \in X$.

ii. ([9, 17]) $f_A = g_B$ if and only if $f_A \tilde{\leq} g_B$ and $g_B \tilde{\leq} f_A$.

iii. ([17]) The complement of the F.S set f_A is denoted by f^c_A and is defined by mapping $f^c : A \to I^X$, such that for each $e \in A$, $f^c(e)$ is the complement of F.S set $f(e)$.

iv. ([9, 17]) $f_A = \Phi_A$ (Null F.S set with respect to A), if for all $e \in A$, $f_e(x) = 0$, $\forall x \in X$.

v. ([9, 17]) $f_A = \tilde{X}_A$ (Absolute F.S set with respect to A), if $\forall x \in X$, $f_e(x) = 1$ for all $e \in A$.
If $A = E$, the null and absolute F.S sets are denoted by Φ and $\tilde{X}$, respectively.

vi. ([9, 17]) The union of two F.S sets f_A and g_B, denoted by $f_A \tilde{\vee} g_B$, is the F.S set $(f \vee g, C)$, where $C = A \cup B$ and $\forall e \in C$, we have $(f \vee g)(e) = f(e) \vee g(e)$ where

$$(f \vee g)_e(x) = \begin{cases} f_e(x) & \text{if } e \in A - B \\ g_e(x) & \text{if } e \in B - A \\ \max\{f_e(x), g_e(x)\} & \text{if } e \in A \cap B. \end{cases}$$

for all $x \in X$.

vii. ([9, 17]) The intersection of two F.S sets f_A and g_B, denoted by $f_A \tilde{\wedge} g_B$, is the F.S set $(f \wedge g, C)$, where $C = A \cap B$ and $\forall e \in C$, we have
$(f \wedge g)(e) = f(e) \wedge g(e)$ where $(f \wedge g)_e(x) = \min\{f_e(x), g_e(x)\}$ for all $x \in X$.

Note that in definition 2.4 (vii), $C = A \cap B$ must be nonempty set.

Definition 2.5 ([6]). Let X_E and $Y_{E'}$ be two F.S spaces over X and Y with regards to parameter sets E and E', respectively and $f_A \in X_E$. Let $u : X \to Y$ and $p : E \to E'$ be ordinary maps. The map $u_p(= H) : X_E \to Y_{E'}$ is called F.S map and is defined as follow:

the image of F.S set f_A under F.S map H, denoted by $(H(f_A), E')$, is a F.S set in $Y_{E'}$ given by

$$H(f_A) : p(E) \subseteq E' \to I^Y$$

such that for any $\beta \in p(E)$ and $y \in Y$ we have

$$[H(f_A)]_\beta(y) = \begin{cases} \sup_{x\in u^{-1}(y)}[\bigvee_{\alpha\in p^{-1}(\beta)\cap A} f(\alpha)](x) & \text{if } u^{-1}(y) \neq \varnothing \text{ and } p^{-1}(\beta) \cap A \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

where $\vee$ is fuzzy union, defined as supremum. $(H(f_A), E')$ is called the F.S image of F.S set f_A.

Definition 2.6 ([6]). Let X_E and $Y_{E'}$ be two F.S spaces over X and Y with regards to parameters sets E and E', respectively. Let $u : X \to Y$ and $p : E \to E'$ be ordinary maps. Suppose $u_p(= H) : X_E \to Y_{E'}$ is a F.S map mentioned in Definition 2.5 and g_B is a F.S set in $Y_{E'}$ where $B \subseteq E'$.
The inverse image of F.S set g_B under F.S map H is denoted by $(H^{-1}(g_B), E)$ which is a F.S set in X_E such that for each $\alpha \in p^{-1}(B) \subseteq E$, and $x \in X$, we have

$$[H^{-1}(g_B)]_\alpha(x) = \begin{cases} g_{p(\alpha)}(u(x)) & \text{if } p(\alpha) \in B \\ 0 & \text{otherwise.} \end{cases}$$

For more details refer to [6, 1, 12, 16].

3. Properties of fuzzy soft sets

In [1], Ahmad and Kharal continued Maji et al.[9] work and studied some properties for fuzzy soft union and intersection. They proved DeMorgan Laws in fuzzy soft set theory regarding to definition of fuzzy soft complement introduced by Maji et al.[9], where the parameter set of the fuzzy soft complement is the negation of parameter set of initial fuzzy soft set. Since in the present paper we consider fuzzy soft complement regarding to work of Tanay and Kandemir[17], see Definition2.4 (iii), DeMorgan Laws are needed to be proved under this definition, whereas in the former papers this issue has not received attention by the authors.

Proposition 3.1 ([17]). *Let f_A, g_B, and h_C be some F.S sets over X where A, B, and C are subsets of the parameter set E. Then the followings are hold:*
(1) $[f_A \tilde{\vee} g_B]\tilde{\wedge} h_C = [f_A \tilde{\wedge} h_C]\tilde{\vee}[g_B \tilde{\wedge} h_C]$
(2) $[f_A \tilde{\wedge} g_B]\tilde{\vee} h_C = [f_A \tilde{\vee} h_C]\tilde{\wedge}[g_B \tilde{\vee} h_C]$

Proof. See [17], Lemma 3.16. □

Proposition 3.2. *(De Morgan Laws) Let X be a universal set, and E be a parameter set. Let $f_E, g_E \in X_E$. Then we have:*
(1) $[f_E \tilde{\vee} g_E]^c = f_E^c \tilde{\wedge} g_E^c$
(2) $[f_E \tilde{\wedge} g_E]^c = f_E^c \tilde{\vee} g_E^c$

Proof. (1) Suppose that $f_E \tilde{\vee} g_E = h_E$ which implies that $[f_E \tilde{\vee} g_E]^c = h_E^c$. So for all $x \in X$ and $e \in E$, Definition 2.4 (iii),(vi), and (vii) imply that

$$\begin{aligned} h_e^c(x) &= 1 - h_e(x) = 1 - \max\{f_e(x), g_e(x)\} \\ &= \min\{1 - f_e(x), 1 - g_e(x)\} = \min\{f_e^c(x), g_e^c(x)\} \\ &= f_e^c(x)\tilde{\wedge} g_e^c(x) \end{aligned}$$

Therefore $[f_E \tilde{\vee} g_E]^c = f_E^c \tilde{\wedge} g_E^c$.
(2) It is similar to (1). □

Proposition 3.3. *Let X be a universal set, E be a parameter set, $A, B \subset E$ and $f_A, g_B \in X_E$. Then*
(1) $[f_A \tilde{\vee} g_B]^c \geq f_A^c \tilde{\wedge} g_B^c$
(2) $[f_A \tilde{\wedge} g_B]^c \leq f_A^c \tilde{\vee} g_B^c$

Proof. (1) Suppose that $f_A \tilde{\vee} g_B = h_C$ where $h = f \vee g$ and $C = A \cup B$. Since $[f_A \tilde{\vee} g_B]^c = h_C^c$, by applying Definition 2.4 (iii) and (vi), we have

$$h_e^c(x) = 1 - h_e(x) = \begin{cases} f_e^c(x) & \text{if } e \in A - B \\ g_e^c(x) & \text{if } e \in B - A \\ \min\{f_e^c(x), g_e^c(x)\} & \text{if } e \in A \cap B. \end{cases}$$

On the other hand, suppose $A \cap B \neq \varnothing$, and let $f_A^c \tilde{\wedge} g_B^c = k_{C'}$ where $k = f^c \wedge g^c$ and $C' = A \cap B$. So for all $e \in C'$ and by using Definition 2.4 (vii) we have $k_e(x) = \min\{f_e^c(x), g_e^c(x)\}$. Since $C' \subseteq C$, Definition 2.4 (i) implies that $[f_A \tilde{\vee} g_B]^c \geq f_A^c \tilde{\wedge} g_B^c$.
(2) It is similar to (1). □

Equality does not hold in general. This is shown in the following example.

Example 3.4. Let $X = \{x, y, z\}$ and $E = \{e_1, e_2, e_3\}$. Let $A = \{e_1, e_3\}$ and $B = \{e_2, e_3\}$ and f_A and g_B be F.S sets over X as below:

$$\begin{aligned} f_A &= \left\{ \left(e_1, \left\{ \frac{0.1}{x} + \frac{0.5}{y} + \frac{0.3}{z} \right\} \right), \left(e_3, \left\{ \frac{0.7}{x} + \frac{0}{y} + \frac{0.4}{z} \right\} \right) \right\} \\ g_B &= \left\{ \left(e_2, \left\{ \frac{0.3}{x} + \frac{0.8}{y} + \frac{0.1}{z} \right\} \right), \left(e_3, \left\{ \frac{0.2}{x} + \frac{0.6}{y} + \frac{0.9}{z} \right\} \right) \right\}. \end{aligned}$$

Thus $f_A^c \tilde{\wedge} g_B^c = \left\{ e_3, \left\{ \frac{0.3}{x} + \frac{0.4}{y} + \frac{0.1}{z} \right\} \right\}$, while

$$\begin{aligned} [f_A \tilde{\vee} g_B]^c &= \left\{ \left(e_1, \left\{ \frac{0.9}{x} + \frac{0.5}{y} + \frac{0.7}{z} \right\} \right), \left(e_2, \left\{ \frac{0.7}{x} + \frac{0.2}{y} + \frac{0.9}{z} \right\} \right), \right. \\ &\qquad \left. \left(e_3, \left\{ \frac{0.3}{x} + \frac{0.4}{y} + \frac{0.1}{z} \right\} \right) \right\}. \end{aligned}$$

4. Fuzzy soft topology

The concept of fuzzy soft topology firstly introduced by Tanay and Kandemir[17]. They defined a fuzzy soft topology over a fuzzy soft set as a collection of F.S subsets of it with different parameter sets. But regarding to Propositions 3.2 and 3.3 since in this case DeMorgan Laws are not hold, Roy and Samanta[12] initiated the concept of fuzzy soft topology over a universal set where the parameter set is supposed fixed everywhere over the space. Here we recall definition of fuzzy soft topology introduced in Roy and Samanta[12] work.

Definition 4.1 ([12])**.** Let X be the universe and E be the parameter set. The fuzzy soft topology τ over X is a collection of F.S subsets of X with fixed parameter set E such that:
i. $\tilde{X}$ and $\Phi \in \tau$.
ii. The union of any numbers of F.S sets in τ belongs to τ.

iii. The intersection of any two F.S sets in τ belongs to τ.

The triplet (X, E, τ) is called a fuzzy soft topological space over X, say F.S topological space, and each element of τ is called a fuzzy soft open set, say F.S open set, in X. The complement of a F.S open set is called fuzzy soft closed set, say F.S closed set.

Example 4.2. Let X be a universal set including objects under consideration and E be the set of parameters. Then the family of all F.S sets over X forms a F.S topology on X which is called discrete F.S topology, while indiscrete or trivial F.S topology on X contains only $\Phi, \tilde{X}$.

Example 4.3. Suppose $X = \mathbb{R}$ be the Real Line with the usual topology τ and $E = (0, 1] \subset \mathbb{R}$. Let $\{U : U \in \tau\}$ be the family of all open intervals $U = (a, b) \subset \mathbb{R}$. Then the family $\{U_E : U \in \tau, E = (0, 1]\}$ forms a F.S topology over $\mathbb{R}$ denoting by $\tau_{F.S}$ where U_E is a F.S set over $\mathbb{R}$ with respect to open interval U and is defined as below:

$$U : E = (0, 1] \to I^{\mathbb{R}}$$

where for each $\alpha \in (0, 1]$, $U(\alpha)$ is defined by characteristic function of open set U i.e., $U(\alpha) = \chi_U$. So

$$U_\alpha(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U. \end{cases}$$

Definition 4.4 ([12])**.** Let (X, E, τ) be a F.S topological space.

i. Fuzzy soft closure, F.S closure, of f_E is denoted by Clf_E and is defined as the intersection of all F.S closed super sets of f_E, i.e.

$$Clf_E = \tilde{\bigwedge}_{g_E \tilde{\geq} f_E} g_E; g_E^c \in \tau.$$

ii. Fuzzy soft interior, F.S interior, of f_E is denoted by $Intf_E$ and is defined as the union of all F.S open subsets of f_E,i.e.

$$Intf_E = \tilde{\bigvee}_{h_E \tilde{\leq} f_E} h_E; h_E \in \tau.$$

Definition 4.5. Let (X, E, τ) and (Y, E', τ') be two F.S topological space and $H : (X, E, \tau) \to (Y, E', \tau')$ be a F.S map. Then H is called

i. F.S continuous map if and only if for each $g_{E'} \in \tau'$, we have $H^{-1}(g_{E'}) \in \tau$.

ii. F.S open map if and only if for each $f_E \in \tau$, we have $H(f_E) \in \tau'$.

Definition 4.6. Let (X, E, τ) be a F.S topological space. The collection $\mathcal{B}$ of F.S open subsets of X is called a F.S base for F.S topology τ if every F.S open sets in τ can be written as a union of members of $\mathcal{B}$.

Definition 4.7 ([4])**.** Let f_A be a F.S set over the universal set X where $X = \{x_1, x_2, \ldots, x_m\}$, $E = \{e_1, e_2, \ldots, e_n\}$, and $A \subseteq E$. Then f_A can be presented by a matrix as in the following form:

$$[a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} = f_{e_j}(x_i); 1 \leq i \leq m, 1 \leq j \leq n$, and

$$a_{ij} = \begin{cases} f_{e_j}(x_i) & \text{if } e_j \in A \\ 0 & \text{otherwise.} \end{cases}$$

This $m \times n$ matrix is called the F.S matrix of F.S set f_A.

Example 4.8. If $X = \{x_1, x_2, \ldots, x_m\}$ and $E = \{e_1, e_2, \ldots, e_n\}$. Then the standard basis for $M_{m\times n}$ ($m \times n$ matrixes space) forms a F.S base for discrete F.S topology τ over X, i.e.,

$$\mathcal{B} = \{E_{ij} = [e_{kh}]_{m\times n} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where $e_{kh} = \begin{cases} 1 & \text{if } k = i, h = j \\ 0 & \text{otherwise} \end{cases}$.

In the below theorem, we present the conditions which make $\mathcal{B}$ as a F.S base for a F.S topology over X.

Theorem 4.9. *Let $\mathcal{B}$ be a collection of F.S subsets of X. Then $\mathcal{B}$ is a base for a F.S topology over X if and only if $\mathcal{B}$ has the following properties:*
(1) $\tilde{\bigvee} f_E = \tilde{X}$ for all $f_E \in \mathcal{B}$,
(2) If $f_{1E}, f_{2E} \in \mathcal{B}$ such that $f_{1E} \tilde{\wedge} f_{2E} \neq \Phi$, then it can be written as a union of members of $\mathcal{B}$.

Proof. ($\Rightarrow$) Let $\mathcal{B}$ be a base for F.S topology τ over X and $f_E \in \mathcal{B}$. Then $\tilde{X} \in \tau$ implies that $\tilde{X} = \tilde{\bigvee} f_E$. Moreover if $f_{1E}, f_{2E} \in \mathcal{B} \subset \tau$, then $f_{1E} \tilde{\wedge} f_{2E} \in \tau$ (since τ is a F.S topology over X). So it can be written as a union of elements of $\mathcal{B}$.

($\Leftarrow$) Let $\mathcal{B}$ has conditions (1),(2). We prove that $\tau_{\mathcal{B}}$ which is defined as the collection of all union of elements of $\mathcal{B}$ is a F.S topology over X.
(1) It is clear that $\Phi, \tilde{X} \in \tau_{\mathcal{B}}$.
(2) Let $\{g_{\alpha E}\} \subseteq \tau_{\mathcal{B}}$ where $\alpha \in I$, an index set, then for each $\alpha \in I$, $g_{\alpha E} = \tilde{\bigvee}_{j\in J} f_{jE}$ where $f_{jE} \in \mathcal{B}$. So the union of all $g_{\alpha E}$ can be written as a union of elements of $\mathcal{B}$. Thus $\tilde{\bigvee} g_{\alpha E} \in \tau_{\mathcal{B}}$.
(3) Let g_E and g'_E be in $\tau_{\mathcal{B}}$. Then there exist $\{f_{iE}\} \subset \mathcal{B}, \{f'_{jE}\} \subset \mathcal{B}$ such that $g_E = \tilde{\bigvee}_{i\in I} f_{iE}$ and $g'_E = \tilde{\bigvee}_{j\in J} f'_{jE}$. Then by applying Proposition 3.1 we have

$$\begin{aligned} g_E \tilde{\wedge} g'_E &= g_E \tilde{\wedge} (\tilde{\bigvee}_j f'_{jE}) = \tilde{\bigvee}_j (g_E \tilde{\wedge} f'_{jE}) \\ &= \tilde{\bigvee}_j ([\tilde{\bigvee}_i f_{iE}] \tilde{\wedge} f'_{jE}) = \tilde{\bigvee}_j (\tilde{\bigvee}_i [f_{iE} \tilde{\wedge} f'_{jE}]) \\ &= \tilde{\bigvee}_j \tilde{\bigvee}_i (f_{iE} \tilde{\wedge} f'_{jE}) = \tilde{\bigvee}_j \tilde{\bigvee}_i k_E \end{aligned}$$

where $k(e) = f_i(e) \wedge f'_j(e)$ for all $e \in E$. Thus $g_E \tilde{\wedge} g'_E$ can be written as a union of elements of $\mathcal{B}$ since $f_i(e) \wedge f'_j(e) \in \mathcal{B}$. So $g_E \tilde{\wedge} g'_E \in \tau_{\mathcal{B}}$. Thus $\tau_{\mathcal{B}}$ is a F.S topology over X. □

Definition 4.10. $\tau_{\mathcal{B}}$ described in above theorem, is called fuzzy soft topology generated by $\mathcal{B}$ and $\mathcal{B}$ is a fuzzy soft base for F.S topology $\tau_{\mathcal{B}}$ such that every element of $\tau_{\mathcal{B}}$ can be written as a union of elements of $\mathcal{B}$.

Definition 4.11. The collection $\mathcal{S}$, of F.S open sets in X_E is called a F.S subbase for F.S topology τ if every F.S open sets in τ can be written as a union of finite intersection of members of $\mathcal{S}$.

5. Product fuzzy soft topology

Definition 5.1. Let X_E and $Y_{E'}$ be two F.S spaces. Let $f_E \in X_E$ and $g_{E'} \in Y_{E'}$. The "Cartesian product" of f_E and $g_{E'}$, denoted by $f_E \tilde{\otimes} g_{E'}$, is a F.S set over $X \times Y$ with regards to parameter set $E \times E'$ defined as below:

$$\begin{array}{rcl} f\tilde{\otimes}g : E \times E' & \longrightarrow & I^X \times I^Y \\ (e, e') & \mapsto & f(e) \times g(e') \end{array}$$

such that $f(e) \times g(e')$ is the fuzzy product of fuzzy sets $f(e)$ and $g(e')$ where

$$\begin{array}{rcl} f(e) \times g(e') : X \times Y & \longrightarrow & [0, 1] \\ (x, y) & \mapsto & \min\{f_e(x), g_{e'}(y)\} \end{array}$$

Proposition 5.2. *Let X_E and $Y_{E'}$ be two F.S spaces. Suppose $f_E \in X_E$ and $g_{E'} \in Y_{E'}$. Then one has the following:*

(1) $\tilde{X}\tilde{\otimes}\tilde{Y} = \widetilde{X \times Y}$ where $\widetilde{X \times Y}$ denotes the absolute F.S set $X \times Y$ with respect to parameter set $E \times E'$.

(2) $f_E\tilde{\otimes}\Phi_{E'} = \Phi_E\tilde{\otimes}g_{E'} = \Phi_{E\times E'} = \Phi_E\tilde{\otimes}\Phi_{E'}$

(3) $\forall e \in E, \forall e' \in E'$, $(f\tilde{\otimes}\tilde{Y})_{(e,e')}(x, y) = f_e(x)$, $(\tilde{X}\tilde{\otimes}g)_{(e,e')} = g_{e'}(y)$, where $x \in X$ and $y \in Y$.

(4) $[f_{1E}\tilde{\wedge}f_{2E}]\tilde{\otimes}[g_{1E'}\tilde{\wedge}g_{2E'}] = [f_{1E}\tilde{\otimes}g_{1E'}]\tilde{\wedge}[f_{2E}\tilde{\otimes}g_{2E'}]$

Proof. (1) Definition 2.4, (v) implies that $\forall (e, e') \in E \times E'$, $\widetilde{X \times Y}_{(e,e')}(x, y) = 1$ for all $(x, y) \in X \times Y$.
On the other hand, we have $\forall e \in E$, $\tilde{X}_e(x) = 1$ for all $x \in X$ and $\forall e' \in E'$, $\tilde{Y}_{e'}(y) = 1$ for all $y \in Y$ [Definition 2.4, (v)]. So $\forall e \in E$ and $\forall e' \in E'$, $\min\{\tilde{X}_e(x), \tilde{Y}_{e'}(y)\} = 1$ for all $x \in X$ and $y \in Y$. By applying Definition 5.1, we have $(\tilde{X}\tilde{\otimes}\tilde{Y})_{(e,e')}(x, y) = 1$ which implies $\tilde{X}\tilde{\otimes}\tilde{Y} = \widetilde{X \times Y}$.

(2) Applying Definition 2.4, (iv) and 5.1.

(3) Let $f_E \in X_E$. Definition 2.4, (v) implies that $\min\{f_e(x), \tilde{Y}_{e'}(y)\} = f_e(x)$, i.e. $(f\tilde{\otimes}\tilde{Y})_{(e,e')}(x, y) = f_e(x)$.

(4) Take $(x, y) \in X \times Y$ and $(e, e') \in E \times E'$. By part (3), we have

$$\begin{array}{rl} & [(f_1\tilde{\otimes}g_1)\tilde{\wedge}(f_2\tilde{\otimes}g_2)]_{(e,e')}(x, y) \\ = & \min\{(f_1\tilde{\otimes}g_1)_{(e,e')}(x, y), (f_2\tilde{\otimes}g_2)_{(e,e')}(x, y)\} \\ = & \min\{\min\{(f_e)_1(x), (g_{e'})_1(y)\}, \min\{(f_e)_2(x), (g_{e'})_2(y)\}\} \\ = & \min\{\min\{(f_e)_1(x), (f_e)_2(x)\}, \min\{(g_{e'})_1(y), (g_{e'})_2(y)\}\} \\ = & \min\{(f_1 \wedge f_2)_e(x), (g_1 \wedge g_2)_{e'}(y)\} \\ = & [(f_1\tilde{\wedge}f_2)\tilde{\otimes}(g_1\tilde{\wedge}g_2)]_{(e,e')}(x, y) \end{array}$$

□

Definition 5.3. Let X_E and $Y_{E'}$ be two F.S spaces. Then

$$X_E \tilde{\otimes} Y_{E'} = \{f_E \tilde{\otimes} g_{E'} : f_E \in X_E, g_{E'} \in Y_{E'}\}$$

is called the F.S cartesian product of F.S spaces sets X_E and $Y_{E'}$.

Definition 5.4. Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two fuzzy soft topological spaces. The F.S topology $\tau^{\otimes}$, generated by $\mathcal{B} = \{f_{E_1} \tilde{\otimes} g_{E_2} : f_{E_1} \in \tau_1, g_{E_2} \in \tau_2\}$, is called F.S product topology over $X_1 \times X_2$.
We denote this new topology by $(X, E, \tau^{\otimes})$ where $X = X_1 \times X_2$ and $E = E_1 \times E_2$.

It is clear that the collection $\mathcal{B}$ mentioned in Definition 5.4, is indeed a F.S base.

Definition 5.5. Let $X = X_1 \times X_2$ and $E = E_1 \times E_2$. Let $\pi_i^X : X \to X_i$, and $\pi_i^E : E \to E_i$ be projection maps. Suppose (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be F.S topological spaces. The F.S map

$$\pi_i^{X,E} : X_{1E_1} \tilde{\otimes} X_{2E_2} \to X_{iE_i}$$

is called the F.S projection map where $\pi_i^{X,E}(f_{1E_1} \tilde{\otimes} f_{2E_2}) = f_{iE_i}$.

Remark 5.6. Let $f_{1E_1} \in X_{1E_1}, f_{2E_2} \in X_{2E_2}$. For $i = 1$ and by Definitions 2.5 and 5.1 we have

$$[\pi_1^{X,E}(f_1 \tilde{\otimes} f_2)]_\alpha(x) = [\pi_1^X(f_1 \tilde{\otimes} f_2)_{\pi_1^E(E_1 \times E_2)}]_\alpha(x) = \sup_z [\bigvee_e (f_1 \tilde{\otimes} f_2)(\alpha, e)](x, z)$$

where $\alpha \in \pi_1^E(E)$, $x \in \pi_1^X(X)$, $z \in X_2$ and $e \in E_2$. Similarly for $i = 2$ we have

$$[\pi_2^{X,E}(f_1 \tilde{\otimes} f_2)]_\beta(y) = [\pi_2^X(f_1 \tilde{\otimes} f_2)_{\pi_2^E(E_1 \times E_2)}]_\beta(y) = \sup_z [\bigvee_e (f_1 \tilde{\otimes} f_2)(e, \beta)](z, y)$$

where $\beta \in \pi_2^E(E)$, $y \in \pi_2^X(X)$, $z \in X_1$ and $e \in E_1$.

Remark 5.7. By applying Definitions 2.6 and 5.1 the inverse of F.S projection map $\pi_i^{X,E} : X_{1E_1} \tilde{\otimes} X_{2E_2} \to X_{iE_i}$ can be defined as below:
Let $f_{1E_1} \in X_{1E_1}, f_{2E_2} \in X_{2E_2}$, $(\alpha, \beta) \in E_1 \times E_2$, and $(x, y) \in X_1 \times X_2$.
For $i = 1$ we have

$$\begin{aligned} [(\pi_1^{X,E})^{-1}(f_{1E_1})]_{(\alpha,\beta)}(x, y) &= [(\pi_1^X)^{-1}(f_1)_{(\pi_1^E)^{-1}(E_1)}]_{(\alpha,\beta)}(x, y) \\ &= f_{1\pi_1^E(\alpha,\beta)}(\pi_1^X(x, y)) = f_{1\alpha}(x) \end{aligned}$$

By Proposition 5.2 (3), we have $(\pi_1^{X,E})^{-1}(f_{1E_1}) = f_{1E_1} \tilde{\otimes} \tilde{X}_2$.
Similarly we have $(\pi_2^{X,E})^{-1}(f_{2E_2}) = \tilde{X}_1 \tilde{\otimes} f_{2E_2}$.

Theorem 5.8. *Let (X_i, E_i, τ_i) , $i = 1, 2$ be F.S topological spaces and $\pi_i^{X,E} : X_{1E_1} \tilde{\otimes} X_{2E_2} \to X_{iE_i}$ be F.S projection maps. Then the following hold.*
(1) *$\pi_i^{X,E}$ is F.S continuous map for each i.*
(2) *$\pi_i^{X,E}$ is F.S open map for each i.*

Proof. (1) Let $f_{E_1} \in \tau_1$. Since $\tilde{X}_2 \in \tau_2$ and $[(\pi_1^{X,E})^{-1}(f_{E_1})] = f_{E_1} \tilde{\otimes} \tilde{X}_2$, we have $[(\pi_1^{X,E})^{-1}(f_{E_1})] \in \mathcal{B}$ (see Definition 5.4). By applying Definition 4.6 it is proved.
(2) It is obvious. (see Definitions 4.6,5.5) □

Theorem 5.9. *Let* (X_1, E_1, τ_1) *and* (X_2, E_2, τ_2) *be two F.S topological spaces. The collection* $\mathcal{S} = \{[(\pi_1^{X,E})^{-1}(f_{E_1})] : f_{E_1} \in \tau_1\} \bigcup \{[(\pi_2^{X,E})^{-1}(g_{E_2})] : g_{E_2} \in \tau_2\}$ *is a F.S subbase for the F.S product topology* $\tau^{\otimes}$ *over* $X_1 \times X_2$ *with regards to parameter set* $E_1 \times E_2$.

Proof. Let $\mathcal{B}$ be the base for $\tau^{\otimes}$, the F.S product topology over $X \times Y$. Suppose that $f_{1E_1} \otimes f_{2E_2} \in \mathcal{B}$, so $f_{1E_1} \in \tau_1$ and $f_{2E_2} \in \tau_2$. Now by applying Theorem 5.8, we have

$$\begin{aligned} \mathcal{S} &= \{[(\pi_1^{X,E})^{-1}(f_{E_1})] : f_{E_1} \in \tau_1\} \bigcup \{[(\pi_2^{X,E})^{-1}(g_{E_2})] : g_{E_2} \in \tau_2\} \\ &= \{f_{E_1} \tilde{\otimes} \tilde{X}_2 : f_{E_1} \in \tau_1\} \bigcup \{\tilde{X}_1 \tilde{\otimes} g_{E_2} : g_{E_2} \in \tau_2\} \end{aligned}$$

implies that $\mathcal{S} \subseteq \tau^{\otimes}$. Moreover by Remark 5.7 we have

$$\begin{aligned} f_{1E_1} \tilde{\otimes} f_{2E_2} &= (f_{1E_1} \tilde{\wedge} \tilde{X}_1) \tilde{\otimes} (f_{2E_2} \tilde{\wedge} \tilde{X}_2) \\ &= (f_{1E_1} \tilde{\otimes} \tilde{X}_2) \tilde{\wedge} (\tilde{X}_1 \tilde{\otimes} f_{2E_2}) \\ &= [(\pi_1^{X,E})^{-1} f_{1E_1}] \tilde{\wedge} [(\pi_2^{X,E})^{-1} f_{2E_2}] \end{aligned}$$

This completes the proof. □

6. Fuzzy soft hausdorff spaces

Although the concept of fuzzy soft point is introduced in the literature (see [13, 16]), these definitions are not free of difficulty. In [13], the definition of fuzzy soft point considers only those cases which are related to one parameter and in[16], it covers the situations in which the membership degree is supposed fixed for all parameters.
Here we introduce a new definition of fuzzy soft point as an extension of both crisp point and fuzzy point by restriction the universal set X to the single set $\{x\} \subset X$.

Definition 6.1.
i. Let x^{λ_e} be a fuzzy point with support $x \in X$ and membership degree $\lambda_e \in (0, 1]$. The fuzzy soft set $\tilde{x}_E$ is called fuzzy soft point, say F.S point, given by the map $\tilde{x} : E \to I^X$ such that for all $z \in X$ and $e \in E$

$$\tilde{x}_e(z) = \begin{cases} \lambda_e & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

So for each $e \in E$, $\tilde{x}(e) = x^{\lambda_e}$, or $\tilde{x}(e) = \lambda_e \chi_{\{x\}}$ where $\chi_{\{x\}}$ is the characteristic function of $\{x\}$. In other words, the F.S point $\tilde{x}_E$, is a fuzzy description of $x \in X$ based on parameter set E.
If $\lambda_e = 1$ for all $e \in E$, we call $\tilde{x}_E$, crisp F.S point and is shown by $\bar{x}_E$ or x^1_E.
ii. The F.S point $\tilde{x}_E$ belongs to F.S set f_E denoting by $\tilde{x}_E \tilde{\in} f_E$, whenever for all $e \in E$ we have $0 < \lambda_e \leq f_e(x)$.

Definition 6.1 is indeed an extension of crisp point x and fuzzy point x^{λ} both, where they can be viewed as a F.S points $\bar{x}_{(0,1]}$ and $\tilde{x}_{(0,1]}$ where $\forall \alpha \in E = (0, 1]$, $\bar{x}(\alpha) = x^1$ and $\tilde{x}(\alpha) = x^{\lambda}$, respectively.

From now, we use notation $\tilde{x}_E$ to show the F.S point x in X where for all $e \in E$ we have $\tilde{x}(e) = x^{\lambda_e}$.

Definition 6.2. Let (X, E, τ) be a F.S topological space. Let $\tilde{x}_E$ and $\tilde{y}_E$ are F.S points over X where

$$\tilde{x}_e(z) = \begin{cases} \lambda_e & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{y}_e(z) = \begin{cases} \gamma_e & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{x}_E$ and $\tilde{y}_E$ are said

i. different if and only if

(1) $x \neq y$ or

(2) Whenever $x = y$, for some $e \in E$ we have $\lambda_e \neq \gamma_e$.

ii. distinct if and only if $\tilde{x}_E \tilde{\wedge} \tilde{y}_E = \Phi$, which implies $x \neq y$.

Definition 6.3 ([13]). Let (X, E, τ) be a F.S topological space. Let $\tilde{x}_E$ be a F.S point over X. The F.S set g_E is called fuzzy soft neighborhood, F.S - N in brief, of F.S point $\tilde{x}_E$ whenever there exists the F.S open set f_E in X such that $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} g_E$.

Definition 6.4. Let (X, E, τ) be a F.S topological space. We say that X is a

i. fuzzy soft T_0, say F.S T_0, if and only if for every two distinct F.S points in X, at least one of them has a F.S open - N which is not intersection with the other.

ii. fuzzy soft T_1 space, say F.S T_1, if and only if for every two distinct F.S points in X such as $\tilde{x}_E$ and $\tilde{y}_E$, there exist two F.S open - N of them like f_E and g_E respectively, such that $\tilde{y}_E \tilde{\wedge} f_E = \Phi$, and $\tilde{x}_E \tilde{\wedge} g_E = \Phi$.

iii. fuzzy soft Hausdorff space, say F.S T_2 or F.S Hausdorff, if and only if for every two distinct F.S points in X such as $\tilde{x}_E$ and $\tilde{y}_E$, there exist two F.S open - N like f_E and g_E, such that $f_E \tilde{\wedge} g_E = \Phi$.

Theorem 6.5. *Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two F.S Hausdorff topological spaces. Then $(X_1 \times X_2, E_1 \times E_2, \tau^{\otimes})$ is also F.S Hausdorff topological space.*

Proof. Let $X = X_1 \times X_2$, $E = E_1 \times E_2$. Suppose that $z = (x_1, y_1) \in X$ and $t = (x_2, y_2) \in X$ and $\tilde{z}_E$ and $\tilde{t}_E$ be two distinct F.S points in F.S topological space $(X, E, \tau^{\otimes})$, such that for any $e \in E$ and $k \in X$

$$\tilde{z}_e(k) = \begin{cases} \lambda_e & \text{if } k = z \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{t}_e(k) = \begin{cases} \gamma_e & \text{if } k = t \\ 0 & \text{otherwise} \end{cases}$$

where $k = (k_1, k_2) \in X_1 \times X_2$ and $e = (\alpha, \beta) \in E_1 \times E_2$. Since $\tilde{z}_E$ and $\tilde{t}_E$ are distinct F.S points in $X_1 \times X_2$, so by Definition 6.2 (ii) we have $\tilde{z}_E \tilde{\wedge} \tilde{t}_E = \Phi_E$. This implies that $z \neq t$ and then $(x_1, y_1) \neq (x_2, y_2)$. Hence $x_1 \neq x_2$ or $y_1 \neq y_2$.

By applying Remark5.7 we can define F.S points $\tilde{x}_{i E_1}$ in $X_{1 E_1}$ and $\tilde{y}_{i E_2}$ in $X_{2 E_2}$ where $i = 1, 2$ as below:

$$\tilde{x}_{1 E_1} = (\pi_1^{X,E})^{-1}(z), \tilde{x}_{2 E_1} = (\pi_1^{X,E})^{-1}(t)$$
$$\tilde{y}_{1 E_2} = (\pi_2^{X,E})^{-1}(z), \tilde{y}_{2 E_2} = (\pi_2^{X,E})^{-1}(t)$$

Now consider two cases.

Case i. If $x_1 \neq x_2$, then $\tilde{x_1}_{E_1} \tilde{\wedge} \tilde{x_2}_{E_1} = \Phi_{E_1}$ in X_{E_1}. Since X_1 is a F.S Hausdorff space, there exist two F.S open - N of $\tilde{x_1}_{E_1}$ and $\tilde{x_2}_{E_1}$ like f_{E_1} and g_{E_1} such that $f_{E_1} \tilde{\wedge} g_{E_1} = \Phi_{E_1}$. Now by applying Proposition 5.2, parts (2) and (4) we obtain

$$(f_{E_1} \tilde{\otimes} \tilde{X}_2) \tilde{\wedge} (g_{E_1} \tilde{\otimes} \tilde{X}_2) \quad = \quad (f_{E_1} \tilde{\wedge} g_{E_1}) \tilde{\otimes} (\tilde{X}_2 \tilde{\wedge} \tilde{X}_2) = \Phi_{E_1} \tilde{\otimes} X_2 = \Phi_E$$

where $f_{E_1} \tilde{\otimes} X_2, g_{E_1} \tilde{\otimes} X_2 \in \tau^{\otimes}$.

Case ii. It is similar to (i). This completes the proof. □

7. Conclusion

Theory of fuzzy soft sets is a new approach to deal with uncertainty. This theory can be seen as a fruitful method to formulate fuzzy information related to some parameters by applying a set-valued map. So it has a wide range of applications in both theory and practice. In this work we study fuzzy soft product topology and consider some related properties of it. So we firstly introduce the concept of fuzzy soft Cartesian product. Then construct a fuzzy soft topology over the Cartesian product of two crisp sets and called this new topological structure "fuzzy soft product topology". The concept of fuzzy soft point is also given and fuzzy soft Hausdorff space is considered. At the end, we show that the Cartesian product of two fuzzy soft Hausdorff spaces is also a fuzzy soft Hausdorff space. This paper may be the beginning for future research on fuzzy soft product topology, fuzzy soft separation axioms, and etc.

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