Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 6, (June 2014), pp. 901–906
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Intuitionistic 2-fuzzy strong, weak continuity and boundedness

Thangaraj Beaula, D. Lilly Esthar Rani

Received 30 July 2013; Revised 18 September 2013; Accepted 12 October 2013

Abstract. This paper defines the concepts- strong fuzzy continuity, weak fuzzy continuity, sequentially fuzzy continuity, strong boundedness and weak boundedness on intuitionistic 2-fuzzy 2-normed linear space and some theorems are established.

2010 AMS Classification: 46B99, 46A19

Keywords: Strongly fuzzy continuous, Weakly fuzzy continuous, Sequentially fuzzy continuous, Strongly bounded, Weakly bounded.

Corresponding Author: Thangaraj Beaula (edwinbeaula@yahoo.co.in)

1. Introduction

The theory of fuzzy sets was introduced by L. A. Zadeh [7] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler [4] in 1964. The notion of 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by R.M.Somasundaram and Thangaraj Beaula [5]. Intuitionistic fuzzy n-normed linear space is briefly established in [6]. The concept of intuitionistic 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a universal set was introduced by Thangaraj Beaula and D.Lilly Esthar Rani [2]. The notion of 2-fuzzy inner product space was developed in a different way in [3].

In this paper strong fuzzy continuity, weak fuzzy continuity, sequentially fuzzy continuity, strong boundedness and weak boundedness are defined for a intuitionistic 2-fuzzy 2-normed linear space. Using these concepts some theorems are proved.

2. Preliminaries

For the sake of completeness, we list the following definitions

Definition 2.1 ([4]). Let X be a real linear space of dimension greater than one and let $||\cdot,\cdot||$ be a real valued function on X $\times$ X satisfying the following conditions:

(1) ||x,y|| = 0 if and only if x and y are linearly dependent,
(2) ||x,y|| = ||y,x||
(3) $||\alpha$x,y|| =$|\alpha|$ ||x,y|| , where α is real,
(4) ||x,y+z|| $\leq$ ||x,y|| + ||x,z||

$||\cdot,\cdot||$ is called a 2-norm on X and the pair (X, $||\cdot,\cdot||$) is called a 2-normed linear space.

Definition 2.2 ([1]). Let X be a linear space over K (the field of real or complex numbers). A fuzzy subset N of X $\times$ R (R,the set of real numbers) is called a fuzzy norm on X if and only if for all x,u $\in$ X and c $\in$ K.

(N1) for all t$\in$R with t$\leq$0, N(x,t)=0
(N2) for all t$\in$R with t>0, N(x,t)=1 if and only if x=0
(N3) for all t$\in$R with t>0, N(cx,t) = N(x, $\frac{t}{|c|}$), if c$\neq$0
(N4) for all s, t$\in$R , x ,u$\in$X, N(x+u , s+t) $\geq$ min { N(x,s),N(u,t)}
(N5) N(x,$\cdot$) is a non decreasing function of R and $\lim_{t\to\infty}$ N(x,t)=1

The pair (X,N) will be referred to as a fuzzy normed linear space.

Definition 2.3 ([5]). Let F(X) be a linear space over the real field K. A fuzzy subset N of F(X)$\times$ R,(R, the set of real numbers) is called a 2-fuzzy norm on F(X) if and only if,

(N1) for all t$\in$ R with t$\leq$0, N(f_1,f_2,t)=0,
(N2) for all t$\in$R with t>0, N (f_1,f_2,t) =1 if and only if f_1 and f_2 are linearly dependent,
(N3) N (f_1, f_2,t) with t$\geq$0, N (f_1, cf_2, t) = N (f_1, f_2,$\frac{t}{|c|}$)if c$\neq$0, c$\in$K (field)
(N4) for all s,t$\in$R, N (f_1 +f_2,s,t)$\geq$min { N(f_1, s),N(f_2,t)} ,
(N5) for all s, t$\in$ R, N (f_1, f_2 + f_3, s +t) $\geq$ min { N(f_1,f_2,s), N (f_1, f_3, t)}
(N6) N (f_1,f_2,.) : (0, ∞) [0,1] is continuous,
(N7) $\lim_{t\to\infty}$ N(f_1, f_2, t)=1.

Then the pair (F(X), N) is a fuzzy 2-normed linear space or (X,N) is a 2-fuzzy 2-normed linear space.

Definition 2.4 ([2]). An intuitionistic fuzzy 2- normed linear space (i.f-2-NLS) is of the form

A = { F(X), N(f_1, f_2, t), M(f_1, f_2, t) / (f_1, f_2) $\in$ F$[(X)]^2$} where F(X) is a linear space over a field K, * is a continuous t-norm, $\lozenge$ is a continuous t-conorm, N and M are fuzzy sets on $[F(X)]^2\times(0,\infty)$ such that N denotes the degree of membership and M denotes the degree of non-membership of (f_1, f_2, t)$\in[F(X)]^2\times$ (0,∞) satisfying the following conditions:

(1) N (f_1, f_2, t) + M (f_1, f_2, t) $\leq$ 1
(2) N(f_1, f_2, t) > 0
(3) N(f_1, f_2, t) = 1 if and only if f_1, f_2 are linearly dependent
(4) N(f_1, f_2, t) is invariant under any permutation of f_1, f_2

(5) N(f_1, f_2, t) : $(0, \infty) \to [0,1]$ is continuous in t.
(6) N(f_1, cf_2, t) = N (f_1, f_2, $\frac{t}{|c|}$), if $c \neq 0$, $c \in K$
(7) N (f_1, f_2, s) * N(f_1, f_3, t) $\leq$ N(f_1, $f_2 + f_3$, s + t) where * is a continuous t-norm
(8) M (f_1, f_2, t)> 0
(9) M(f_1, f_2, t) = 0 if and only if f_1, f_2 are linearly dependent
(10) M (f_1, f_2, t) is invariant under any permutation of f_1, f_2
(11) M (f_1, cf_2, t) = M (f_1, f_2, $\frac{t}{|c|}$) if $c \neq 0$, $c \in k$
(12) M (f_1, f_2, s) $\Diamond$ M (f_1, f_3, t) $\geq$ M (f_1, f_2 + f_2, s + t) where $\Diamond$ is a continuous t-co-norm .
(13) M (f_1, f_2, t) : $(0, \infty) \to [0,1]$ is continuous in t.

3. Intuitionistic 2-fuzzy strongly weakly continuity and boundedness

Definition 3.1. A mapping T from (A, N_1, M_1) to (B, N_2, M_2) is said to be intuistionistic 2-fuzzy continuous at $f_0 \in A$ if for given $\in > 0$, $\alpha \in (0, 1)$, there exists $\delta = \delta(\alpha, \varepsilon) > 0, \beta = \beta(\alpha, \varepsilon) \in (0, 1)$, such that for every f $\in$ A, $N_1(f - f_0, g_i, \delta) > 1 - \beta$ and $M_1(f - f_0, g_i, \delta) < \beta$ implies $N_2(Tf - Tf_0, g_i, \varepsilon) > 1 - \alpha$ and $M_2(Tf - Tf_0, g_i, \varepsilon) < \alpha$ where g_i are linearly independent for i=1,2.

Definition 3.2. A linear operator T : A $\to$ B where (A, N_1, M_1), and (B, N_2, M_2) are IF 2-Banach spaces is said to be intuitionistic 2-fuzzy strongly continuous at $f_0 \in A$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every f $\in$ A
$N_2(Tf - Tf_0, g_i, \varepsilon) \geq N_1(f - f_0, g_i, \delta)$ and
$M_2(Tf - Tf_0, g_i, \varepsilon) \leq M_1(f - f_0, g_i, \delta)$

Definition 3.3. A mapping T : A $\to$ B is said to be weakly fuzzy continuous at $f_0 \in A$ if for given $\varepsilon > 0$, $\alpha \in (0, 1)$ there exists $\delta = \delta(\alpha, \varepsilon) > 0$ such that for every $f \in A$, $N_1(f - f_0, g_i, \delta) \geq 1 - \beta$ and $M_1(f - f_0, g_i, \delta) < \beta$ implies
$N_2(Tf - Tf_0, g_i, \varepsilon) \geq 1 - \alpha$ and $M_2(Tf - Tf_0, g_i, \varepsilon) < \alpha$ where g_i are linearly dependent for i=1,2.

Definition 3.4. A mapping T : A $\to$ B is said to be intuitionistic sequentially 2-fuzzy continuous at $f_0 \in A$ if for any sequence $\{ f_n \}$ in A,
$\lim_{n\to\infty} N_1(f_n - f_0, g_i, t) = 1$ and $\lim_{n\to\infty} M_1(f_n - f_0, g_i, t) = 0$ for all $t > 0$ implies $\lim_{n\to\infty} N_2(Tf_n - Tf_0, g_i, t) = 1$ and $\lim_{n\to\infty} M_2(Tf_n - Tf_0, g_i, t) = 0$ for all $t > 0$ If T is intuisionistic sequentially 2-fuzzy continuous at each point of A then T is said to be intuisionistic sequentially 2-fuzzy continuous on A.

Theorem 3.5. *Let $T : (A, N_1, M_1) \to (B, N_2, M_2)$ be a mapping where (A, N_1, M_1) & (B, N_2, M_2) are intuitionistic 2-fuzzy normed linear spaces. If T is intuitionastic 2-fuzzy strongly contionous then it is intuitionistic sequentially 2-fuzzy continuous.*

Proof. This is the proof of Theorem 3.5. Suppose that T is intutionistic 2-fuzzy strongly continuous at $f_0 \in A$.
Thus for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $f \in A$.

$$N_2(Tf - Tf_0, g_i, \varepsilon) \geq N_1(f - f_0, g_i, \delta) \text{and} \quad M_2(Tf - Tf_0, g_i, \varepsilon) \geq M_1(f - f_0, g_i, \delta) \tag{3.1}$$

where g_i are linearly independent for i = 1,2 Let $\{ f_n\}$ be a sequence in A such that $f_n \to f_0$ that is

$$\lim_{n\to\infty} N_1(f_n - f_0, g_i, \varepsilon) = 1 \text{and} \lim_{n\to\infty} M_1(f_n - f_0, g_i, \varepsilon) = 0 \tag{3.2}$$

for all $t > 0$ and g_i are linear independent for i = 1, 2 Now from (3.1) we have , $N_2(Tf_n - Tf_0, g_i, \varepsilon) \geq N_1(f_n - f_0, g_i, \delta)$, for n=1, 2, ...
$\implies \lim_{n\to\infty} N_2(Tf_n - Tf_0, g_i, \varepsilon) \geq \lim_{n\to\infty} N_1(f_n - f_0, g_i, \delta)$
$\implies \lim_{n\to\infty} N_2(Tf_n - Tf_0, g_i, \varepsilon) = 1$ by (3.2) Since ε is arbitrary small number it follows that $Tf_n \to Tf_0$. □

Theorem 3.6. *Let $T : (A, N_1, M_1) \to (B, N_2, M_2)$ be a mapping where (A, N_1, M_1) & (B, N_2, M_2) are intutionistic 2-fuzzy normed linear spaces. Then T is intuitionistic 2-fuzzy continuous if and only if it is intuitionistic sequentially 2-fuzzy continuous.*

Proof. This is the proof of Theorem 3.6. Suppose T is Intutionistic 2-fuzzy continuous at $f_0 \in A$. Let $\{ f_n\}$ be a sequence in A such that $f_n \to f_0$.

Let $\varepsilon > 0$ be given. Choose $\alpha \in$ (0,1). Since T is Intutionistic 2-fuzzy continuous at f_0 there exists $\delta = \delta(\alpha, \varepsilon) > 0$ and $\beta = \beta(\alpha, \varepsilon)$ such that for every $f \in A$.

$N_1(f - f_0, g_i, \delta) > 1 - \beta$ and $M_1(f - f_0, g_i, \delta) < \beta$

implies $N_2(Tf - Tf_0, g_i, \varepsilon) > 1 - \alpha$ and $M_2(Tf - Tf_0, g_i, \varepsilon) < \alpha$.

where g_i are linearly independent for i=1,2 since $f_n \to f_0$ in A there exists a positive integer n_0 such that $N_1(f_n f_0, g_i, \varepsilon) > 1 - \beta$ and $M_1(f_n - f_0, g_i, \varepsilon) < \beta$ for all $n \geq n_0$

Then $N_2(Tf_n - Tf_0, g_i, \varepsilon) > 1 - \alpha$ and $M_2(Tf_n - Tf_0, g_i, \varepsilon) < \alpha$ for all $n \geq n_0$.

This implies $\lim_{n\to\infty} N_2(Tf_n - Tf_0, g_i, \varepsilon) \leq 1$ and $\lim_{n\to\infty} M_2(Tf_n - Tf_0, g_i, \varepsilon) = 0$ Thus $Tf_n \to Tf_0$ in (A, N_2, M_2) since $\varepsilon > 0$ is arbitrary.

Next suppose T is intutionistic sequentially 2-fuzzy continuous at f_0, there exists $\varepsilon > 0$ and $\alpha > 0$ such that for any $\delta > 0$ and $\beta \in$ (0,1) there exists h (depending on α, β) such that

$$N_1(f_0 - h, g_i, \delta) > 1 - \beta \text{ and } M_1(f_0 - h, g_i, \delta) < \beta \tag{3.3}$$

But $N_2(Tf - Th, g_i, \varepsilon) < 1 - \alpha$ and $M_2(Tf - Th, g_i, \varepsilon) > \alpha$.

Thus for $\beta = \frac{1}{(n+1)}, \delta = fracn(n+1)$, for n=1, 2,...

There exists hn such that $N_1(f_0 - hn, g_i, \frac{n}{(n+1)}) > 1 - \frac{1}{(n+1)}$,

$M_1(f_0 - hn, g_i, \frac{1}{(n+1)}) < \frac{1}{(n+1)}$

But $N_2(Tf_0 - Th_n, g_i, \varepsilon) \leq 1 - \alpha$ and $M_2(Tf_0 - Th_n, g_i, \varepsilon) > \alpha$

Taking $\delta > 0$,there exists n_0 such that $\frac{1}{(n+1)} < \delta$

for all $n \geq n_0$

then

$N_1(f_0 - hn, g_i, \delta) \geq N_1(f_0 - hn, g_i, \frac{n}{(n+1)}) > 1 - \frac{1}{(n+1)}$

$$\implies \lim_{n\to\infty} N_1(f_0 - h_n, g_i, \delta) \to 1 \tag{3.4}$$

But from (1) $N_2(Tf_0 - Th_n, g_i, \varepsilon) < 1 - \alpha$ So

$$N_2(Tf_0 - Th_n, g_i, \varepsilon) \nrightarrow 1 asn \to \infty \tag{3.5}$$

Also $M_1(f_0 - hn, g_i, \delta) \leq M_2(f_0 - hn, g_i, \frac{n}{(n+1)}) < \frac{n}{(n+1)}$, for all $n \geq N$.

$\therefore \lim_{n\to\infty} M_1(f_0 - h_n, g_i, \delta) < \lim_{n\to\infty} \frac{n}{(n+1)}$

$$i.e. \lim_{n\to\infty} M_1(f_0 - h_n, g_i, \delta) \to 0 \tag{3.6}$$

But from (3.3)

$$M_2(Tf_0 - Th_n, g_i, \varepsilon) > \frac{n}{(n+1)} \nrightarrow 0 as n \to \infty \tag{3.7}$$

Thus combining (3.4) and (3.6) we get $h_n \to f_0$ but combining (3.5) and (3.7) we get $Th_n \nrightarrow Tf_0$ which is a contradiction to our assumption. Hence T is Intuitionistic 2-fuzzy continuous at f_0. □

Definition 3.7. Let$T : (A, N_1, M_1) \to (B, N_2, M_2)$ be a linear operator where (A, N_1, M_1) and (B, N_2, M_2) are Intuitionistic 2-fuzzy normed linear spaces.

T is said to be intuitionistic 2-fuzzy strongly bounded if and only if there exists a positive real number M such that for every $f \in A$ and for every $t \in R$.,

$N_2(Tf, g_i, t) \geq N_1(f, g_i, \frac{t}{M})$ and $M_2(Tf, g_i, t) \geq M_1(f, g_i, \frac{t}{M})$ where g_i are linearly independent

Definition 3.8. $T : (A, N_1, M_1) \to (B, N_2, M_2)$ be a linear operator where (A, N_1, M_1) and (B, N_2, M_2) are Intuitionistic 2-fuzzy normed linear spaces.

T is said to be Intuitionistic 2-fuzzy weakly bounded on A if for any $\alpha \in$ (0,1) there exists $M_\alpha > 0$ such that for every $f \in A$, for all $t \in R$ $N_1(f, g_i, \frac{t}{M_\alpha}) > 1 - \alpha$

$\implies N_2(Tf, g_i, t) > 1 - \alpha$ and $M_1(f, g_i, \frac{t}{M_\alpha}) < \alpha$

$\implies M_2(Tf, g_i, \frac{t}{M_\alpha}) < \alpha$ where gi are linearly independent.

Theorem 3.9. *let $T : (A, N_1, M_1) \to (B, N_2, M_2)$ be a linear operator where (A, N_1, M_1) and (B, N_2, M_2) are Intuitionistic 2-fuzzy normed linear space. If T is intutionistic strongly 2-fuzzy bounded then it is Intuitionistic 2-weakly fuzzy bounded.*

Proof. This is the proof of Theorem 3.9. suppose T is intuitionistic 2-fuzzy strongly bounded , there exists $M > 0$ such that for $f \in A$ and for all $t \in R$ we have,

$N_2(Tf, g_i, t) \geq N_1(f, g_i, \frac{t}{M})$ and $M_2(Tf, g_i, t) \leq M_1(f, g_i, \frac{t}{M})$

Thus for any $\alpha \in (0, 1)$, there exists $M_\alpha (= M) > 0$ such that

$N_1(f, g_i, \frac{t}{M_\alpha}) \geq 1 - \alpha$

$\implies N_2(Tf, g_i, t) > 1 - \alpha$ and $M_1(f, g_i, \frac{t}{M_\alpha}) < \alpha$

$\implies M_2(Tf, g_i, \frac{t}{M_\alpha}) < \alpha$ for all $f \in A$ and $t \in R$.

This implies T is intuitionistic 2-fuzzy weakly bounded. □

Theorem 3.10. *Let (A, N_1, M_1) and (B, N_2, M_2) be two Intuitionistic 2-fuzzy normed linear spaces. Let T be a linear operator from A to B. Then*

i. *T is intuitionistic strongly 2-fuzzy continuous on A of T is intuitionistic strongly 2-fuzzy continuous at $f_0 \in A$.*

ii. *T is intuitionistic strongly 2-fuzzy continuous if and only if T is intuitionistic strongly 2-fuzzy bounded.*

Proof. This is the proof of Theorem 3.10. Since T is intuitionistic strongly 2-fuzzy continuous at $f_0 \in A$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $f \in A$, we have

$N_2(Tf - Tf_0, g_i, \varepsilon) \geq N_1(f - f_0, g_i, \delta)$ and $M_2(Tf - Tf_0, g_i, \varepsilon) \leq M_1(f - f_0, g_i, \delta)$

Taking any $h \in A$ and replacing f by $f + f_0 - h$

we get, $N_2(T(f+f_0-h)-Tf_0,g_i,\varepsilon) \geq N_1(f+f_0-h-f_0,g_i,\delta)$
$\implies N_2(Tf+Tf_0-Th-Tf_0,g_i,\varepsilon) \geq N_1(f-h,g_i,\delta)$
$\implies N_2(Tf-Th,g_i,\varepsilon) \geq N_1(f-h,g_i,\delta)$
and $M_2(T(f+f_0-h)-Tf_0,g_i,\varepsilon) \leq M_1(f+f_0-h-f_0,g_i,\delta)$
$\implies M_2(Tf+Tf_0-Th-Tf_0,g_i,\varepsilon) \leq M_1(f-h,g_i,\delta)$
$\implies M_2(Tf-Th,g_i,\varepsilon) \leq M_1(f-h,g_i,\delta)$
Since h is arbitrary, it follows that T is intuitionistic strongly 2-fuzzy continuous on A. □

References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear space, J. Fuzzy Math. 11(3) (2003) 687–705.
[2] T. Beaula and D. Lilly Esthar Rani, Some aspects of intuitionistic 2-fuzzy 2-normed linear spaces, J. Fuzzy Math. 20(2) (2012) 371–378.
[3] T. Beaula and R. Angeline Sarguna Gifta, Some aspects of 2-fuzzy inner product space, Ann. Fuzzy Math. Inform. 4(2) (2012) 335–342.
[4] S. Gahler, Lineare 2-normierte Raume, Math. Nachr. 28 (1964) 1–43.
[5] R. M. Somasundaram and Thangaraj Beaula, Some aspects of 2-fuzzy 2-normed linear spaces, Bull. Malays. Math. Sci. Soc. (2) 32 (2009) 211–221.
[6] N. Thillaigovindan, S. Anita Shanthi and Y. B. Jun, On lacunary statistical convergence in intuitionistic fuzzy n-normed linear space, Ann. Fuzzy Math. Inform. 1(1) (2011) 119–131.
[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Thangaraj Beaula (edwinbeaula@yahoo.co.in)
Department of Mathematics, Faculty of Arts and Science, TBML College, Porayar, Tamil Nadu, India

D. Lilly Esthar Rani (jestharrani@gmail.com)
Department of Mathematics, Faculty of Arts and Science, TBML College, Porayar, Tamil Nadu, India