Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 6, (June 2014), pp. 891–900
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

New method for solving fuzzy linear programming problem

Mohsen Hekmatnia, Mehdi Allahdadi, Hossein Zehtab

Received 23 June 2013; Revised 6 September 2013; Accepted 4 November 2013

Abstract. This study investigates possibilistic linear programming and offer a new method to achieve optimal value of the necessary degree of constraints for Decision Maker in fuzzy linear programming with fuzzy technological coefficients and solve problem by this value. In the proposed algorithm, fuzzy decision set algorithm have been used that is based on the definition of fuzzy decision. Yet in possibilistic programming problem there were not any method to establish optimum value of necessary degree. When possibilistic linear programming is used for solving fuzzy linear programming problem with fuzzy technological coefficients, the decision maker must establish necessary degree of constraints, there is a need for a method which is able to achieve optimal value of necessary degree and solve the problem.

2010 AMS Classification: 90C70, 90B50

Keywords: Fuzzy decision set algorithm, Fuzzy linear programming, Optimal decision, Possibilistic programming.

Corresponding Author: Mehdi Allahdadi (m_allahdadi@math.usb.ac.ir)

1. Introduction

After Zadeh [9] there has been much research on the possibility theory. Possibilistic decision making models have provided an important aspect in handling practical decision making problems. This study proposes an extension of the solution of possibilistic linear programming problems with fuzzy number parameters which is introduced by Buckley [2]. The solutions of these linear program produce a convex possibility distribution of the method relies on α-cut of the fuzzy number parameters to generate a succession of pairs of classical linear programs. The optimal objective values are corresponding to the necessity levels of the decision variables. Gasimov [3] offer modified sub gradient method to solving fuzzy linear programming problem. This study investigates possibilistic programming and offers a new method to achieve optimal value of the necessary degree of constraints for Decision Maker in fuzzy linear programming by fuzzy technological coefficients and solve the problem by this value. In the proposed algorithm, fuzzy decision set algorithm have been used that is based on the definition of fuzzy decision. Yet in possibilistic programming problem there were not any method to establish optimum value of necessary degree. When possibilistic linear programming is used for solving fuzzy linear programming problem with fuzzy technological coefficients, the decision maker must establish the necessary degree of constraints, there is a need for a method which is able to achieve optimal value of necessary degree and solve the problem

2. Preliminaries

Throughout this work, X is a collection of objects denoted generically by x then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{\{(x,\ \mu_A(x)\}\,|x \in X\}\,.$$

$\mu_A\,(x)$ is called the membership function or grade of membership.

Definition 2.1 ([5])**.** Let $\tilde{A}$ be a fuzzy set, and $\alpha \in [0,\ 1]$. The α-cut of the fuzzy set$\tilde{A}$ is the crisp set $\tilde{A}_\alpha$ given by

$$\tilde{A}_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}.$$

Definition 2.2 ([7, 8])**.** A fuzzy set $\widetilde{A}$ is convex if all α -cuts of $\widetilde{A}$ are convex.

Definition 2.3 ([5])**.** Let $\tilde{A}$ be a fuzzy set, the height h($\tilde{A}$) of $\widetilde{A}$ is defined as:

$$h\left(\tilde{A}\right) = \sup_{x \in \tilde{A}} \mu_{\tilde{A}}(x).$$

if $h(\tilde{A}) = 1$ then the fuzzy set $\widetilde{A}$ is called a normal fuzzy set, otherwise it is called sub normal.

Definition 2.4. A fuzzy number $\tilde{A}$ is a normal and convex fuzzy set with a piecewise continuous membership function.

Definition 2.5. A fuzzy number $\tilde{A}$ is called LR if its membership function is defined as follows:

$$\mu_{\tilde{A}}\,(x) = \begin{cases} l\left(\frac{m-x}{\alpha}\right) : x \leq m, \alpha > 0 \\ R\left(\frac{x-m}{\beta}\right) : x \geq m, \beta > 0. \end{cases}$$

LR Fuzzy number $\tilde{A}$ is shown as follows:

$$\tilde{A} = (m, \alpha, \beta)_{LR}.$$

Where m, α, β are the middle, the left and the right width respectively.

Definition 2.6. A triangular fuzzy number is a LR number if the L (X) and R(X) functions are as follows:

$$L\,(x) = R\,(x) = max\,\{0, 1 - |x|\}\,.$$

If $\alpha = \beta$, then it is called symmetric triangular fuzzy number.

Definition 2.7 ([1]). (Bellman and Zadeh fuzzy decision) Bellman and Zadeh defined the following three concepts:

1- Fuzzy goal (G): It is a fuzzy set that is specified by its membership function as follows:

$$\mu_G\left(x\right) : X \to \left[0, 1\right].$$

2-Fuzzy constraint (C): It is a fuzzy set, its membership function is specified as follows:

$$\mu_C\left(x\right) : X \to [0, 1].$$

3- Fuzzy decision (D): It is expressed as a result of fuzzy goal and fuzzy constraint.

$$\mu_D\left(x\right) : X \to [0, 1]\, D = G \cap C,$$

$$\mu_D\left(x\right) = \min(\mu_G\left(x\right), \mu_C\left(x\right)).$$

Definition 2.8 ([4]). The possibility degree of the variable α that is defined by the distribution facility $\mu_{\tilde{A}}$ be in fuzzy set $\tilde{B}$.

$$pos(\alpha \in \tilde{B}) = sup_r \min\left(\mu_{\tilde{A}}\left(r\right), \mu_{\tilde{B}}\left(r\right)\right).$$

Definition 2.9. Necessity degree of the variable α that is defined by the distribution facility $\mu_{\tilde{A}}$ be in fuzzy set $\tilde{B}$.

$$nes(\alpha \in \tilde{B}) = \inf_r \max(1 - \mu_{\tilde{A}}(r), \mu_{\tilde{B}}(r)).$$

Let $\tilde{B}$ be a crisp set, in this case $B = (-\infty, g]\, or B = [g, \infty)$.

A - Suppose $B = (-\infty, g]$, Since B is a real set, so the membership function is:

$$\mu_B\left(r\right) = C_B\left(x\right) = \begin{cases} 1 & : r \leq g \\ 0 & : r > g, \end{cases}$$

and

$$pos\left(r \in B\right) = sup_{r \leq g}\left(\mu_{\tilde{A}}\left(r\right)\right).$$

B -Suppose $B = [g, \infty)$, since B is a real set, so its membership function is:

$$\mu_B\left(r\right) = C_B\left(x\right) = \begin{cases} 1 & : r \geq g \\ 0 & : r < g, \end{cases}$$

$$pos\left(r \in B\right) = sup_{r \geq g}\left(\mu_{\tilde{A}}\left(r\right)\right),$$

and

$$nes\left(r \in B\right) = 1 - sup_{r < g}\left(\mu_{\tilde{A}}\left(r\right)\right).$$

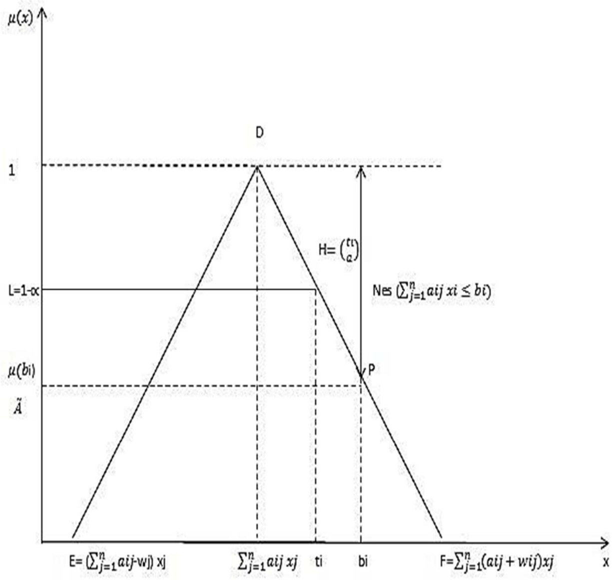

Figure 1. Necessity degree of constraints

3. Solving fuzzy linear programming

Consider the following fuzzy linear programming problem:

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j \\ &s.t. \sum_{j=1}^{n} a_{ij} x_j \leq b_i \ i = 1, \ldots, m \\ &\qquad x_j \geq 0. \end{aligned} \tag{1}$$

Where c_j, a_{ij}, b_i are possibilistic variables limited by symmetric triangular fuzzy numbers $\tilde{B}_i, \tilde{A}_{ij}, \tilde{C}_j$, we have:

$$\tilde{B}_i = (b_i, w_{b_i}), \tilde{A}_{ij} = (a_{ij}, w_{a_{ij}}),$$

$$\tilde{C}_j = (c_j, w_{c_j}).$$

Note that in solving fuzzy linear programming problems with symmetric triangular fuzzy numbers, w_{b_i}, w_{aij} and w_{c_j} are predetermined.

Suppose that the minimum necessity degree of constraints, established by the decision maker is equal to α, as seen in figure 1, H is in $\overline{DF}$ and $t_i \leq b_i$. Below is the equation of the line $\overline{DF}$.

$$\overline{DF} : y - 1 = \frac{1}{-\sum_{j=1}^{n} w_{a_{ij}} x_j} \left(x - \sum_{j=1}^{n} a_{ij} x_j \right).$$

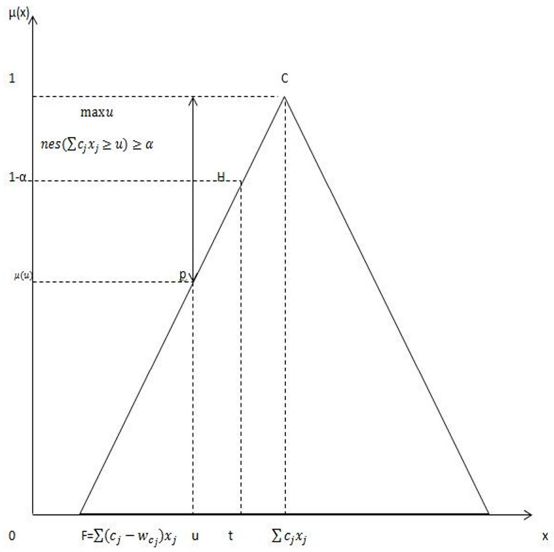

FIGURE 2. Necessity degree of objective function

By putting the coordinates of point H on it we will have:

$$\sum_{j=1}^{n}(a_{ij}+w_{a_{ij}}\alpha)x_j \le b_i.$$

[6]

If the decision maker wants to maximize the object function of his problem with a minimum of necessary degree α, it means:

$$\begin{aligned} &\max u \\ s.t.\quad & nes\left(\sum_{j=1}^{n} c_j x_j \ge u\right) \ge \alpha. \end{aligned} \tag{2}$$

Since $u \le t$, as seen in figure 2, we know that H is in $\overline{CF}$, therefore, by putting its coordinate in $\overline{CF}$ we will have:

$$t = \sum (c_j - \alpha w_{c_j})x_j,$$

and therefore (2) is equal to:

$$\begin{aligned} &\max \sum_{j=1}^{n}(c_j - w_{c_j}\alpha)x_j \\ &s.t. \sum_{j=1}^{n}(a_{ij}+w_{a_{ij}}\alpha)x_j \le b_i \\ &\qquad x \ge 0. \end{aligned} \tag{3}$$

So the problem (2) with a known amount of α becomes the conventional linear programming (3) [4].

3.1. **Fuzzy Decision set algorithm for obtaining the best necessary degree [3].**

Fuzzy decision set algorithm is a bisection method for obtaining the best necessary degree of constraints. We use this method because it is absolutely convergent.

Step 1. Set α =1 and test whether a feasible set exists or not using phase one of the simplex method.

If a feasible set exists, set α =1 Otherwise, set $\alpha^l = 0$, $\alpha^R = 1$ and go to the next step.

Step 2. For the value of $\alpha = \frac{\alpha^l+\alpha^R}{2}$, update the value of α^l and α^R by using the bisection method as follows:

1. $\alpha = \alpha^l$ If feasible set is nonempty for α.
2. $\alpha = \alpha^R$ If feasible set is empty for α.

Consequently, for each α; test whether a feasible set exists or not.

Using phase one of the Simplex method and determine the maximum value α^* which is satisfying the constraints of the problem.

Theorem 3.1. *Bisection method is convergent.*

Proof. If achieved value of necessity degree in Nth iteration be α_n, and real value of necessity degree be α therefor calculation error is given by following inequality:

$$|\alpha_n - \alpha| \leq \frac{1}{2^n}.$$

so when $n \to \infty$ therefor calculation error $\to 0$ therefor this method is convergent. □

3.2. **Optimal value of necessary degree for decision maker in possibilistic programming.**

Consider fuzzy linear programming problem with fuzzy technological coefficients:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i \\ & x_j \geq 0. \end{aligned} \qquad (4)$$

Where $\tilde{a}_{ij} = \left(a_{ij}, w_{a_{ij}}\right)$ is symmetric triangular fuzzy number. If the decision maker wants to hold the constraints with a minimum of necessary degree α, $\alpha \in [0, 1]$, problem (4) by using (3) and replace w_{c_j} with 0, becomes problem(5).

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j \\ &s.t. \ \sum_{j=1}^{n} (a_{ij} + \alpha w_{a_{ij}}) x_j \le b_i \\ &\quad x_j \ge 0, \alpha \in [0,1]. \end{aligned} \tag{5}$$

But to get the maximum value α that the problem stays feasible and achieved solution be optimal for fixed α, we do as follows:

$$\sum_{j=1}^{n} (a_{ij} + \alpha w_{a_{ij}}) x_j \le b_i.$$

Then for the feasibility of the problem , achieved value for the decision variables, should satisfy this constraint. We know that the optimal solutions of (5), are among the optimal solutions of the two problems (6) and (7)[3].

$$\begin{aligned} z_1 = \ &max \ \sum_{j=1}^{n} c_j x_j \\ &s.t. \sum_{j=1}^{n} a_{ij} x_j \le b_i \\ &\quad x_j \ge 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} z_2 = \ &max \ \sum_{j=1}^{n} c_j x_j \\ &s.t. \sum_{j=1}^{n} (a_{ij} + w_{a_{ij}}) x_j \le b_i \\ &\quad x_j \ge 0. \end{aligned} \tag{7}$$

Let $z_u = \max\{z_1, z_2\}, z_l = min\{z_1, z_2\}$ consider fuzzy goal (G) with following membership function:

$$\mu_G(x) = \begin{cases} 1 & : \sum_{j=1}^{n} c_j x_j \ge z_u \\ \frac{\sum_{j=1}^{n} c_j x_j - z_l}{z_u - z_l} & : z_l \le \sum_{j=1}^{n} c_j x_j < z_u \\ 0 & : o/w. \end{cases}$$

Since $\mu_G(x) \in [0,1]$ and $\alpha \in [0,1]$, Therefor there exist a feasible solution such that:

$$\frac{\sum_{j=1}^{n} c_j x_j - z_l}{z_u - z_l} \ge \alpha$$

and so:

$$\alpha(z_u - z_l) - \sum_{j=1}^{n} c_j x_j + z_l \le 0.$$

This condition ensures that the object function, for fixed α, did not worsen. Hence, to determine the maximum value of α, problem that satisfies the above inequality should be as following:

$$
\begin{aligned}
&\max \quad \alpha \\
s.t.\quad & \alpha(z_u - z_l) - \sum_{j=1}^{n} c_j x_j + z_l \leq 0 \\
& \sum_{j=1}^{n} (a_{ij} + \alpha w_{a_{ij}}) x_j \leq b_i \\
& x_j \geq 0, \alpha \in [0,1] .
\end{aligned} \tag{8}
$$

4. Examples

Example 4.1. Let the following fuzzy linear programming problem:

$$
\begin{aligned}
&Max \quad x_1 + 2x_2 \\
s.t.\quad & \widetilde{a_{11}} x_1 + \widetilde{a_{12}} x_2 \leq 6 \\
& \widetilde{a_{21}} x_1 + \widetilde{a_{22}} x_2 \leq 10 \\
& x_2 \geq 0 \\
& x_1 \geq 0.
\end{aligned}
$$

Where

$$\tilde{a}_{11} = (3,1) \ , \ \tilde{a}_{12} = (2,2),$$

$$\tilde{a}_{21} = (5,1) \ , \ \tilde{a}_{22} = (3,2).$$

By (5) we have:

$$
\begin{aligned}
z_1 = & \max x_1 + 2x_2 \\
s.t.\quad & 3x_1 + 2x_2 \leq 6 \\
& 5x_1 + 3x_2 \leq 10 \\
& x_1 \geq 0, x_2 \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
z_2 = & \max x_1 + 2x_2 \\
s.t.\quad & 4x_1 + 4x_2 \leq 6 \\
& 6x_1 + 5x_2 \leq 10 \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

The optimal solutions of the sub problems are:

$$
\begin{aligned}
& z_1 = 6 \ z_2 = 3, \\
& x_1^* = 0 \ x_1^* = 0, \\
& x_2^* = 3 \ x_2^* = 1.5.
\end{aligned}
$$

To maximize α by (8) we have:

$$
\begin{aligned}
& max \quad \alpha \\
s.t.\quad & (\alpha + 3)\, x_1 + (2\alpha + 2)\, x_2 \leq 6 \\
& (\alpha + 5) x_1 + (2\alpha + 3)\, x_2 \leq 10 \\
& -x_1 - 2x_2 + 3\alpha + 3 \leq 0 \\
& x_1 \geq 0, x_2 \geq 0, \alpha \in [0,1] .
\end{aligned}
$$

By solving this model with fuzzy decision set algorithm, mentioned in section 3, we have:

$$\alpha^* = 0.4142136.$$

By setting this value in (5) we will have:

$$\begin{aligned} & z = \max x_1 + 2x_2 \\ & s.t.\ \ 3.4142136x_1 + 2.8284272x_2 \leq 6 \\ & \quad\ \ 5.4142136x_1 + 3.8284272x_2 \leq 10 \\ & \quad\ \ x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

The optimal value and the optimal solutions are:
$x_1^* = 0$ and $x_2^* = 2.121320$ and $z = 4.242641$.
Which are equal to the obtained solutions by the modified sub gradient methods[3].

Example 4.2. Let the following fuzzy linear programming problem:[6]

$$\begin{aligned} & z = \ \max\ 2x_1 + 3x_2 \\ & s.t.\ \ \tilde{1}x_1 + \tilde{2}x_2 \leq 4 \\ & \quad\ \ \tilde{3}x_1 + \tilde{1}x_2 \leq 6 \\ & \quad\ \ x_1\ ,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} & \tilde{1} = (1,1)\,, \tilde{2} = (2,3)\,, \tilde{3} = (3,2) \\ & , \tilde{1} = (1,3). \end{aligned}$$

By (5) we have:

$$\begin{aligned} & z_1 = \ \max 2x_1 + 3x_2 \\ & s.t.\ \ x_1 + 2x_2 \leq 4 \\ & \quad\ \ 3x_1 + x_2 \leq 6 \\ & \quad\ \ x_1\ ,\ x_2 \geq 0, \end{aligned}$$

and

$$\begin{aligned} & z_2 = \max 2x_1 + 3x_2 \\ & s.t.\ \ 2x_1 + 5x_2 \leq 4 \\ & \quad\ \ 5x_1 + 4x_2 \leq 6 \\ & \quad\ \ x_1, x_2 \geq 0. \end{aligned}$$

The optimal solutions and the optimal value are:
$z_1 = 6.8$, ${x_1}^* = 1.6$ and ${x_2}^* = 1.2$, $z_2 = 3.06$, ${x_1}^* = 0.82$, ${x_2}^* = 0.47$.
To maximize α by (8) we have:

$$\begin{aligned} & \quad\ \max\ \ \alpha \\ & s.t.\ \ 2x_1 + 3x_2 \geq 3.06 + 3.74\alpha \\ & \quad\ \ (1+\alpha)\,x_1 + (2+3\alpha)\,x_2 \leq 4 \\ & \quad\ \ (3+2\alpha)\,x_1 + (1+3\alpha)\,x_2 \leq 6 \\ & \quad\ \ \alpha \in [0,1]\,x_1, x_2 \geq 0. \end{aligned}$$

By solving the above problem by fuzzy decision sets algorithm and after 13 iteration we have:

$$\alpha^* = 0.397323.$$

By setting this value in (5) we will have:

$$\begin{aligned} & z = \ \max 2x_1 + 3x_2 \\ & s.t.\ \ 1.397323x_1 + 3.191969x_2 \leq 4 \\ & \quad\ \ 3.794646x_1 + 2.191969x_2 \leq 6 \\ & \quad\ \ x_1, x_2 \geq 0. \end{aligned}$$

The optimal solutions and the optimal value are:

$$x_1{}^* = 1.1475877 \quad x_2{}^* = 0.75147$$
$$z \; = 4.5474.$$

5. Conclusions

Yet in possibilistic programming problem there were not any method to establish optimum value of necessary degree. By using the fuzzy decision set algorithm and establishing the optimal value of α, relations between possibilistic programming problem and sub gradient problem have been proved. Best α for the decision maker in possibilistic programming is equal to λ in solving corresponding nonlinear programming problem by the sub gradient method and both problems when $\lambda = \alpha$ have same solutions. It is useful that instead solving nonlinear programming problem, using this method to find the optimal value of α because:

A) Fuzzy decision set algorithm is a bisection method that is absolutely convergence.

B) Solving nonlinear programming problem is harder than linear programming problem.

References

[1] R. E. Bellman and L. A. Zadeh, Decision-making in a fuzzy environment, Management Sci. 17 (1970/71), B141–B164.

[2] C. Buckley, Possibilistic programming, Fuzzy Sets and Systems 31 (1989) 329–341.

[3] R. Gasimov and K. Yenilmez, Solving fuzzy linear programming problems with linear membership functions, Turkish J. Math. 26 (2002) 375–396.

[4] M. Inuiguchi and J. Ramik, Possibilistic linear programming, A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, Fuzzy Sets and systems 11 (2000) 3–28.

[5] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic-Theory and Applications, Prentice-Hall Inc, 1995.

[6] Y. G.Lai and Hwang, C-L Fuzzy Mathematical Programming, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag 1992.

[7] A. A. Noura, A. Payan and E. Poodineh, Credibility approach to solve fuzzy multiple objective minimum cost flow problem, Ann. Fuzzy Math. Inform. 28 (2011) 221–235.

[8] M. Sakawa and H. Yana, Interactive decision making for multi-objective linear fractional programming problems with fuzzy parameters, Cybernet. Systems Anal. 16 (1985) 377–397.

[9] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 13–28.

Mohsen Hekmatnia (mohsenhekmatnia68@gmail.com)
Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran

Mehdi Allahdadi (m_allahdadi@math.usb.ac.ir)
Mathematics Faculty, University of Sistan and Baluchestan, Zahedan, Iran

Hossein Zehtab (h.zehtab@yahoo.com)
Faculty of Veterinarian research part, Agriculture Research and Natural Resources Center, Golestan, Iran