Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 6, (June 2014), pp. 879–890
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Homomorphism theorems in the new view of fuzzy rings

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Received 2 August 2013; Revised 20 September 2013; Accepted 14 October 2013

Abstract. In this paper, we give an example of fuzzy binary operation, fuzzy group, a new fuzzy binary operation on a nonempty set, and a new fuzzy ring. Also, we give homomorphism theorems between two fuzzy rings and some related properties are investigated.

2010 AMS Classification: 16S99, 16Y99, 03E72, 08A72

Keywords: Fuzzy binary operation, Fuzzy ring, Fuzzy ideal, Fuzzy homomorphism.

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1. Introduction

In 1965, L. A. Zadeh [7] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Then, in 1971, A. Rosenfeld [5] used the notion of a fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group. Rosenfeld's paper inspired the development of fuzzy abstract algebra. After these studies, many mathematicians studied these subject. For more details, see [2].

In 2004, X. Yuan and E. S. Lee [6] presented a new kind of fuzzy group based on fuzzy binary operation. Recently, H. Aktaş and N. Çağman [1] considered, by the use of X. Yuan and E. S. Lee's definition of fuzzy group based on fuzzy binary operations, a new type of fuzzy ring. Also, in 2010, M. A. Öztürk, Y. B. Jun and H. Yazarlı [3] introduced a new kind of fuzzy gamma ring and discussed related results (see [4]).

In this paper, an example of a new kind of fuzzy group [6] and a new kind of fuzzy ring [1] is presented. Moreover, homomorphism theorems between two fuzzy rings are proved.

2. Preliminaries

In this section, we summarize the preliminary definitions that will be required in this paper.

Definition 2.1 ([6])**.** Let G be a nonempty set and R be a fuzzy subset of $G\times G\times G$. R is called a fuzzy binary operation on G if
(i) For all $a,b\in G$, there exists $c\in G$ such that $R(a,b,c)>\theta$;
(ii) For all $a,b,c_1,c_2\in G$, $R(a,b,c_1)>\theta$ and $R(a,b,c_2)>\theta$ imply $c_1=c_2$,
where $\theta\in[0,1)$ is fixed real number.

Let R be a fuzzy binary operation on G. Then we have a mapping $R:F(G)\times F(G)\to F(G)$ defined by $R(A,B)$ where

$$F(G)=\{A\,|A:G\to[0,1]\ \text{ is a mapping}\}$$

and

$$R(A,B)(c)=\bigvee_{a,b\in G}(A(a)\wedge B(b)\wedge R(a,b,c)) \tag{2.1}$$

for all $c\in G$.

Let $A=\{a\}$, $B=\{b\}$ and let $R(A,B)$ be denoted as $a\circ b$. Then

$$(a\circ b)(c)=R(a,b,c) \tag{2.2}$$

for all $c\in G$. Also,

$$((a\circ b)\circ c)(z)=\bigvee_{d\in G}(R(a,b,d)\wedge R(d,c,z)) \tag{2.3}$$

and

$$(a\circ(b\circ c))(z)=\bigvee_{d\in G}(R(b,c,d)\wedge R(a,d,z)) \tag{2.4}$$

for all $z\in G$.

Definition 2.2 ([6])**.** Let G be a nonempty set and R be a fuzzy binary operation on G. Then (G,R) is called a fuzzy group if the following conditions are true:

(G_1) For all $a,b,c,z_1,z_2\in G$, $((a\circ b)\circ c)(z_1)>\theta$ and $(a\circ(b\circ c))(z_2)>\theta$ imply $z_1=z_2$;

(G_2) There exists $e_\circ\in G$ such that $(e_\circ\circ a)(a)>\theta$ and $(a\circ e_\circ)(a)>\theta$ for all $a\in G$ ($e_\circ$ is called the identity element of G);

(G_3) There exists $b\in G$ such that $(a\circ b)(e_\circ)>\theta$ and $(b\circ a)(e_\circ)>\theta$ for all $a\in G$ (b is called the inverse of a and denoted as a^{-1}).

Remark 2.3 ([6])**.** Let (G,R) be a fuzzy group. Then

$$R(a,b,c)>\theta\Rightarrow R\left(c,b^{-1},a\right)>\theta \text{ and } R\left(a^{-1},c,b\right)>\theta.$$

Let H be a nonempty set and R a fuzzy binary operation on H. Then R is a fuzzy subset of $H\times H\times H$. Let S be a fuzzy binary operation on H. In this case, we may regard S as the mapping

$$S:F(H)\times F(H)\to F(H)\,,\ \ (A,B)\mapsto S(A,B)$$

where

$$F(H) = \{A \,|A : H \to [0,1] \text{ is a mapping}\}$$

and

$$S(A,B)(c) = \bigvee_{a,b\in H} (A(a) \wedge B(b) \wedge S(a,b,c)) \tag{2.5}$$

for all $c \in H$.

Let $A = \{a\}$ and $B = \{b\}$. Let $R(A,B)$ and $S(A,B)$ be denoted as $a \circ b$ and $a * b$, respectively. We will use the following notation to simplify the calculations:

$$(a \circ b)(c) = R(a,b,c) \tag{2.6}$$

and

$$(a * b)(c) = S(a,b,c) \tag{2.7}$$

for all $c \in H$. Also,

$$((a \circ b) \circ c)(z) = \bigvee_{d\in H} (R(a,b,d) \wedge R(d,c,z)), \tag{2.8}$$

$$(a \circ (b \circ c))(z) = \bigvee_{d\in H} (R(b,c,d) \wedge R(a,d,z)), \tag{2.9}$$

$$((a * b) * c)(z) = \bigvee_{d\in H} (S(a,b,d) \wedge S(d,c,z)), \tag{2.10}$$

$$(a * (b * c))(z) = \bigvee_{d\in H} (S(b,c,d) \wedge S(a,d,z)), \tag{2.11}$$

$$(a * (b \circ c))(z) = \bigvee_{d\in H} (R(b,c,d) \wedge S(a,d,z)), \tag{2.12}$$

$$((a * b) \circ (a * c))(z) = \bigvee_{d,e\in H} (S(a,b,d) \wedge S(a,c,e) \wedge R(d,e,z)), \tag{2.13}$$

$$((a \circ b) * c)(z) = \bigvee_{d\in H} (R(a,b,d) \wedge S(d,c,z)), \tag{2.14}$$

$$((a * c) \circ (b * c))(z) = \bigvee_{d,e\in H} (S(a,c,d) \wedge S(b,c,e) \wedge R(d,e,z)) \tag{2.15}$$

for all $z \in H$.

Definition 2.4 ([1])**.** Let H be a nonempty set, R and S fuzzy binary operations on H, all with the same value of θ. Then (H,R,S) is called a fuzzy ring if the following conditions hold:

(H_1) (H,R) is an abelian fuzzy group;

(H_2) For all $a,b,c,z_1,z_2 \in H$, $((a * b) * c)(z_1) > \theta$ and $(a * (b * c))(z_2) > \theta$ imply $z_1 = z_2$;

(H_3) For all $a, b, c, z_1, z_2 \in H$,
(i) $(a * (b \circ c))(z_1) > \theta$ and $((a * b) \circ (a * c))(z_2) > \theta$ imply $z_1 = z_2$,
(ii) $((a \circ b) * c)(z_1) > \theta$ and $((a * c) \circ (b * c))(z_2) > \theta$ imply $z_1 = z_2$.

The identity element of the fuzzy group (H, R) is called the zero element of (H, R, S) and denoted by $e_\circ$. Let (H, R, S) be a fuzzy ring and H' be a nonempty subset of H. Let $R_{H'}(a, b, c) = R(a, b, c)$ and $S_{H'}(a, b, c) = S(a, b, c)$ for all $a, b, c \in H'$. In this case, we have

$$(a \circ b)(c) = R_{H'}(a, b, c) = R(a, b, c)$$

and

$$(a * b)(c) = S_{H'}(a, b, c) = S(a, b, c)$$

for all $a, b, c \in H'$.

Definition 2.5 ([1])**.** Let (H, R, S) be a fuzzy ring and H' be a nonempty subset of H for which

(i) $(a \circ b)(c) > \theta$ implies $c \in H'$ and $(a * b)(c) > \theta$ implies $c \in H'$ for all $a, b \in H'$ and $c \in H$;

(ii) (H', R, S) is a fuzzy ring.

Then (H', R, S) is called a fuzzy subring of (H, R, S).

Proposition 2.6 ([1])**.** *Let (H, R, S) be a fuzzy ring and H' be a nonempty subset of H. Then (H', R, S) is a fuzzy subring of H if and only if*

(i) $(a \circ b)(c) > \theta$ implies $c \in H'$ and $(a * b)(c) > \theta$ implies $c \in H'$ for all $a, b \in H'$ and $c \in H$;

(ii) $a^{-1} \in H'$ for all $a \in H'$.

Definition 2.7 ([1])**.** Let (H, R, S) be a fuzzy ring. A nonempty subset I of H is called a left (right) fuzzy ideal of H if

(i) $(a \circ b)(c) > \theta$ implies $c \in I$ for all $a, b \in I$ and $c \in H$;

(ii) $a^{-1} \in I$ for all $a \in I$;

(iii) $(h * a)(c) > \theta$ implies $c \in I$ ($(a * h)(c) > \theta$ implies $c \in I$) for all $a \in I$, $h \in H$ and $c \in H$.

Also, a nonempty subset I of a fuzzy ring (H, R, S) is called a fuzzy ideal of H if I is both a left and a right fuzzy ideal of H.

Remark 2.8 ([1])**.** From the definition of a fuzzy left (right) ideal of (H, R, S), I is a fuzzy subring of (H, R, S).

Proposition 2.9 ([1])**.** *Let I_i $(i \in \Lambda)$ be a fuzzy ideal of fuzzy ring (H, R, S), where Λ is an index set. Then $\bigcap_{i \in \Lambda} I_i$ is a fuzzy ideal of H.*

Let I be a fuzzy ideal of fuzzy ring (H, R, S) and $\Delta = \{a \circ I \mid a \in H\}$. Define a relation over Δ by

$$a_1 \circ I \sim a_2 \circ I \Leftrightarrow \exists u \in I \text{ such that } R\left(a_1^{-1}, a_2, u\right) > \theta.$$

The fuzzy relation " $\sim$ " on the set Δ is a fuzzy equivalence relation by [6, Theorem 4.1]. Let $[a \circ I] = \{a' \circ I \mid a' \circ I \sim a \circ I\}$, $\overline{a} = \{a' \mid a' \in H \text{ and } a' \circ I \sim a \circ I\}$ and $H/I = \{[a \circ I] \mid a \in H\}$. Also, (I, R) is a fuzzy subgroup of (H, R) and since (H, R) is abelian, (I, R) is a normal fuzzy group of (H, R) by [6, Theorem 3.1]. Hence, H/I

denotes the set of all coset $[a \circ I]$ such that $(a \circ I)(u) = \bigvee_{x \in I} R(a, x, u)$ for all $a \in H$, $(H/I, \overline{R})$ is a commutative fuzzy group by [6, Theorem 4.2], where

(2.16)
$$([a \circ I] \oplus [b \circ I])(c \circ I) = \overline{R}([a \circ I], [b \circ I], [c \circ I]) = \bigvee_{(a',b',c') \in \overline{a} \times \overline{b} \times \overline{c}} R(a', b', c').$$

Theorem 2.10 ([1])**.** *Let (H, R, S) be a fuzzy ring and I be a fuzzy ideal of H. Then the fuzzy quotient group $(H/I, \overline{R})$ is a fuzzy ring with*

$$\begin{aligned}([a \circ I] \otimes [b \circ I])([c \circ I]) &= \overline{S}([a \circ I], [b \circ I], [c \circ I]) \\ &= \bigvee_{(a',b',c') \in \overline{a} \times \overline{b} \times \overline{c}} S(a', b', c').\end{aligned}$$

Definition 2.11 ([1])**.** If (H, R, S) is a fuzzy ring and I is a fuzzy ideal of H, then the fuzzy ring $\left(H/I, \overline{R}, \overline{S}\right)$ is called the fuzzy quotient ring of H by I.

Definition 2.12 ([1])**.** Let (H_1, R_1, S_1) and (H_2, R_2, S_2) be two fuzzy rings and φ be a function from H_1 into H_2. Then φ is called a fuzzy homomorphism of H_1 into H_2 if

(i) $R_1(a, b, c) > \theta$ implies $R_2(\varphi(a), \varphi(b), \varphi(c)) > \theta$ for all $a, b, c \in H_1$;
(ii) $S_1(a, b, c) > \theta$ implies $S_2(\varphi(a), \varphi(b), \varphi(c)) > \theta$ for all $a, b, c \in H_1$.

Definition 2.13 ([1])**.** A homomorphism φ of a fuzzy ring H_1 into a fuzzy ring H_2 is called

(i) a monomorphism if φ is one to one;
(ii) an epimorphism if φ is onto;
(iii) an isomorphism if φ is both one to one and onto.

If φ is an isomorphism of H_1 onto H_2, then two fuzzy rings H_1 and H_2 are called isomorphic and we denote this by $H_1 \cong H_2$.

Theorem 2.14 ([1])**.** *Let (H_1, R_1, S_1) and (H_2, R_2, S_2) be two fuzzy rings and let φ be a fuzzy homomorphism of H_1 into H_2. Then*

(i) $\varphi(e_\circ) = e'_\circ$, *where $e'_\circ$ is the zero of H_2.*
(ii) $\varphi\left(a^{-1}\right) = (\varphi(a))^{-1}$ *for all $a \in H_1$.*

Theorem 2.15 ([1])**.** *Let (H_1, R_1, S_1) and (H_2, R_2, S_2) be two fuzzy rings and let φ be a fuzzy homomorphism of H_1 into H_2. Then*

(i) $Ker\varphi = \{a \in H_1 \mid \varphi(a) = e'_\circ\}$ *is a fuzzy ideal of H_1.*
(ii) $\operatorname{Im}\varphi = \{\varphi(a) \mid a \in H_1\}$ *is a fuzzy subring of H_2.*

Theorem 2.16 ([1])**.** *Let $\varphi : (H_1, R_1, S_1) \to (H_2, R_2, S_2)$ be a fuzzy epimorphism. Then $H_1/N \cong H_2$, where $N = Ker\varphi$.*

3. An example of a new view of fuzzy rings

In [6], Yuan and Lee introduced concept of fuzzy binary operation and fuzzy group. In [1], Aktaş and Çağman introduced a new fuzzy binary operation on H and a new fuzzy ring. In this section, we give an example of fuzzy binary operation, fuzzy group, a new fuzzy binary operation on H and a new fuzzy ring.

Example 3.1. Let

$$H = \{ \begin{bmatrix} \bar{0} & \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} & \bar{0} \end{bmatrix} \} \subset (\mathbb{Z}_2)_{1\times 3}$$

be a nonempty set. We will use the following notation to simplify the calculations:

$$a = \begin{bmatrix} \bar{0} & \bar{0} & \bar{0} \end{bmatrix}, b = \begin{bmatrix} \bar{1} & \bar{0} & \bar{0} \end{bmatrix}, c = \begin{bmatrix} \bar{1} & \bar{1} & \bar{0} \end{bmatrix}, d = \begin{bmatrix} \bar{0} & \bar{1} & \bar{0} \end{bmatrix}.$$

Let us define R and S fuzzy binary operations on H, all with the same value of $\theta = 0.7$, as follows:

$R(a,a,a) = 0.9$	$R(b,a,a) = 0.2$	$R(c,a,a) = 0.2$	$R(d,a,a) = 0.4$
$R(a,a,b) = 0.3$	$R(b,a,b) = 0.9$	$R(c,a,b) = 0.3$	$R(d,a,b) = 0.1$
$R(a,a,c) = 0.2$	$R(b,a,c) = 0.4$	$R(c,a,c) = 0.9$	$R(d,a,c) = 0.2$
$R(a,a,d) = 0.0$	$R(b,a,d) = 0.2$	$R(c,a,d) = 0.2$	$R(d,a,d) = 0.8$
$R(a,b,a) = 0.2$	$R(b,b,a) = 0.8$	$R(c,b,a) = 0.4$	$R(d,b,a) = 0.1$
$R(a,b,b) = 0.8$	$R(b,b,b) = 0.2$	$R(c,b,b) = 0.1$	$R(d,b,b) = 0.4$
$R(a,b,c) = 0.2$	$R(b,b,c) = 0.1$	$R(c,b,c) = 0.9$	$R(d,b,c) = 0.3$
$R(a,b,d) = 0.2$	$R(b,b,d) = 0.3$	$R(c,b,d) = 0.4$	$R(d,b,d) = 0.9$
$R(a,c,a) = 0.1$	$R(b,c,a) = 0.4$	$R(c,c,a) = 0.8$	$R(d,c,a) = 0.2$
$R(a,c,b) = 0.3$	$R(b,c,b) = 0.9$	$R(c,c,b) = 0.1$	$R(d,c,b) = 0.1$
$R(a,c,c) = 0.9$	$R(b,c,c) = 0.4$	$R(c,c,c) = 0.0$	$R(d,c,c) = 0.3$
$R(a,c,d) = 0.1$	$R(b,c,d) = 0.1$	$R(c,c,d) = 0.3$	$R(d,c,d) = 0.9$
$R(a,d,a) = 0.4$	$R(b,d,a) = 0.2$	$R(c,d,a) = 0.2$	$R(d,d,a) = 0.8$
$R(a,d,b) = 0.1$	$R(b,d,b) = 0.9$	$R(c,d,b) = 0.1$	$R(d,d,b) = 0.4$
$R(a,d,c) = 0.3$	$R(b,d,c) = 0.3$	$R(c,d,c) = 0.9$	$R(d,d,c) = 0.2$
$R(a,d,d) = 0.9$	$R(b,d,d) = 0.1$	$R(c,d,d) = 0.3$	$R(d,d,d) = 0.1$

(H, R) is a fuzzy group, since the following conditions are true:

- (G_1) $((a \circ b) \circ c)(z_1) > 0.7$ and $(a \circ (b \circ c))(z_2) > 0.7$ imply $z_1 = z_2$ for all $a, b, c, z_1, z_2 \in H$;
- (G_2) There exists $e_\circ \in H$ such that $(e_\circ \circ a)(a) > 0.7$ and $(a \circ e_\circ)(a) > 0.7$ for all $a \in H$ ($e_\circ$ is called an identity element of H);
- (G_3) There exists $b \in H$ such that $(a \circ b)(e_\circ) > 0.7$ and $(b \circ a)(e_\circ) > 0.7$ for all $a \in H$ (b is called an inverse element of a and denoted as a^{-1}).

Furthermore,

$S(a,a,a) = 0.8$	$S(b,a,a) = 0.9$	$S(c,a,a) = 0.9$	$S(d,a,a) = 0.8$
$S(a,\alpha,b) = 0.1$	$S(b,a,b) = 0.3$	$S(c,a,b) = 0.2$	$S(d,a,b) = 0.1$
$S(a,\alpha,c) = 0.3$	$S(b,a,c) = 0.1$	$S(c,a,c) = 0.4$	$S(d,a,c) = 0.3$
$S(a,\alpha,d) = 0.5$	$S(b,a,d) = 0.2$	$S(c,a,d) = 0.3$	$S(d,a,d) = 0.2$
$S(a,b,a) = 0.9$	$S(b,b,a) = 0.8$	$S(c,b,a) = 0.8$	$S(d,b,a) = 0.9$
$S(a,b,b) = 0.2$	$S(b,b,b) = 0.4$	$S(c,b,b) = 0.3$	$S(d,b,b) = 0.2$
$S(a,b,c) = 0.4$	$S(b,b,c) = 0.2$	$S(c,b,c) = 0.2$	$S(d,b,c) = 0.4$
$S(a,b,d) = 0.1$	$S(b,b,d) = 0.1$	$S(c,b,d) = 0.4$	$S(d,b,d) = 0.1$
$S(a,c,a) = 0.9$	$S(b,c,a) = 0.8$	$S(c,c,a) = 0.8$	$S(d,c,a) = 0.9$
$S(a,c,b) = 0.1$	$S(b,c,b) = 0.5$	$S(c,c,b) = 0.1$	$S(d,c,b) = 0.3$
$S(a,c,c) = 0.2$	$S(b,c,c) = 0.1$	$S(c,c,c) = 0.2$	$S(d,c,c) = 0.5$
$S(a,c,d) = 0.4$	$S(b,c,d) = 0.2$	$S(c,c,d) = 0.3$	$S(d,c,d) = 0.2$
$S(a,d,a) = 0.8$	$S(b,d,a) = 0.9$	$S(c,d,a) = 0.9$	$S(d,d,a) = 0.8$

$S(a,d,b)=0.4 \quad S(b,d,b)=0.3 \quad S(c,d,b)=0.4 \quad S(d,d,b)=0.1$
$S(a,d,c)=0.3 \quad S(b,d,c)=0.1 \quad S(c,d,c)=0.1 \quad S(d,d,c)=0.3$
$S(a,d,d)=0.1 \quad S(b,d,d)=0.4 \quad S(c,d,d)=0.3 \quad S(d,d,d)=0.4$

(H,R,S) is a fuzzy ring, since the following conditions are hold:

(H_1) (H,R) is an abelian fuzzy group;

(H_2) For all $a,b,c,z_1,z_2 \in H$, $((a*b)*c)(z_1) > 0.7$ and $(a*(b*c))(z_2) > 0.7$ imply $z_1 = z_2$;

(H_3) For all $a,b,c,z_1,z_2 \in H$,

(i) $(a*(b\circ c))(z_1) > 0.7$ and $((a*b)\circ(a*c))(z_2) > 0.7$ imply $z_1=z_2$,

(ii) $((a\circ b)*c)(z_1) > 0.7$ and $((a*c)\circ(b*c))(z_2) > 0.7$ imply $z_1=z_2$.

4. Homomorphism theorems

Theorem 4.1. *(H_1,R_1,S_1) and (H_2,R_2,S_2) be fuzzy rings and let φ be a fuzzy homomorphism of H_1 into H_2. Then*

(i) *If J is a fuzzy ideal of H_2, then $\varphi^{-1}(J)$ is a fuzzy ideal of H_1 containing $Ker\varphi$.*

(ii) *If I is a fuzzy ideal of H_1 containing $Ker\varphi$, then $\varphi^{-1}(\varphi(I)) = I$.*

(iii) *If φ is surjective and I is a fuzzy ideal of H_1, then $\varphi(I)$ is a fuzzy ideal of H_2.*

(iv) *φ induces a one-one inclusion preserving correspondence between the fuzzy ideals of H_1 containing $Ker\varphi$, and the fuzzy ideals of H_2 in such a way that if I is a fuzzy ideal of H_1 containing $Ker\varphi$, then $\varphi(I)$ is the corresponding fuzzy ideal of H_2 and if J is a fuzzy ideal of H_2, then $\varphi^{-1}(J)$ is the corresponding fuzzy ideal of H_1.*

Proof. (i) Let J be a fuzzy ideal of H_2. Since $\varphi(e_\circ) = e'_\circ \in J$, $e_\circ \in \varphi^{-1}(J)$ and so $\varphi^{-1}(J) \neq \varnothing$. Let $x_1, x_2 \in \varphi^{-1}(J)$ and $h \in H_1$ such that $R_1(x_1,x_2,h) > \theta$. Since φ is a fuzzy homomorphism, we have $R_2(\varphi(x_1), \varphi(x_2), \varphi(h)) > \theta$. Therefore, from the hypothesis, $\varphi(h) \in J$ and so $h \in \varphi^{-1}(J)$.

Let $x \in \varphi^{-1}(J)$. Then $\varphi(x) \in J$ and so $(\varphi(x))^{-1} \in J$ from the hypothesis. In that case, since φ is a fuzzy homomorphism, we get $\varphi\left(x^{-1}\right) \in J$. Thus $x^{-1} \in \varphi^{-1}(J)$.

On the other hand, let $x \in \varphi^{-1}(J)$ and $h,h' \in H_1$ such that $S_1(x,h',h) > \theta$. Since φ is a fuzzy homomorphism, we obtain $S_2(\varphi(x), \varphi(h'), \varphi(h)) > \theta$. From the hypothesis and $\varphi(x) \in J$, $\varphi(h) \in J$ and so $h \in \varphi^{-1}(J)$. Similarly, if $x \in \varphi^{-1}(J)$ and $S_1(h',x,h) > \theta$ for all $h,h' \in H_1$, then $h \in \varphi^{-1}(J)$. Therefore $\varphi^{-1}(J)$ is a fuzzy ideal of H_1. Moreover, let $x \in Ker\varphi$. Since J is a fuzzy ideal of H_2, $\varphi(x) = e'_\circ \in J$ and so $x \in \varphi^{-1}(J)$. Hence $Ker\varphi \subseteq \varphi^{-1}(J)$.

(ii) Since $\varphi(x) \in \varphi(I)$ for all $x \in I$, we have $x \in \varphi^{-1}(\varphi(I))$. Thus $I \subseteq \varphi^{-1}(\varphi(I))$.

Let $x \in \varphi^{-1}(\varphi(I))$. Therefore $\varphi(x) \in \varphi(I)$ and so there exists $a \in I$ such that $\varphi(x) = \varphi(a)$. Since $\varphi(x) = \varphi(a)$, we get $R_2(\varphi(a), (\varphi(x))^{-1}, e'_\circ) > \theta$. Since H_1 is a fuzzy ring and $a, x \in H_1$, there exists $c \in H_1$ such that $R_1\left(a, x^{-1}, c\right) > \theta$. Since φ is a fuzzy homomorphism, we obtain

$$R_2\left(\varphi(a), \varphi\left(x^{-1}\right), \varphi(c)\right) = R_2(\varphi(a), (\varphi(x))^{-1}, \varphi(c)) > \theta.$$

Hence $\varphi(c) = e'_\circ$, that is, $c \in Ker\varphi$. Since $Ker\varphi \subseteq I$, we have $c \in I$. Since I is a fuzzy ideal of H_1 and $R_1\left(a, x^{-1}, c\right) > \theta$, we get $x \in I$. Hence $\varphi^{-1}(\varphi(I)) \subseteq I$.

(iii) Let φ is surjective and I is a fuzzy ideal of H_1. Since $e_\circ \in I$ and $\varphi(e_\circ) = e'_\circ$, $e'_\circ \in \varphi(I)$ and so $\varphi(I) \neq \varnothing$. Let $y_1, y_2 \in \varphi(I)$. Then there exist $x_1, x_2 \in I$ such that $y_1 = \varphi(x_1)$ and $y_2 = \varphi(x_2)$. Since I is a fuzzy ideal of H_1, there exists $x \in I$ such that $R_1(x_1, x_2, x) > \theta$. Since φ is a fuzzy homomorphism, we obtain $R_2(\varphi(x_1), \varphi(x_2), \varphi(x)) > \theta$ and $\varphi(x) \in \varphi(I)$.

Let $y \in \varphi(I)$. Thus there exists $x \in I$ such that $y = \varphi(x)$. Since I is a fuzzy ideal of H_1, $x^{-1} \in I$ and so $R_1\left(x, x^{-1}, e_\circ\right) > \theta$. In that case, since φ is a fuzzy homomorphism, we have

$$R_2\left(\varphi(x), \varphi\left(x^{-1}\right), \varphi(e_\circ)\right) = R_2\left(\varphi(x), \varphi\left(x^{-1}\right), e'_\circ\right) > \theta$$

from Theorem 2.14. Then $y^{-1} = \varphi\left(x^{-1}\right) \in \varphi(I)$.

Let $y \in \varphi(I)$ and $h, h' \in H_2$ such that $S_2(h', y, h) > \theta$. Since $y \in \varphi(I)$ and φ is surjective, there exist $x \in I$ and $k' \in H_1$ such that $y = \varphi(x)$ and $h' = \varphi(k')$. Since I is a fuzzy ideal of H_1, there exists $k \in I$ such that $S_1(k', x, k) > \theta$. Thus $\varphi(k) \in \varphi(I)$. Also, since φ is a fuzzy homomorphism, we get $S_2(\varphi(k'), \varphi(x), \varphi(k)) > \theta$. Then $h = \varphi(k) \in \varphi(I)$ since S_2 is a fuzzy binary operation on H_2. Hence $\varphi(I)$ is a left fuzzy ideal of H_2. Similarly, it can be shown that $\varphi(I)$ is a right fuzzy ideal of H_2. Therefore $\varphi(I)$ is a fuzzy ideal of H_2.

(iv) Let N_1 be the set of all fuzzy ideals of H_1 containing $Ker\varphi$ and N_2 be the set of all fuzzy ideals of H_2. Let ψ be a mapping of N_1 into N_2 defined by $\psi(I) = \varphi(I)$ for all $I \in N_1$. Since φ is well-defined, ψ is well-defined. Let $\psi(I_1) = \psi(I_2)$ for $I_1, I_2 \in N_1$. Now, $\varphi(I_1) = \varphi(I_2)$ for $I_1, I_2 \in N_1$. Since $\varphi^{-1}(\varphi(I_1)) = \varphi^{-1}(\varphi(I_2))$, we obtain $I_1 = I_2$ from (ii). Therefore ψ is one-one. Let $J \in N_2$. Then $\varphi^{-1}(J) \in N_1$ from (i). Let $y \in \varphi\left(\varphi^{-1}(J)\right)$. Thus there exists $x \in \varphi^{-1}(J)$ such that $y = \varphi(x)$. Since $y = \varphi(x)$, that is, $\varphi(x) \in J$, we have $\varphi\left(\varphi^{-1}(J)\right) \subseteq J$. On the other hand, let $x \in J$. Since $x \in J \subseteq H_2 = \varphi(H_1)$, there exists $a \in H_1$ such that $\varphi(a) = x$. Therefore $\varphi(a) \in J$ and so $a \in \varphi^{-1}(J)$. Hence $x = \varphi(a) \in \varphi\left(\varphi^{-1}(J)\right)$ and so $J \subseteq \varphi\left(\varphi^{-1}(J)\right)$. Thus $\psi\left(\varphi^{-1}(J)\right) = \varphi\left(\varphi^{-1}(J)\right) = J$. Hence ψ is surjective.

Let I_1 and I_2 be fuzzy ideals of H_1 such that $I_1 \subset I_2$. Then $\varphi(I_1) \subseteq \varphi(I_2)$, that is, $\psi(I_1) \subseteq \psi(I_2)$. If $\psi(I_1) = \psi(I_2)$, then $I_1 = I_2$ since the mapping $\psi : N_1 \to N_2$ is one-one. This is a contradiction. Hence $\psi(I_1) \neq \psi(I_2)$. Therefore $\psi(I_1) \subset \psi(I_2)$.

Conversely, let $\psi(I_1) \subset \psi(I_2)$, that is, $\varphi(I_1) \subset \varphi(I_2)$. Therefore $\varphi^{-1}(\varphi(I_1)) \subseteq \varphi^{-1}(\varphi(I_2))$. From (ii), $I_1 \subseteq I_2$. Since $\varphi(I_1) \subset \varphi(I_2)$, we get $I_1 \neq I_2$. Hence $I_1 \subset I_2$. □

Let (H, R, S) be a fuzzy ring, I_1 and I_2 be fuzzy ideals of H. Then

$$I_1 \circ I_2 := \{c \in H \mid (a_1 \circ a_2)(c) > \theta, \forall a_1 \in I_1 \text{ and } \forall a_2 \in I_2\}.$$

Lemma 4.2. *Let (H, R, S) be a fuzzy ring, I_1 and I_2 be fuzzy ideals of H. Then $I_1 \circ I_2$ is a fuzzy ideal of H.*

Proof. Since I_1 and I_2 are fuzzy ideals of H, $e_\circ \in I_1$ and $e_\circ \in I_2$. Since $(e_\circ \circ e_\circ)(e_\circ) > \theta$, $e_\circ \in I_1 \circ I_2$ and so $I_1 \circ I_2 \neq \varnothing$.

(i) For all $c_1, c_2 \in I_1 \circ I_2$ and $c \in H$, $(c_1 \circ c_2)(c) > \theta$ implies $c \in I_1 \circ I_2$:

Let $c_1, c_2 \in I_1 \circ I_2$. Then there exist $a_1, b_1 \in I_1$ and $a_2, b_2 \in I_2$ such that $(a_1 \circ a_2)(c_1) > \theta$ and $(b_1 \circ b_2)(c_2) > \theta$. Hence $R(a_1, a_2, c_1) > \theta$ and $R(b_1, b_2, c_2) > \theta$.

Since I_1 and I_2 are fuzzy ideals of H, there exist $d_1 \in I_1$ and $d_2 \in I_2$ such that $R(a_1, b_1, d_1) > \theta$ and $R(a_2, b_2, d_2) > \theta$. Let $t \in H$ such that $R(d_1, d_2, t) > \theta$. Since $d_1 \in I_1$ and $d_2 \in I_2$, $t \in I_1 \circ I_2$.

Since $R(a_1, b_1, d_1) > \theta$, we obtain $R\left(d_1, b_1^{-1}, a_1\right) > \theta$ from Remark 2.3. Let $t_1, t_2 \in H$ such that $R(a_2, d_1, t_1) > \theta$ and $R\left(t_1, b_1^{-1}, t_2\right) > \theta$. Then we have

$$\left(a_2 \circ \left(d_1 \circ b_1^{-1}\right)\right)(c_1) \geq R\left(d_1, b_1^{-1}, a_1\right) \wedge R(a_1, a_2, c_1) > \theta$$

since (H, R) is an abelian fuzzy group, and

$$\left((a_2 \circ d_1) \circ b_1^{-1}\right)(t_2) \geq R(a_2, d_1, t_1) \wedge R\left(t_1, b_1^{-1}, t_2\right) > \theta.$$

Thus $c_1 = t_2$ and so $R(c_1, b_1, t_1) > \theta$.

Let $c_3 \in H$ such that $R(t_1, b_2, c_3) > \theta$. Then we get

$$(c_1 \circ (b_1 \circ b_2))(c) \geq R(b_1, b_2, c_2) \wedge R(c_1, c_2, c) > \theta$$

and

$$((c_1 \circ b_1) \circ b_2)(c_3) \geq R(c_1, b_1, t_1) \wedge R(t_1, b_2, c_3) > \theta.$$

Therefore $c = c_3$ and so $R(t_1, b_2, c) > \theta$. Also, since

$$(b_2 \circ (a_2 \circ d_1))(c) \geq R(a_2, d_1, t_1) \wedge R(t_1, b_2, c) > \theta$$

and

$$((b_2 \circ a_2) \circ d_1)(t) \geq R(a_2, b_2, d_2) \wedge R(d_1, d_2, t) > \theta,$$

$c = t$. Hence $c \in I_1 \circ I_2$.

(ii) For all $c \in I_1 \circ I_2$, $c^{-1} \in I_1 \circ I_2$:

Since $c \in I_1 \circ I_2$, there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $R(a_1, a_2, c) > \theta$. Since I_1 and I_2 are fuzzy ideals of H, we obtain $a_1^{-1} \in I_1$ and $a_2^{-1} \in I_2$. Since $R\left(a_1^{-1}, a_2^{-1}, c^{-1}\right) > \theta$, we have $c^{-1} \in I_1 \circ I_2$.

(iii) For all $c \in I_1 \circ I_2$ and $h \in H$, $(c * h)(k) > \theta$ $(k \in H)$ implies $k \in I_1 \circ I_2$:

Since $c \in I_1 \circ I_2$, there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $R(a_1, a_2, c) > \theta$. Since I_1 and I_2 are fuzzy ideals of H, there exist $k_1 \in I_1$ and $k_2 \in I_2$ such that $S(a_1, h, k_1) > \theta$ and $S(a_2, h, k_2) > \theta$ for all $h \in H$. Let $t \in H$ such that $R(k_1, k_2, t) > \theta$. Since $k_1 \in I_1$ and $k_2 \in I_2$, we get $t \in I_1 \circ I_2$. Then

$$((a_1 \circ a_2) * h)(k) \geq R(a_1, a_2, c) \wedge S(c, h, k) > \theta$$

and

$$\begin{aligned}((a_1 * h) \circ (a_2 * h))(t) &> S(a_1, h, k_1) \wedge S(a_2, h, k_2) \\ &\wedge R(k_1, k_2, t) > \theta.\end{aligned}$$

Thus $t = k$ and so $k \in I_1 \circ I_2$. Therefore $I_1 \circ I_2$ is a right fuzzy ideal of H. Similarly, it can be shown that $I_1 \circ I_2$ is a left fuzzy ideal of H. Hence $I_1 \circ I_2$ is a fuzzy ideal of H. □

Definition 4.3. Let (H, R, S) be a fuzzy ring, I_1 and I_2 be fuzzy ideals of H. Then

$$I_1 \circ I_2 = \{c \in H \mid (a_1 \circ a_2)(c) > \theta, \forall a_1 \in I_1 \text{ and } \forall a_2 \in I_2\}$$

is called a fuzzy addition of two fuzzy ideals of H.

Theorem 4.4. *Let (H,R,S) be a fuzzy ring and I be a fuzzy ideal of H. Then the mapping $\Pi : H \to H/I$ by $\Pi(a) = [a \circ I]$ for all $a \in H$ is a fuzzy homomorphism, called the fuzzy canonical homomorphism.*

Proof. Let $a,b,c \in H$ such that $R(a,b,c) > \theta$. Therefore

$$\begin{aligned}\overline{R}(\Pi(a),\Pi(b),\Pi(c)) &= \overline{R}([a\circ I],[b\circ I],[c\circ I]) \\ &= ([a\circ I] \oplus [b\circ I])(c\circ I) \\ &= \bigvee_{(a',b',c')\in\overline{a}\times\overline{b}\times\overline{c}} R(a',b',c') \\ &\geq R(a,b,c) > \theta\end{aligned}$$

by 2.16.

Let $a,b,c \in H$ such that $S(a,b,c) > \theta$. Then

$$\begin{aligned}\overline{S}(\Pi(a),\Pi(b),\Pi(c)) &= \overline{S}([a\circ I],[b\circ I],[c\circ I]) \\ &= ([a\circ I] \otimes [b\circ I])(c\circ I) \\ &= \bigvee_{(a',b',c')\in\overline{a}\times\overline{b}\times\overline{c}} S(a',b',c') \\ &\geq S(a,b,c) > \theta\end{aligned}$$

from Theorem 2.10. Thus Π is a fuzzy homomorphism from Definition 2.12. □

Theorem 4.5. *Let (H,R,S) be a fuzzy ring, I_1 and I_2 be fuzzy ideals of H. Then $(I_1 \circ I_2)/I_2 \cong I_1/(I_1 \cap I_2)$.*

Proof. For all $a \in I_2$, $R(e_\circ, a, a) > \theta$. Thus $I_2 \subseteq I_1 \circ I_2$ since $e_\circ \in I_1$. Therefore I_2 is a fuzzy ideal of $I_1 \circ I_2$.

Let $\varphi : I_1 \to (I_1 \circ I_2)/I_2$, $\varphi(a) = [a \circ I_2]$. It is clear that φ is surjective.

(1) Let $a = b$ for $a,b \in I_1$. Since $R(a, b^{-1}, e_\circ) > \theta$ and $e_\circ \in I_2$, we obtain $a \circ I_2 \sim b \circ I_2$ and so $[a \circ I_2] = [b \circ I_2]$. Thus φ is well-defined.

(2) Let $a,b,c \in I_1$ such that $R(a,b,c) > \theta$. Then we have

$$\begin{aligned}\overline{R}([a\circ I_2],[b\circ I_2],[c\circ I_2]) &= \bigvee_{(a',b',c')\in\overline{a}\times\overline{b}\times\overline{c}} R(a',b',c') \\ &\geq R(a,b,c) > \theta.\end{aligned}$$

Thus $\overline{R}(\varphi(a),\varphi(b),\varphi(c)) > \theta$.

(3) Let $a,b,c \in I_1$ such that $S(a,b,c) > \theta$. Then we get

$$\begin{aligned}\overline{S}([a\circ I_2],[b\circ I_2],[c\circ I_2]) &= \bigvee_{(a',b',c')\in\overline{a}\times\overline{b}\times\overline{c}} S(a',b',c') \\ &\geq S(a,b,c) > \theta.\end{aligned}$$

Thus $\overline{S}(\varphi(a),\varphi(b),\varphi(c)) > \theta$.

(4)

$$\begin{aligned} Ker\varphi &= \{a \in I_1 \mid \varphi(a) = [e_\circ \circ I_2]\} \\ &= \{a \in I_1 \mid [a \circ I_2] = [e_\circ \circ I_2]\} \\ &= \{a \in I_1 \mid a \circ I_2 \sim e_\circ \circ I_2\} \\ &= \left\{a \in I_1 \mid R\left(a^{-1}, e_\circ, h\right) > \theta, \text{ for some } h \in I_2\right\} \\ &= \{a \in I_1 \mid a \in I_2\} \\ &= I_1 \cap I_2. \end{aligned}$$

Therefore we obtain $(I_1 \circ I_2)/I_2 \cong I_1/(I_1 \cap I_2)$ from Theorem 2.16. □

Theorem 4.6. *Let (H, R, S) be a fuzzy ring, I_1 and I_2 be fuzzy ideals of H such that $I_1 \subseteq I_2$. Then $(H/I_1)/(I_2/I_1) \cong H/I_2$.*

Proof. Let $\varphi : H/I_1 \to H/I_2$, $\varphi([a \circ I_1]) = [a \circ I_2]$. It is clear that φ is surjective.

(1) Let $[a \circ I_1] = [b \circ I_1]$ for $a, b \in H$. There exists $h \in I_1$ such that $R\left(a^{-1}, b, h\right) > \theta$ since $a \circ I_1 \sim b \circ I_1$. Since $I_1 \subseteq I_2$, $h \in I_2$. Hence $a \circ I_2 \sim b \circ I_2$ and so $[a \circ I_2] = [b \circ I_2]$.

(2) Let $\overline{R}([a \circ I_1], [b \circ I_1], [c \circ I_1]) > \theta$. We have that there exist $a_1 \in \overline{a}$, $b_1 \in \overline{b}$, $c_1 \in \overline{c}$ such that $R(a_1, b_1, c_1) > \theta$. Therefore $a_1 \circ I_1 \sim a \circ I_1$, $b_1 \circ I_1 \sim b \circ I_1$ and $c_1 \circ I_1 \sim c \circ I_1$, and so there exist $h_1, h_2, h_3 \in I_1$ such that $R(a_1, h_1, a) > \theta$, $R(b_1, h_2, b) > \theta$ and $R(c_1, h_3, c) > \theta$. Also, $h_1, h_2, h_3 \in I_2$ since $I_1 \subseteq I_2$.

Let $u \in H$ such that $R(a, b, u) > \theta$. Similar to the proof of [6, Theorem 4.2], we get $h \in I_2$ such that $R(c_1, h, u) > \theta$. Then $c \circ I_2 \sim u \circ I_2$ and consequently, $\overline{R}([a \circ I_2], [b \circ I_2], [u \circ I_2]) = \overline{R}([a \circ I_2], [b \circ I_2], [c \circ I_2]) > \theta$.

(3) Let $\overline{S}([a \circ I_1], [b \circ I_1], [c \circ I_1]) > \theta$. Similar to the proof of (2), we obtain $\overline{S}([a \circ I_2], [b \circ I_2], [c \circ I_2]) > \theta$.

(4)

$$\begin{aligned} Ker\varphi &= \{[a \circ I_1] \in H/I_1 \mid \varphi([a \circ I_1]) = [e_\circ \circ I_2]\} \\ &= \{[a \circ I_1] \in H/I_1 \mid [a \circ I_2] = [e_\circ \circ I_2]\} \\ &= \{[a \circ I_1] \in H/I_1 \mid a \circ I_2 \sim e_\circ \circ I_2\} \\ &= \left\{[a \circ I_1] \in H/I_1 \mid R\left(a^{-1}, e_\circ, h\right) > \theta, \text{ for some } h \in I_2\right\} \\ &= \{[a \circ I_1] \in H/I_1 \mid a \in I_2\} \\ &= I_2/I_1. \end{aligned}$$

Thus we have $(H/I_1)/(I_2/I_1) \cong H/I_2$ from Theorem 2.16. □

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