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On pairwise fuzzy Volterra spaces

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Abstract. In this paper the concepts of pairwise fuzzy Volterra spaces and pairwise fuzzy weakly Volterra spaces are introduced and characterizations of pairwise fuzzy Volterra spaces and pairwise fuzzy weakly Volterra spaces are studied.

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Keywords: Pairwise fuzzy nowhere dense set, Pairwise fuzzy first category set, Pairwise fuzzy second category set, Pairwise fuzzy Baire space, Pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces.

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1. Introduction

The fundamental concept of a fuzzy set introduced by L.A.Zadeh [14] in 1965, provides a natural foundation for building new branches of fuzzy mathematics. The theory of fuzzy topological spaces was introduced by C.L.Chang [4] in 1968. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. In 1989, Kandil [9] introduced the concept of fuzzy bitopological spaces. The concepts of Volterra spaces have been studied extensively in classical topology in [3], [5], [6], [7] and [8]. The concept of Volterra spaces in fuzzy setting was introduced and studied by G.Thangaraj and S.Soundararajan in [12]. In this paper, we introduce the concepts of pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces. Also we discuss several characterizations of pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are fuzzy topologies on the non-empty set X. The complement λ' of a fuzzy set λ is defined by $\lambda'(x) = 1 - \lambda(x),\ x \in X$.

Definition 2.1. Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu : X \to [0, 1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, (1-\mu) \in T\}$.

For any fuzzy set λ in a fuzzy topological space (X, T), it is easy to see that $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$. [1]

Definition 2.3 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a *fuzzy G_δ-set* if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ for each $\lambda_i \in T$.

Definition 2.4 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a *fuzzy F_σ-set* if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ for each $(1 - \lambda_i) \in T$.

Definition 2.5. A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy open set* if $\lambda \in T_i,\ (i = 1, 2)$.

Definition 2.6. Let (X, T_1, T_2) be a fuzzy bitopological space and λ be a fuzzy set in X. Then λ is called a *pairwise fuzzy G_δ-set* if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$, where λ_i's are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.7. Let (X, T_1, T_2) be a fuzzy bitopological space and λ be a fuzzy set in X. Then λ is called a *pairwise fuzzy F_σ-set* if $\lambda = \vee_{i=1}^{\infty} \lambda_i$, where λ_i's are pairwise fuzzy closed sets in (X, T_1, T_2).

Definition 2.8 ([11])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy dense* if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.9 ([10])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy dense set* if $cl_{T_1} cl_{T_2}(\lambda) = cl_{T_2} cl_{T_1}(\lambda) = 1$.

Definition 2.10 ([11])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called *fuzzy nowhere dense* if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.11 ([13])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy nowhere dense set* if $int_{T_1} cl_{T_2}(\lambda) = int_{T_2} cl_{T_1}(\lambda) = 0$.

Definition 2.12 ([13])**.** Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy first category* if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). A fuzzy set which is not of fuzzy first category is said to be a *pairwise fuzzy second category set* in (X, T_1, T_2).

Definition 2.13 ([13])**.** If λ is a pairwise fuzzy first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a *pairwise fuzzy residual set* in (X, T_1, T_2).

Definition 2.14 ([13]). A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy Baire space* if $int_{T_i}\Big(\vee_{k=1}^{\infty}(\lambda_k)\Big) = 0,\ (i = 1, 2)$ where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

Lemma 2.15 ([1]). *For a family $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case $\mathscr{A}$ is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.*

3. Pairwise fuzzy volterra spaces

Motivated by the classical concept introduced in [7] we shall now define :

Definition 3.1. A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy Volterra space* if $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, $(i = 1, 2)$, where λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Example 3.2. Let $X = \{a, b, c\}$. The fuzzy sets α, β, δ and μ are defined on X as follows :

$\alpha : X \to [0, 1]$ is defined as $\alpha(a) = 0.6;\ \ \alpha(b) = 0.9;\ \ \alpha(c) = 0.8$
$\beta : X \to [0, 1]$ is defined as $\beta(a) = 0.7;\ \ \beta(b) = 0.8;\ \ \beta(c) = 0.9$
$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.8;\ \ \delta(b) = 0.6;\ \ \delta(c) = 0.7$
$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.7;\ \ \mu(b) = 0.5;\ \ \mu(c) = 0.9$.

Clearly $T_1 = \{0, \alpha, \beta, \delta, \alpha\vee\beta, \alpha\vee\delta, \beta\vee\delta, \alpha\wedge\beta, \alpha\wedge\delta, \beta\wedge\delta, \alpha\vee(\beta\wedge\delta), \delta\vee(\alpha\wedge\beta), \beta\wedge(\alpha\vee\delta), \alpha\vee\beta\vee\delta, 1\}$ and $T_2 = \{0, \alpha, \mu, \delta, \alpha\vee\mu, \alpha\vee\delta, \mu\vee\delta, \alpha\wedge\mu, \alpha\wedge\delta, \mu\wedge\delta, \alpha\vee(\mu\wedge\delta), \mu\vee(\alpha\wedge\delta), \delta\vee(\alpha\wedge\mu), \alpha\wedge(\mu\vee\delta), \mu\wedge(\alpha\vee\delta), \delta\wedge(\alpha\vee\mu), \alpha\wedge\mu\wedge\delta, \alpha\vee\mu\vee\delta, 1\}$ are fuzzy topologies on X. The fuzzy sets $\alpha, \delta, \alpha\vee\beta, \alpha\vee\delta, \alpha\vee\mu, \alpha\wedge\delta, \beta\wedge\delta, \delta\wedge(\alpha\vee\mu), \alpha\vee(\beta\wedge\delta), \alpha\vee(\mu\wedge\delta), \alpha\vee\beta\vee\delta, \alpha\vee\mu\vee\delta, 1$ are pairwise fuzzy open sets in (X, T_1, T_2). Now $\lambda = \alpha\wedge(\alpha\vee\beta)\wedge(\alpha\vee\delta)\wedge[\alpha\vee(\beta\wedge\delta)]$, $\eta = \delta\wedge(\alpha\wedge\delta)\wedge(\beta\wedge\delta)$ and $\gamma = (\alpha\vee\mu)\wedge[\delta\wedge(\alpha\vee\mu)]\wedge[\alpha\vee(\beta\wedge\delta)]\wedge(\alpha\vee\mu\vee\delta)$ are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Also, we have $cl_{T_1}cl_{T_2}(\lambda) = cl_{T_2}cl_{T_1}(\lambda) = 1$; $cl_{T_1}cl_{T_2}(\eta) = cl_{T_2}cl_{T_1}(\eta) = 1$ and $cl_{T_1}cl_{T_2}(\gamma) = cl_{T_2}cl_{T_1}(\gamma) = 1$. Hence the fuzzy sets λ, η and γ are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Now $cl_{T_i}(\lambda\wedge\eta\wedge\gamma) = 1,\ (i = 1, 2)$, implies that the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Volterra space.

Theorem 3.3 ([13]). *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proposition 3.4. *Let (X, T_1, T_2) be a fuzzy bitopological space. If $int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) = 0,\ \ (i = 1, 2)$, where λ_k's are pairwise fuzzy nowhere dense and pairwise fuzzy F_σ-sets in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Suppose that $int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) = 0,\ \ (i = 1, 2)$, where λ_k's are pairwise fuzzy nowhere dense and pairwise fuzzy F_σ-sets in a fuzzy bitopological space (X, T_1, T_2). Now $1 - int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) = 1,\ \ (i = 1, 2)$. Then $cl_{T_i}\big(1 - \vee_{k=1}^{N}(\lambda_k)\big) = 1,\ \ (i = 1, 2)$. This implies that $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) = 1,\ \ (i = 1, 2)$. Since λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2), by Theorem 3.3, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Also since λ_k's are pairwise fuzzy F_σ-sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence we have $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) =$

$1, \ (i = 1, 2)$, where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Proposition 3.5. *If the pairwise fuzzy nowhere dense set λ in a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy F_σ-set, then λ is a pairwise fuzzy first category set in (X, T_1, T_2).*

Proof. Suppose that λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2) such that $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy closed sets in (X, T_1, T_2). Then $cl_{T_i}(\lambda) = cl_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) \geq \vee_{k=1}^{\infty} cl_{T_i}(\lambda_k), \ (i = 1, 2)$, by Lemma 2.15. Since the fuzzy sets λ_k's are pairwise fuzzy closed sets in (X, T_1, T_2), $cl_{T_i}(\lambda_k) = \lambda_k, \ (i = 1, 2)$. Hence we have $\vee_{k=1}^{\infty}(\lambda_k) \leq cl_{T_i}(\lambda), \ (i = 1, 2)$. That is, $\vee_{k=1}^{\infty}(\lambda_k) \leq cl_{T_1}(\lambda)$ and $\vee_{k=1}^{\infty}(\lambda_k) \leq cl_{T_2}(\lambda)$. Then we have $int_{T_2}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) \leq int_{T_2}(cl_{T_1}(\lambda))$ and $int_{T_1}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) \leq int_{T_1}(cl_{T_2}(\lambda))$. Since λ is a pairwise fuzzy nowhere dense set in $(X, T_1, T_2), int_{T_1} cl_{T_2}(\lambda) = int_{T_2} cl_{T_1}(\lambda) = 0$. Then we have $int_{T_1}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$ and $int_{T_2}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$.

Since $\vee_{k=1}^{\infty}\big(int_{T_2}(\lambda_k)\big) \leq int_{T_2}\big(\vee_{k=1}^{\infty}(\lambda_k)\big)$ and $\vee_{k=1}^{\infty}\big(int_{T_1}(\lambda_k)\big) \leq int_{T_1}\big(\vee_{k=1}^{\infty}(\lambda_k)\big)$, $\vee_{k=1}^{\infty}\big(int_{T_2}(\lambda_k)\big) = 0$ and $\vee_{k=1}^{\infty}\big(int_{T_1}(\lambda_k)\big) = 0$. This implies that $int_{T_2}(\lambda_k) = 0$ and $int_{T_1}(\lambda_k) = 0$. Hence $int_{T_1} cl_{T_2}(\lambda_k) = 0$ and $int_{T_2} cl_{T_1}(\lambda_k) = 0$. Then λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Therefore $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2), implies that λ is a pairwise fuzzy first category set in (X, T_1, T_2). □

Theorem 3.6 ([13]). *Let (X, T_1, T_2) be a fuzzy bitopological space. Then the following are equivalent :*

1. *(X, T_1, T_2) is a pairwise fuzzy Baire space.*
2. *$int_{T_i}(\lambda) = 0, \ (i = 1, 2)$ for every pairwise fuzzy first category set λ in (X, T_1, T_2).*
3. *$cl_{T_i}(\mu) = 1, \ (i = 1, 2)$ for every pairwise fuzzy residual set μ in (X, T_1, T_2).*

Proposition 3.7. *If every pairwise fuzzy nowhere dense set λ in a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy F_σ-set, then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proof. Suppose that the pairwise fuzzy nowhere dense set λ in (X, T_1, T_2) is a pairwise fuzzy F_σ-set. Then, by proposition 3.5, λ is a pairwise fuzzy first category set in (X, T_1, T_2). Now $\lambda \leq cl_{T_1}(\lambda)$ and $\lambda \leq cl_{T_2}(\lambda)$. Then we have $int_{T_2}(\lambda) \leq int_{T_2} cl_{T_1}(\lambda)$ and $int_{T_1}(\lambda) \leq int_{T_1} cl_{T_2}(\lambda)$. Since λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2), $int_{T_1} cl_{T_2}(\lambda) = 0$ and $int_{T_2} cl_{T_1}(\lambda) = 0$. Then $int_{T_1}(\lambda) = 0$ and $int_{T_2}(\lambda) = 0$. That is, $int_{T_i}(\lambda) = 0, \ (i = 1, 2)$, for a pairwise fuzzy first category set λ in (X, T_1, T_2). Hence, by theorem 3.6, (X, T_1, T_2) is a pairwise fuzzy Baire space. □

Proposition 3.8. *If every pairwise fuzzy nowhere dense set λ in a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy F_σ-set, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Suppose that the pairwise fuzzy nowhere dense set λ in a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy F_σ-set. Then, by Proposition 3.7, (X, T_1, T_2) is a pairwise fuzzy Baire space. Hence $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0,\ (i = 1, 2)$ where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Then we have $1 - int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = cl_{T_i}\big(\wedge_{k=1}^{\infty}(1-\lambda_k)\big) = 1,\ (i = 1, 2)$. Since λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2), by Theorem 3.3, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Also since λ_k's are pairwise fuzzy F_σ-sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Now $cl_{T_i}\big(\wedge_{k=1}^{\infty}(1 - \lambda_k)\big) \leq cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big)$, implies that $1 \leq cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big)$. That is, $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) = 1,\ (i = 1, 2)$, where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Definition 3.9. A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy P-space* if every non-zero pairwise fuzzy G_δ-set in (X, T_1, T_2), is a pairwise fuzzy open set in (X, T_1, T_2). That is, if (X, T_1, T_2) is a pairwise fuzzy P-space if $\lambda \in T_i,\ (i = 1, 2)$ for $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy open sets in (X, T_1, T_2).

Theorem 3.10 ([13]). *If $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$, where λ_k's are T_i-fuzzy dense and T_i-fuzzy open sets in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proposition 3.11. *If λ_k is a T_i-fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2), then λ_k is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proof. Let λ_k be a T_i-fuzzy dense set in (X, T_1, T_2). Then $cl_{T_1}(\lambda_k) = 1$ and $cl_{T_2}(\lambda_k) = 1$. Now $cl_{T_1}cl_{T_2}(\lambda_k) = cl_{T_1}(1) = 1$ and $cl_{T_2}cl_{T_1}(\lambda_k) = cl_{T_2}(1) = 1$. Hence λ_k is a pairwise fuzzy dense set in (X, T_1, T_2). □

Proposition 3.12. *If $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$, where λ_k's are pairwise fuzzy G_δ-sets in a fuzzy bitopological space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Suppose that $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$, where λ_k's are pairwise fuzzy G_δ-sets in a fuzzy bitopological space (X, T_1, T_2). Now $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq \wedge_{k=1}^{\infty}cl_{T_i}(\lambda_k)$, implies that $1 \leq \wedge_{k=1}^{\infty}cl_{T_i}(\lambda_k)$. That is, $\wedge_{k=1}^{\infty}cl_{T_i}(\lambda_k) = 1$. Then we have $cl_{T_i}(\lambda_k) = 1,\ (i = 1, 2)$. Then, by proposition 3.11, λ_k's are pairwise fuzzy dense sets in (X, T_1, T_2). Now $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big)$, implies that $1 \leq cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big)$. That is, $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$, where λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Proposition 3.13. *If $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$, where λ_k's are pairwise fuzzy G_δ-sets in a fuzzy bitopological P-space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proof. Since (X, T_1, T_2) is a pairwise fuzzy P-space, every pairwise fuzzy G_δ-set in (X, T_1, T_2), is a pairwise fuzzy open set in (X, T_1, T_2). Hence $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1, (i = 1, 2)$, where λ_k's are pairwise fuzzy dense dense(from the proof of proposition 3.12) and pairwise fuzzy open sets in (X, T_1, T_2). Then, by theorem 3.10 and proposition 3.11, (X, T_1, T_2) is a pairwise fuzzy Baire space. □

4. Pairwise fuzzy weakly volterra spaces

Motivated by the classical concept introduced in [7] we shall now define :

Definition 4.1. A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy weakly Volterra space* if $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Proposition 4.2. *If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Volterra space, then (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy Volterra space. Let λ_k's $(k = 1 \ to \ N)$ be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Suppose that $\wedge_{k=1}^{N}(\lambda_k) = 0$. Then $cl_{T_i}\big(\wedge_{k=1}^{N} (\lambda_k)\big) = 0$, $(i = 1, 2)$ which is a contradiction to (X, T_1, T_2) being a pairwise fuzzy Volterra space. Thus $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Hence (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space. □

Proposition 4.3. *If $\big(\vee_{k=1}^{N} (\lambda_k)\big) = 1$, where λ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy F_σ-sets in a pairwise fuzzy weakly Volterra space (X, T_1, T_2), then $int_{T_1} int_{T_2}(\lambda_k) \neq 0$ and $int_{T_2} int_{T_1}(\lambda_k) \neq 0$ for atleast one λ_k, $(k = 1 \ to \ N)$.*

Proof. Suppose that $int_{T_1} int_{T_2}(\lambda_k) = 0$ and $int_{T_2} int_{T_1}(\lambda_k) = 0$ for all pairwise fuzzy F_σ-sets λ_k, $(k = 1 \ to \ N)$ in a pairwise fuzzy weakly Volterra space (X, T_1, T_2). Then we have $1 - int_{T_1} int_{T_2}(\lambda_k) = 1$ and $1 - int_{T_2} int_{T_1}(\lambda_k) = 1$. Then $cl_{T_1} cl_{T_2}\big(1 - (\lambda_k)\big) = cl_{T_2} cl_{T_1}\big(1 - (\lambda_k)\big) = 1$. Hence $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Since λ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy F_σ-sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Now consider $\big(\wedge_{k=1}^{N} (1 - \lambda_k)\big) = 1 - \vee_{k=1}^{N}\big((\lambda_k)\big) = 1 - 1 = 0$. Hence we have $\big(\wedge_{k=1}^{N} (1 - \lambda_k)\big) = 0$ where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). But this is a contradiction (X, T_1, T_2) being a pairwise fuzzy weakly Volterra space. Hence we must have $int_{T_1} int_{T_2}(\lambda_k) \neq 0$ and $int_{T_2} int_{T_1}(\lambda_k) \neq 0$ for atleast one λ_k, $(k = 1 \ to \ N)$. □

Proposition 4.4. *If $\big(\vee_{k=1}^{N} (\lambda_k)\big) = 1$, where λ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy F_σ-sets in a fuzzy bitopological space (X, T_1, T_2) such that $int_{T_1} int_{T_2}(\lambda_k) \neq 0$ and $int_{T_2} int_{T_1}(\lambda_k) \neq 0$ for atleast one λ_k, $(k = 1 \ to \ N)$, then (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Suppose that $\big(\wedge_{k=1}^{N} (\mu_k)\big) = 0$, where μ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Then $1 - \big(\wedge_{k=1}^{N} (\mu_k)\big) = 1$, implies that $\vee_{k=1}^{N}\big(1 - \mu_k\big) = 1$. Since μ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy G_δ-sets in (X, T_1, T_2), $(1 - \mu_k)$'s are pairwise fuzzy F_σ-sets in (X, T_1, T_2). Since μ_k's $(k = 1 \ to \ N)$ are pairwise fuzzy dense sets $cl_{T_1} cl_{T_2}(\mu_k) = cl_{T_2} cl_{T_1}(\mu_k) = 1$. Then we have $int_{T_1} int_{T_2}(1 - \mu_k) = 0$ and $int_{T_2} int_{T_1}(1 - \mu_k) = 0$ for all $k = 1 \ to \ N$. Let $1 - \mu_k = \lambda_k$. Thus, for the pairwise fuzzy F_σ-sets λ_k in (X, T_1, T_2), we have $\big(\vee_{k=1}^{N} (\lambda_k)\big) = 1$ and $int_{T_1} int_{T_2}(\lambda_k) = 0$ and $int_{T_2} int_{T_1}(\lambda_k) = 0$ for all $k = 1 \ to \ N$. But this is a contradiction to the hypothesis. Hence we must have $\big(\wedge_{k=1}^{N} (\mu_k)\big) \neq 0$. Therefore, (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space. □

Proposition 4.5. *If a fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy second category space and the pairwise fuzzy nowhere dense sets in (X, T_1, T_2) are pairwise fuzzy F_σ-sets, then (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy second category space. Then $1 \neq \vee_{k=1}^{\infty}(\lambda_k)$, where the λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Now $1 - \vee_{k=1}^{\infty}(\lambda_k) \neq 0$, implies that $\wedge_{k=1}^{\infty}(1 - \lambda_k) \neq 0$. Since λ_k's ($k = 1$ *to* ∞) are pairwise fuzzy F_σ-sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Also, since the λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2), by theorem 3.3, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Thus for the pairwise fuzzy dense and pairwise fuzzy G_δ-sets $(1 - \lambda_k)$ in (X, T_1, T_2), $[\wedge_{k=1}^{\infty}(1 - \lambda_k)] \neq 0$. Now $[\wedge_{k=1}^{\infty}(1 - \lambda_k)] \leq [\wedge_{k=1}^{N}(1 - \lambda_k)]$, implies that $[\wedge_{k=1}^{N}(1 - \lambda_k)] \neq 0$. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space. □

Definition 4.6. A non-zero fuzzy set in a fuzzy bitopological space (X, T_1, T_2) is called a *pairwise somewhere fuzzy dense set* if $int_{T_1} cl_{T_2}(\lambda_k) \neq 0$ and $int_{T_2} cl_{T_1}(\lambda_k) \neq 0$.

Proposition 4.7. *If $\wedge_{k=1}^{N}(\lambda_k)$ is a pairwise somewhere fuzzy dense set in (X, T_1, T_2) where λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let $\wedge_{k=1}^{N}(\lambda_k)$ be a pairwise somewhere fuzzy dense set in (X, T_1, T_2). Then $int_{T_1} cl_{T_2}\left(\wedge_{k=1}^{N}(\lambda_k)\right) \neq 0$ and $int_{T_2} cl_{T_1}\left(\wedge_{k=1}^{N}(\lambda_k)\right) \neq 0$. Suppose that $\wedge_{k=1}^{N}(\lambda_k) = 0$, where the λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Then $int_{T_1} cl_{T_2}\left(\wedge_{k=1}^{N}(\lambda_k)\right) = 0$ and $int_{T_2} cl_{T_1}\left(\wedge_{k=1}^{N}(\lambda_k)\right) = 0$, a contradiction. Hence we must have $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Therefore (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space. □

5. Conclusions

In this paper we introduced the concept of pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces. Characterizations of pairwise fuzzy Volterra, pairwise fuzzy Baire and pairwise fuzzy weakly Volterra spaces are studied.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika (Prague) 31(5) (1995) 459–464.
[3] J. Cao and D. Gauld, On Volterra spaces revisited, J. Aust. Math. Soc. 79 (2005) 61–76.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] D. Gauld, S. Greenwood and Z. Piotrowski, On Volterra spaces. II. Papers on general topology and applications (Gorham, ME, 1995), 169–173, Ann. New York Acad. Sci., 806, New York Acad. Sci., New York, 1996.
[6] D. Gauld, S. Greenwood and Z. Piotrowski, On Volterra spaces. III. Topological operations. Proceedings of the 1998 Topology and Dynamics Conference (Fairfax, VA). Topology Proc. 23 (1998), Spring, 167–182.
[7] D. B. Gauld and Z. Piotrowski, On Volterra spaces, Far East J. Math. Sci. 1(2) (1993) 209–214
[8] G. Gruenhage and D. Lutzer, Baire and Volterra spaces, Proc. Amer. Math. Soc. 128(10) (2000) 3115–3124.

[9] A. Kandil and M. E. El-Shafee, Biproximities and fuzzy bitopological spaces, Simon Stevin 63(1) (1989) 45–66.
[10] G. Thangaraj, On pairwise fuzzy resolvable and fuzzy irresolvable spaces, Bull. Calcutta Math. Soc. 101 (2010) 59–68.
[11] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11(3) (2003) 725–736.
[12] G. Thangaraj and S. Soundararajan, On fuzzy Volterra spaces, J. Fuzzy Math. 21(4) (2013) 895–904.
[13] G. Thangaraj and S. Sethuraman, On pairwise fuzzy Baire spaces, Gen. Math. Notes 20(2) (2014) 12–21.
[14] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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