On \((\in, \in \lor q_k)\)-fuzzy filters of CI-algebras

Ameneh Namdar, Arsham Borumand Saeid, Ghazanfar Jabbari

Received 17 July 2013; Accepted 20 October 2013

Abstract. In this paper, by considering the relationship between a fuzzy point and a fuzzy set we introduce the notion of \((\in, \in \lor q_k)\)-fuzzy filter in CI-algebras. We state and prove some theorems in \((\in, \in \lor q_k)\)-fuzzy filter of CI-algebras also we study relationship between \((\in, \in \lor q_k)\)-fuzzy filters and fuzzy filters.

2010 AMS Classification: 06F35, 03G25, 08A72

Keywords: CI-algebra, Fuzzy CI-algebra, \((\in, \in \lor q)\)-fuzzy filter.

Corresponding Author: Ameneh Namdar (namdar.amene@gmail.com)

1. Introduction

In 1966, Imai and Iseki defined a class of algebras of type (2,0) called BCK-algebras. There exist several generalizations of BCK-algebras such as BCH-algebras, BCC-algebras, BH-algebras, d-algebras, BE-algebras, etc.

In 2009, B. L. Meng introduced the notion of a CI-algebra as a generalization of BE-algebra and dual BCK/BCI/BCH-algebras\[3\]. For further study see \[2, 4, 5\].

The theory of fuzzy sets was first developed by Zadeh\[7\] and has been applied to many branches in mathematics. A new type of fuzzy subgroup, that is, the \((\in, \in \lor q)\)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das\[1\] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets which was introduced by Pu and Liu\[6\].

In this paper, we introduce the concept of \((\in, \in \lor q_k)\)-fuzzy filter of CI-algebras and deal with related properties. We investigate the relationship between \((\in, \in \lor q_k)\)-fuzzy filter and fuzzy filters. We give some conditions for a fuzzy set \(\mu\) to be an \((\in, \in \lor q_k)\)-fuzzy filter. We establish characterizations of an \((\in, \in \lor q_k)\)-fuzzy filter.
2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2,0)$.

**Definition 2.1.** An element $X \in K(\tau)$ is called a CI-algebra if it satisfies the following axioms:

(a1) $x \ast x = 1$,
(a2) $1 \ast x = x$,
(a3) $x \ast (y \ast z) = y \ast (x \ast z)$,

for all $x, y, z \in X$. If a CI-algebra $X$ satisfies:

(a4) $x \ast 1 = 1$ for all $x \in X$,

Then we say that $X$ is a $BE$-algebra.

We can define a partial ordering $\leq$ on $X$ by

$(\forall x, y \in X)(x \leq y \iff x \ast y = 1)$.

In a CI-algebra $X$, the following hold (see\[3]):

(b1) $y \ast ((y \ast x) \ast x) = 1$
(b2) $(x \ast 1) \ast (y \ast 1) = (x \ast y) \ast 1$
(b3) $1 \leq x \Rightarrow x = 1$,

for all $x, y \in X$.

A CI-algebra $X$ is said to be self-distributive if $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$, for all $x, y, z \in X$. A CI-algebra $X$ is said to be transitive (see\[3]) if it satisfies:

$(\forall x, y, z \in X)((y \ast z) \ast ((x \ast y) \ast (x \ast z)) = 1)$.

A subset $F$ of a CI-algebra $X$ is called a filter of $X$ (sec\[3]) if it satisfies:

(F1) $1 \in F$
(F2) $(\forall x, y \in X)(x \ast y \in F, x \in F \Rightarrow y \in F)$.

**Definition 2.2** ([4]). A Fuzzy set $\mu$ in a CI-algebra $X$ is called a fuzzy filter of $X$ if it satisfies:

(F3) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(F4) $(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x \ast y), \mu(x)\})$,

For any fuzzy set $\mu$ in $X$ and $t \in (0,1]$, the set

$$U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}$$

is called a level subset of $X$. A fuzzy set $\mu$ in a set $X$ of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $(x, t)$.

For a fuzzy point $(x, t)$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu\([6]\) introduced the symbol $(x, t)\mu q\alpha$, where $\alpha \in \{\in, \in \lor q, \in \land q\}$. We say that $(x, t) \in \mu(\text{resp. } (x, t)q\mu\alpha$, it means $\mu(x) \geq t(\text{resp. } \mu(x) + t > 1)$, and in this case, $(x, t)$ is said to belong to (resp. be quasi-coincident with) a fuzzy set $\mu$. To say that $(x, t) \in \mu q\alpha(\text{resp. } (x, t)q\alpha)$, we mean $(x, t) \in \mu$ or $(x, t)q\alpha\mu(\text{resp. } (x, t)q\alpha\mu)$. For $(x, t)q\alpha\mu$, we mean $(x, t) \in \mu$ or $(x, t)q\alpha\mu(\text{resp. } (x, t)q\alpha\mu)$.

3. Main results

In what follows, $X$ is a CI-algebra and $\kappa \in [0,1]$ unless otherwise specified.

Also we define for $\alpha \in \{\in, \in \lor q\}$...
A fuzzy set in X is called an \((\varepsilon, \in \bot q)\)-fuzzy filter of X if it satisfies

\((c_1)\) \((x, t) \in \mu \Rightarrow (1, t) \in q_k\mu,
\((c_2)\) \((x, t) \in \mu \text{ and } (x * y, r) \in \mu \Rightarrow (y, \min \{t, r\}) \in q_k\mu \text{ for all } x, y \in X\) and \(t, r \in [0, 1] \).

An \((\varepsilon, \in \bot q)\)-fuzzy filter of X with \(k = 0\) is called \((\varepsilon, \in \bot q)\)-fuzzy filter of X.

**Example 3.2.** Let \(X = \{1, a, b, c\}\) be a set with the following table:


<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a CI-algebra. Define a fuzzy set \(\mu : X \rightarrow [0, 1]\) on X, by 
\(\mu(1) = 0.8, \mu(a) = \mu(b) = 0.4\) and \(\mu(c) = 0.3\). Then \(\mu\) is a \((\varepsilon, \in \bot q)\)-fuzzy filter of X.

**Theorem 3.3.** A fuzzy set \(\mu\) in X is an \((\varepsilon, \in \bot q)\)-fuzzy filter of X if and only if it satisfies two conditions:

\((c_3)\) \((\forall x \in X, \mu(x) \geq \min \{\mu(x), \frac{1-k}{2}\}\),
\((c_4)\) \((\forall x, y \in X, \mu(x, y) \geq \min \{\mu(x), \mu(x * y), \frac{1-k}{2}\})\).

**Proof.** Assume that \(\mu\) is an \((\varepsilon, \in \bot q)\)-fuzzy filter of X, if \((c_3)\) is not valid, then 
\(\mu(1) < t_a \leq \min \{\mu(a), \frac{1-k}{2}\}\), for some \(a \in X\) and \(t_a \in (0, \frac{1-k}{2})\). Thus \((a, t_a) \in \mu\) but \((1, t_a) \notin q_k\mu\). Also, \(\mu(1) + t_a < 2t_a \leq 1 - k\), i.e. \((1, t_a) \notin \bot q_k\mu\). Therefore \((1, t_a) \in \bot q_k\mu\), which is a contradiction. Consequently, \(\mu(1) \geq \min \{\mu(x), \frac{1-k}{2}\}\) for all \(x \in X\).

Assume that \((c_4)\) is not valid. Then there exist \(a, b \in X\) such that \(\mu(b) \leq t \leq \min \{\mu(a), \mu(a * b), \frac{1-k}{2}\}\). If \(\min \{\mu(a), \mu(a * b)\} = \frac{1-k}{2}\), then 
\(\mu(b) < t \leq \min \{\mu(a), \mu(a * b)\} = \frac{1-k}{2}\). Hence \(a \in \mu\) and \(a * b \in \mu\) but \((y, t) \notin \mu\). Moreover, \(\mu(b) + t < 2t \leq 1 - k\), and so \((b, t) \notin \bot q_k\mu\). Therefore \((b, t) \in \bot q_k\mu\), which is a contradiction, if \(\min \{\mu(a), \mu(a * b)\} = \frac{1-k}{2}\) then \((a, \frac{1-k}{2}) \in \mu\) and \((a + b, \frac{1-k}{2}) \in \mu\) but \((b, \frac{1-k}{2}) \notin \mu\).

Conversely, let \(\mu\) be a fuzzy set in X satisfying \((c_3)\) and \((c_4)\). Let \(x \in X\) and \(t \in (0, 1]\) be such that \((x, t) \in \mu\). Then \(\mu(x) \geq t\), and so \(\mu(1) \geq \min \{\mu(x), \frac{1-k}{2}\} \geq \min \{t, \frac{1-k}{2}\}\). If \(t \leq \frac{1-k}{2}\), then \(\mu(1) \geq t\), i.e. \((1, t) \in \mu\). If \(t > \frac{1-k}{2}\), then \(\mu(1) \geq \frac{1-k}{2}\). Thus \(\mu(1) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k\), i.e. \((1, t) \notin \bot q_k\mu\). Hence \((1, t) \in \bot q_k\mu\), which proves \((c_1)\). Let \(x, y \in X\) and \(t, r \in (0, 1]\) be such that \((x, t) \in \mu\) and \((x * y, r) \in \mu\). Then \(\mu(x) \geq t\) and \(\mu(x * y) \geq r\). It follows from \((c_4)\) that \(\mu(y) \geq \min \{\mu(x), \mu(x * y), \frac{1-k}{2}\}\)

\[ \geq \min \{t, r, \frac{1-k}{2}\} = \begin{cases} \min \{t, r\} & \text{if } t \leq \frac{1-k}{2} \text{ or } r \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \text{ or } r > \frac{1-k}{2}. \end{cases} \]
The case $\mu(y) \geq \min\{t, r\}$ implies that $(y, \min\{t, r\}) \in \mu$. From the $\mu(y) \geq \frac{1-k}{2}$ we have $\mu(y) + \min\{t, r\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$,
\[ \Rightarrow \begin{cases} \mu(y) + \min\{t, r\} > 1 - k, \\
i.e., (y, \min\{t, r\}) \in \mu. \end{cases} \]
Hence $(y, \min\{t, r\}) \in \mu$. Therefore the condition $(c_2)$ is valid. Consequently, $\mu$ is an $(\mu, \in \mu)$-fuzzy filter of $L$. \hfill \Box

**Theorem 3.4.** A fuzzy set $\mu$ in $L$ is an $(\mu, \in \mu)$-fuzzy filter of $L$ if and only if it satisfies:
\[ (*): \forall t \in (0, \frac{1-k}{2}) U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t) \text{ is a filter of } L. \]

**Proof.** Let $\mu$ be an $(\mu, \in \mu)$-fuzzy filter of $L$. Let $t \in (0, \frac{1-k}{2})$ be such that $U(\mu; t) \neq \emptyset$. Obviously, $1 \in U(\mu; t)$ for all $t \in (0, \frac{1-k}{2})$. Let $x, y \in X$ be such that $x \in U(\mu; t)$ and $x \ast y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(x \ast y) \geq t$. It follows from $(c_4)$ that $\mu(x \ast y) \geq \min\{\mu(x), \mu(x \ast y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t$ so that $y \in U(\mu; t)$. Hence $U(\mu; t)$ is a filter of $X$.

Conversely, let $\mu$ be a fuzzy set in $X$ in which $(*)$ is valid. If there exists a $a \in X$ such that $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$ for some $t_a \in (0, \frac{1-k}{2})$. Thus $(a, t_a) \in \mu$ but $(1, t_a) \not\in \mu$. Also, $\mu(1) + t_a < 2t_a \leq 1 - k$, i.e., $(1, t_a) \not\in \mu$. Hence $(1, t_a) \in \mu$, which is a contradiction. Therefore $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in X$. Assume that exist $a, b \in X$ such that $\mu(b) < \min\{\mu(a), \mu(a \ast b), \frac{1-k}{2}\}$. Then $\mu(b) < t \leq \min\{\mu(a), \mu(a \ast b), \frac{1-k}{2}\}$ for some $t \in (0, \frac{1-k}{2})$, and so $a \in U(\mu; t)$ and $a \ast b \in U(\mu; t)$, but $b \not\in U(\mu; t)$. Since $U(\mu; t)$ is a filter of $X$, it is a contradiction. Therefore $\mu(y) \geq \min\{\mu(x), \mu(x \ast y), \frac{1-k}{2}\}$ for all $x, y \in X$. Consequently, we conclude that $\mu$ is an $(\mu, \in \mu)$-fuzzy filter of $X$ by Theorem 3.3. \hfill \Box

**Corollary 3.5.** A fuzzy set $\mu$ in $X$ is an $(\mu, \in \mu)$-fuzzy filter of $X$ if and only if it satisfies:
\[ \forall t \in (0, 0.5), U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t) \text{ is a filter of } X. \]

**Theorem 3.6.** If $A$ is a filter of $X$, then a fuzzy set $\mu$ in $X$ defined by
\[ \mu : X \rightarrow [0, 1], x := \begin{cases} t_1 & \text{if } x \in A, \\
t_2 & \text{otherwise} \end{cases} \]
where $t_1 \in [\frac{1-k}{2}, 1]$ and $t_2 \in (0, \frac{1-k}{2})$, is an $(\mu, \in \mu)$-fuzzy filter of $X$.

**Proof.** Note that
\[ U(\mu; r) := \begin{cases} A & \text{if } r \in (t_2, \frac{1-k}{2}), \\
X & \text{if } r \in (0, t_2] \end{cases} \]
which is a filter of $X$. It follows from Theorem 3.4.
That $\mu$ is an $(\mu, \in \mu)$-fuzzy filter of $X$. \hfill \Box

**Corollary 3.7.** If $A$ is a filter of $X$, then a fuzzy set $\mu$ in $X$ defined by
\[ \mu : L \rightarrow [0, 1], x := \begin{cases} t_1 & \text{if } x \in A, \\
t_2 & \text{otherwise} \end{cases} \]
where $t_1 \in [0.5, 1]$ and $t_2 \in (0, 0.5)$, is an $(\mu, \in \mu)$-fuzzy filter of $X$.

**Theorem 3.8.** Every fuzzy filter is an $(\mu, \in \mu)$-fuzzy filter.

We show that the converse of above theorem is not true in general.

**Example 3.9.** Let $X = \{1, a, b, c\}$ be a set with the following table:

---

Assume that

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a CI-algebra. Define a fuzzy set \(\mu : X \to [0, 1]\) on \(X\), by \(\mu(1) = \mu(a) = \mu(b) = 0.6\) and \(\mu(c) = 0.5\) and \(k = 0.2\). Then \(\mu\) is a \((\in, \vee \in q_k)\)-fuzzy filter of \(X\) but is not a fuzzy filter because \(\mu(c)\) is not \(\leq \min\{\mu(b), \mu(c * b)\}\).

**Theorem 3.10.** If \(\mu\) is an \((\in, \vee \in q_k)\)-fuzzy filter satisfying \(\mu(1) < \frac{1-k}{2}\), then \(\mu\) is a fuzzy filter.

**Proof.** Let \(\mu\) be an \((\in, \vee \in q_k)\)-fuzzy filter of \(X\) such that \(\mu(1) < \frac{1-k}{2}\), using (c3) we have \(\min\{\mu(x), \frac{1-k}{2}\} \leq \mu(1) < \frac{1-k}{2}\) and so \(\mu(x) \leq \frac{1-k}{2}\) for all \(x \in X\). It follows from (c3) and (c4) that \(\mu(1) \geq \mu(x)\) and \(\mu(y) \geq \min\{\mu(x), \mu(y * x)\}\) for all \(x, y \in X\). Hence \(\mu\) is a fuzzy filter of \(X\). \(\square\)

**Corollary 3.11.** If \(\mu\) is an \((\in, \vee \in q)\)-fuzzy filter satisfying \(\mu(1) < 0.5\), then \(\mu\) is a fuzzy filter.

**Proposition 3.12.** For any \(k_1, k_2 \in (0, 1]\) with \(k_1 < k_2\), every \((\in, \vee \in q_{k_1})\)-fuzzy filter is an \((\in, \vee \in q_{k_2})\)-fuzzy filter.

**Example 3.13.** Let \(X = \{1, a, b, c\}\) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a CI-algebra. Define a fuzzy set \(\mu : X \to [0, 1]\) on \(X\), by \(\mu(1) = \mu(a) = \mu(b) = 0.6\) and \(\mu(c) = 0.5\) and \(k_1 = 0.1\). Then \(\mu\) is a \((\in, \vee \in q_{k_1})\)-fuzzy filter of \(X\). Also by \(\mu(1) = \mu(a) = \mu(b) = 0.6\) and \(\mu(c) = 0.4\) and \(k_2 = 0.4\), then \(\mu\) is a \((\in, \vee \in q_{k_2})\)-fuzzy filter of \(X\), but \((\in, \vee \in q_{k_2})\)-fuzzy filter is not an \((\in, \vee \in q_{k_1})\)-fuzzy filter.

For any fuzzy set \(\mu\) in \(X\) and any \(t \in (0, 1]\), we consider for subsets:

\[Q(\mu; t) := \{x \in X \mid (x, t) \in \mu\}, [\mu]_t := \{x \in X \mid (x, t) \in \mu\}\]

\[Q^{k}(\mu; t) := \{x \in X \mid (x, t) \in \mu^{k}\}, [\mu]^{k}_t := \{x \in X \mid (x, t) \in \mu^{k}\}\]

It is clear that \([\mu]_t = U(\mu; t) \cup Q(\mu; t)\) and \([\mu]^{k}_t = U(\mu; t) \cup Q^{k}(\mu; t)\).

**Theorem 3.14.** If \(\mu\) is an \((\in, \vee \in q)\)-fuzzy filter of \(X\), then \(Q^{k}(\mu; t)\) is a filter of \(X\) whenever \(Q^{k}(\mu; t) \neq \emptyset\), for all \(t \in (\frac{1-k}{2}, 1]\).

**Proof.** Assume that \(\mu\) is an \((\in, \vee \in q)\)-fuzzy filter of \(X\) and let \(t \in (\frac{1-k}{2}, 1]\) be such that \(Q^{k}(\mu; t) \neq \emptyset\). Then there exists \(x \in Q^{k}(\mu; t)\), and so \(\mu(x) + t + k > 1\). It follows (c3) that \(\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{1 - t - k, \frac{1-k}{2}\} = 1 - t - k\).
Assume that \( x, t \in (0,1] \) and \( \{0,1\} \) is a filter of \( X \).

Corollary 3.15. If \( \mu \) is an \((\epsilon, \in \vee q)\)-fuzzy filter of \( X \), then \( \forall t \in (0,1], Q(\mu; t) \neq \emptyset \Rightarrow Q(\mu; t) \) is a filter of \( X \).

Proof. It is obvious by taking \( k = 0 \) in Theorem 3.14.

Corollary 3.16. Let \( k, r \in (0,1] \) with \( k < r \). If \( \mu \) is an \((\epsilon, \in \vee q_k)\)-fuzzy filter of \( X \), then \( Q' (\mu; t) \) is an filter of \( X \) whenever \( Q'(\mu; t) \neq \emptyset \) for all \( t \in (\frac{1}{k+1}, 1] \).

Proof. It is clear by Proposition 3.12 and Theorem 3.14.

Theorem 3.17. For any fuzzy \( \mu \) in \( X \), the following are equivalent:

1. \( \mu \) is an \((\epsilon, \in \vee q_k)\)-fuzzy filter of \( X \).

2. \( \forall t \in (0,1]; [\mu]^k \neq \emptyset \Rightarrow [\mu]^k \) is a filter of \( X \).

Proof. Assume that \( \mu \) is an \((\epsilon, \in \vee q_k)\)-fuzzy filter of \( X \) and let \( t \in (0,1] \) be such that \( [\mu]^k \neq \emptyset \). Then there exists \( x \in [\mu]^k = U(\mu; t) \cup Q^k(\mu; t) \), and so \( x \in U(\mu; t) \) or \( x \in Q^k(\mu; t) \). If \( x \in U(\mu; t) \), then (c4) implies that

\[
\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}
\]

Thus \( 1 \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]^k \).

Assume that \( x \in Q^k(\mu; t) \). Then \( (x,t)q_k\mu \), i.e., \( \mu(x) + t + k > 1 \). Thus if \( t > \frac{1-k}{2} \), then

\[
\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} = \left\{ \begin{array}{ll} \mu(x) > 1 - t - k & \text{if } \mu(x) < \frac{1-k}{2}, \\
\frac{1-k}{2} > 1 - k - t & \text{if } \mu(x) \geq \frac{1-k}{2} \end{array} \right.
\]

and so \( 1 \in Q^k(\mu; t) \leq [\mu]^k \). If \( t \leq \frac{1-k}{2} \), then

\[
\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} = \left\{ \begin{array}{ll} \mu(x) > 1 - t - k & \text{if } \mu(x) < \frac{1-k}{2}, \\
\frac{1-k}{2} > t & \text{if } \mu(x) \geq \frac{1-k}{2} \end{array} \right.
\]

which implies that \( 1 \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]^k \). Let \( x, y \in X \) be such \( x \in [\mu]^k \) and \( x + y \in [\mu]^k \). Then \( \mu(x) \geq t \) or \( \mu(x) + t + k > 1 \) and \( \mu(x) + t + k > 1 \) we can consider four cases:

1. \( \mu(x) \geq t \) and \( \mu(x + y) \geq t \)
2. \( \mu(x) \geq t \) and \( \mu(x + y) + t + k > 1 \),
3. \( \mu(x) + t + k > 1 \) and \( \mu(x + y) \geq t \),
4. \( \mu(x) + t + k > 1 \) and \( \mu(x + y) + t + k > 1 \).

For the first case, (c4) implies that

\[
\mu(y) \geq \min\{\mu(x), \mu(x + y), \frac{1-k}{2}\} \\
\geq \min\{t, \frac{1-k}{2}\} = \left\{ \begin{array}{ll} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\
t & \text{if } t \leq \frac{1-k}{2} \end{array} \right.
\]

856
and so \( \mu(y) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \), i.e., \((y, t)q_k\mu\), or \(y \in U(\mu; t)\). Therefore \(y \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]^k_t\). For the case (3), \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \min\{\mu(x * y), \frac{1-k}{2}\} > 1 - t - k\).

we have \(\min\{\mu(x * y), \frac{1-k}{2}\} \leq \mu(x)\); and \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \mu(x)\geq t\)

whenever \(\min\{\mu(x * y), \frac{1-k}{2}\} \leq \mu(x)\); and \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \mu(x)\geq t\)

whenever \(\min\{\mu(x * y), \frac{1-k}{2}\} > \mu(x)\). Thus \(y \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]^k_t\). Suppose that \(t < \frac{1-k}{2}\). Then \(1 - t > \frac{1-k}{2}\), which implies that \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \min\{\mu(x), \mu(x * y)\} > 1 - t - k\).

we have \(\min\{\mu(x * y), \frac{1-k}{2}\} \leq \mu(x)\); and \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \mu(x)\geq t\)

whenever \(\min\{\mu(x * y), \frac{1-k}{2}\} \leq \mu(x)\); and \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \mu(x)\geq t\)

whenever \(\min\{\mu(x * y), \frac{1-k}{2}\} > \mu(x)\). Thus \(y \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]^k_t\). We have similar result of the case (4), for the final case , if \(t > \frac{1-k}{2}\), then \(1 - t - k < 1 - t < \frac{1-k}{2}\). Hence \(\mu(y) \geq \min\{\mu(x), \mu(x * y), \frac{1-k}{2}\} = \min\{\mu(x), \mu(x * y)\} > 1 - t - k\).

Conversely, let \(\mu\) be a fuzzy set in \(X\) such that \([\mu]^k_t\) is a filter of \(X\) for all \(t \in (0, 1]\). If there exists \(a \in X\) such that \(\mu(1) < \min\{\mu(a), 1/k\}\), then \(\mu(1) < t_a \leq \min\{\mu(a), 1/k\}\) for some \(t_a \in (0, 1/k]\). It follows that \(a \in U(\mu; t_a) \subseteq [\mu]^k_{t_a}\) but \(1 \notin U(\mu; t_a)\). Also, \(\mu(1) + t_a < 2t_a \leq 1 - k\), and so \((1, t_a) \notin \mu\), i.e, \(1 \notin Q^k(\mu; t_a)\). Therefore \(1 \notin [\mu]^k_{t_a}\), which is a contradiction.

Hence \(\mu(1) \geq \min\{\mu(x), 1/k\}\) for all \(x \in X\). Suppose that there exist \(a, b \in X\) such that \(\mu(b) < \min\{\mu(a), \mu(a * b), 1/k\}\). Then

\[(5) \mu(b) < t_b \leq \min\{\mu(a), \mu(a * b), 1/k\}\]

for some \(t_b \in (0, 1/k]\), which implies \(a, a * b \in U(\mu; t_b) \subseteq [\mu]^k_{t_b}\) so from (b2) that \(b \in [\mu]^k_{t_b} = U(\mu; t_b) \cup Q^k(\mu; t_b)\) since \([\mu]^k_{t_b}\) is a filter of \(X\). But , (5) implies that \(b \notin U(\mu; t_b)\) and \(\mu(b) + t_b < 2t_b \leq 1 - k\) i.e, \(b \notin Q^k(\mu; t_b)\). Which is a contradiction. Therefore \(\mu(y) \geq \min\{\mu(x), \mu(x * y), 1/k\}\) for all \(x, y \in X\). By Theorem \ref{3.3} we conclude that \(\mu\) is an \((\in, e)\)-fuzzy filter of \(X\).

\[\square\]

\textbf{Corollary 3.18.} For any fuzzy set \(\mu\) in \(X\), the following conditions are equivalent

1. \(\mu\) is an \((\in, e)\)-fuzzy filter of \(X\),
2. \(\forall \in [0, 1]; [\mu]^1_1 \neq \emptyset \Rightarrow [\mu]^1_1\) is a filter of \(X\).

4. Conclusions

In this paper, we defined some types of \((\in, e)\)-fuzzy filter of CI-algebra and investigated the relationship between filters and \((\in, e)\)-fuzzy filter which seems to be a good support for future study. The other direction of future study is an investigation of \((\in, e)\)-fuzzy filter on interval valued.

857
Acknowledgements. The authors wish to thank the reviewers for their excellent suggestions that have been incorporated into this paper.

References


Ameneh Namdar (namdar.amene@gmail.com)
Department of Mathematics, Zarrin Dasht Branch, Islamic Azad University, Zarrin Dasht, Iran

Arsham Borumand Saeid (arsham@mail.uk.ac.ir)
Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

Ghazanfar Jabbari (Mm-Jabbari@gmail.com)
Department of Mathematics, Zarrin Dasht Branch, Islamic Azad University, Zarrin Dasht, Iran