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Fuzzy generalized open sets

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Abstract. In general topology *Á. Császár* [8] introduced generalized open sets. Using this concept B. Roy [16] introduced generalized μ-closed sets in general topology. In this paper we introduce and study a new class of fuzzy sets called fuzzy generalized μ-closed (briefly, fg_μ-closed) sets in fuzzy generalized topological spaces. The class of all fg_μ-closed sets is strictly larger than the class of all fuzzy μ-closed sets. Some of their properties are investigated here. In [12] fuzzy ψ-closed set has been introduced and studied. This type of set is also a fuzzy μ-closed set. In the last section, some characterizations of $f\mu_g$-regular and $f\mu_g$-normal spaces which are the generalizations of μ_g-regular and $\mu_g$normal spaces introduced by B. Roy [16] have been studied here.

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1. Introduction

From the very beginning, different types of fuzzy open sets have been introduced and studied by many researchers. This paper plays a significant role to generalize some of these fuzzy open sets. If one studies these fuzzy open sets minutely, it will be observed that the corresponding definitions have many features in common. This paved a new direction to the author to introduce generalized open sets in fuzzy topology in the sense of Chang [7].

Let X be a non-empty set and I^X denote the set of all fuzzy sets [17] in X. We call a class $\mu \subseteq I^X$, a fuzzy generalized topology (briefly, FGT) if $0_X \in \mu$ and μ is closed under arbitrary union. Then (X, μ) is called a fuzzy generalized topological space (briefly, FGTS). The support of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [15] with the singleton support $x \in X$ and the value α $(0 < \alpha \leq 1)$ at x will be denoted by x_α. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement of a fuzzy set [17] A in X will be denoted by $1_X \setminus A$. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B [15] if $A(x) + B(x) > 1$, for some $x \in X$; the negation of it is denoted by $A\overline{q}B$. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure and fuzzy interior of A in X [17]. A fuzzy set A in X is called fuzzy regular open [1] if $A = intclA$. A fuzzy set A in X is said to be fuzzy semiopen [1] if there exists a fuzzy open set U in X such that $U \leq A \leq clU$, or equivalently, if $A \leq clintA$. The fuzzy θ-closure [14] (resp., fuzzy δ-closure [9]) of a fuzzy set A in an fts (X, τ) is the union of all those fuzzy points x_α such that $clUqA$ whenever $x_\alpha qU \in \tau$ (resp., UqA whenever $x_\alpha qU$ where U is fuzzy regular open set in X). A fuzzy set A is called fuzzy θ-closed [14] (resp., fuzzy δ-closed [9])if $A = \theta clA$ (resp., $A = \delta clA$) and the complement of fuzzy θ-closed (resp., fuzzy δ-closed) set is known as a fuzzy θ-open (resp., fuzzy δ-open) set. A fuzzy set A in an fts (X, τ) is called fuzzy preopen [13] (resp., fuzzy δ-preopen [3], fuzzy α-open [6], fuzzy β-open [2]) if $A \leq intclA$ (resp., $A \leq int\delta clA$, $A \leq intclintA$, $A \leq clintclA$). We note that for an fts (X, τ), the collection of all fuzzy open (resp., fuzzy preopen, fuzzy semiopen, fuzzy δ-open, fuzzy δ-preopen, fuzzy α-open, fuzzy β-open) sets is denoted by τ (resp. FPO(X), FSO(X), FδO(X), FδPO(X), FαO(X), FβO(X)). Each of these collections is an FGT.

For an FGTS (X, μ), the elements of μ are called fuzzy μ-open sets and the complements of fuzzy μ-open sets are called fuzzy μ-closed sets. For $A \in I^X$, we denote by $c_\mu(A)$, the infimum of all fuzzy μ-closed sets B with $A \leq B$, i.e., $c_\mu(A) = inf\{B : A \leq B, B \in \mu^c\}$; and by $i_\mu(A)$, the supremum of all fuzzy μ-open sets B with $B \leq A$, i.e., $i_\mu(A) = sup\{B : B \leq A, B \in \mu\}$. In an fts (X, τ), if one takes τ as the FGT, then c_μ becomes the usual fuzzy closure operator. Similarly, c_μ becomes fuzzy pcl (resp., fuzzy scl, fuzzy δcl, fuzzy δpcl, fuzzy βcl, fuzzy αcl) if μ stands for FPO(X) (resp., FSO(X), FδO(X), FδPO(X), FβO(X), FαO(X)).

It is to be observed that i_μ and c_μ are idempotent and monotonic where $\gamma : I^X \to I^X$ is said to be idempotent iff for any two fuzzy sets A, B in X, $A \leq B \Rightarrow \gamma(\gamma(A)) = \gamma(A)$ and monotonic if $\gamma(A) \leq \gamma(B)$.

2. fg_μ-closed set and its properties

Result 2.1. *Let (X, μ) be an FGTS and $A \in I^X$ and x_α be a fuzzy point in X. Then $x_\alpha \in c_\mu(A)$ iff for every fuzzy μ-open set M in X q-coincident with x_α, MqA.*

Proof. Let x_α be a fuzzy point in X such that $x_\alpha \in c_\mu(A)$ where $A \in I^X$. Let M be a fuzzy μ-open set q-coincident with x_α. Then $M(x) + \alpha > 1$ ... (1)$\Rightarrow \alpha > 1 - M(x)$. Again, $x_\alpha \in c_\mu(A) \Rightarrow B(x) \geq \alpha$, for all fuzzy μ-closed set B with $A \leq B$......(2).
If possible, let $M\overline{q}A$. Then $M(x) + A(x) \leq 1$, for all $x \in X \Rightarrow A(x) \leq 1 - M(x)$, for all $x \in X \Rightarrow 1_X \setminus M$ is a fuzzy μ-closed set with $A \leq 1_X \setminus M$. Then $1 - M(x) \geq \alpha$ by (2). Therefore, $M(x) + \alpha \leq 1$ which contradicts (1). Hence MqA.

Conversely, let $B \in \mu^c$ be such that $A \leq B$. we claim that $B(x) \geq \alpha$. Now $1_X \setminus B \leq 1_X \setminus A$ and $1_X \setminus B \in \mu$. If possible, let $B(x) < \alpha$. Then $1 - B(x) > 1 - \alpha$ $\Rightarrow 1 - B(x) + \alpha > 1 \Rightarrow x_\alpha q(1_X \setminus B)$ and so by hypothesis, $(1_X \setminus B)qA \Rightarrow$ there exists $y \in X$ such that $(1_X \setminus B)(y) + A(y) > 1 \Rightarrow (1_X \setminus B)(y) > (1_X \setminus A)(y)$ for some $y \in X$ which contradicts $1_X \setminus B \leq 1_X \setminus A$. Therefore, $B(x) \geq \alpha \Rightarrow x_\alpha \in c_\mu(A)$. □

Result 2.2. *For any two fuzzy μ-open sets A and B in an FGTS (X, μ), $A\overline{q}B \Rightarrow c_\mu(A)\overline{q}B$ and $A\overline{q}c_\mu(B)$.*

Proof. Let A and B be two fuzzy μ-open sets in an FGTS (X, μ) with $A\overline{q}B$. If possible, let $c_\mu(A)qB$. Then there exists $x \in X$ such that $c_\mu(A)(x) + B(x) > 1$. Let $c_\mu(A)(x) = \alpha$. Then $x_\alpha \in c_\mu(A)$ and $x_\alpha qB$. Then by Result 2.1, BqA, a contradiction. Similarly, we can prove that $A\overline{q}c_\mu(B)$. □

Result 2.3. *Let (X, μ) be an FGTS and $A \in I^X$. Then $c_\mu(1_X \setminus A) = 1_X \setminus i_\mu(A)$.*

Proof. Let $A \in I^X$ and $x_\alpha \in c_\mu(1_X \setminus A)$. Then $B(x) \geq \alpha$, for all fuzzy μ-closed set B with $1_X \setminus A \leq B$ ... (1). We have to show that $(1_X \setminus i_\mu(A))(x) \geq \alpha$. Now $1_X \setminus B \leq A$ where $1_X \setminus B$ is fuzzy μ-open set in X. Let $U \in \mu$ be such that $U \leq A$. Then $1_X \setminus A \leq 1_X \setminus U$, where $1_X \setminus U \in \mu^c$. Then by (1), $(1_X \setminus U)(x) \geq \alpha$ and so $U(x) \leq 1 - \alpha$, for all $U \in \mu$ with $U \leq A$. Then $i_\mu(A)(x) \leq 1 - \alpha \Rightarrow (1_X \setminus i_\mu(A))(x) \geq \alpha$. Therefore, $c_\mu(1_X \setminus A) \leq 1_X \setminus i_\mu(A)$....(i).

Conversely, let $y_\alpha \in 1_X \setminus i_\mu(A)$. Then $1 - i_\mu(A)(y) \geq \alpha \Rightarrow i_\mu(A)(y) \leq 1 - \alpha \Rightarrow U(y) \leq 1 - \alpha$, for all $U \leq A$, $U \in \mu \Rightarrow 1 - U(y) \geq \alpha$, for all $1_X \setminus U$ $(\in \mu^c)$ $\geq 1_X \setminus A$ ... (2). Let $V \in \mu^c$ be such that $1_X \setminus A \leq V$. Then by (2), $V(y) \geq \alpha$. Therefore, $inf\{V(y) : 1_X \setminus A \leq V, V \in \mu^c\} \geq \alpha$. Therefore $c_\mu(1_X \setminus A)(y) \geq \alpha \Rightarrow y_\alpha \in c_\mu(1_X \setminus A)$. And so $1_X \setminus i_\mu(A) \leq c_\mu(1_X \setminus A)$... (ii).
Combining (i) and (ii), we get, $c_\mu(1_X \setminus A) = 1_X \setminus i_\mu(A)$. □

Definition 2.4. Let (X, μ) be an FGTS. Then $A \in I^X$ is called a fuzzy generalized μ-closed set (or fg_μ-closed set, for short) if $c_\mu(A) \leq U$ whenever $A \leq U \in \mu$. The complement of an fg_μ-closed set is called a fuzzy generalized μ-open set (or fg_μ-open set, for short).

Result 2.5. *In an FGTS (X, μ), every fuzzy μ-closed set is fg_μ-closed. Indeed, if A is a fuzzy μ-closed set with $A \leq U \in \mu$, then $c_\mu(A) = A \leq U$ implies that A is fg_μ-closed set.*

Remark 2.6. The converse of Result 2.5 is not true in general as it shown from the following example.

Example 2.7. Let $X = \{a, b\}$, $\mu = \{0_X, 1_X, B\}$ where $B(a) = 0.5$, $B(b) = 0.4$. Then (X, μ) is an FGTS. Now $C(a) = 0.6$, $C(b) = 0.3$ is a fuzzy set in X which is not fuzzy μ-closed. But the only fuzzy μ-open set containing C is 1_X and $C_\mu(C) = 1_X$ and hence C is fg_μ-closed set in (X, μ).

The next two examples show that the union (resp. intersection) of two fg_μ-closed sets is not fg_μ-closed, in general.

Example 2.8. Let $X = \{a, b\}$, $\mu = \{0_X, 1_X, B\}$ where $B(a) = 0.5$, $B(b) = 0.4$. Then (X, μ) is an FGTS. Let C, D be two fuzzy sets in X defined by $C(a) = 0.5$, $C(b) = 0.7$, $D(a) = 0.6$, $D(b) = 0.4$. Then $(C \bigwedge D)(a) = 0.5$, $(C \bigwedge D)(b) = 0.4$. Clearly, C and D are fg_μ-closed sets. But $C \bigwedge D$ is not fg_μ-closed. Indeed, $C \bigwedge D = B \in \mu$ and $c_\mu(C \bigwedge D) = 1_X \setminus B \not\leq B$.

Example 2.9. Let $X = \{a, b\}$, $\mu = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.45$, $A(b) = 0.62$, $B(a) = 0.58$, $B(b) = 0.4$, $C(a) = 0.58$, $C(b) = 0.62$. Then (X, μ) is an FGTS. Let U and V be two fuzzy sets in X defined by $U(a) = 0.4$, $U(b) = 0.6$, $V(a) = 0.5$, $V(b) = 0.3$. Then U and V are clearly fg_μ-closed sets in (X, μ). Now $(U \bigvee V)(a) = 0.5$, $(U \bigvee V)(b) = 0.6$ and $U \bigvee V \leq C \in \mu$, but $c_\mu(U \bigvee V) = 1_X \not\leq C$. Hence $U \bigvee V$ is not fg_μ-closed in (X, μ).

Theorem 2.10. *Let (X, μ) be an FGTS and $A \in I^X$. Then A is fg_μ-closed in (X, μ) iff for any fuzzy μ-closed set F, $A\overline{q}F \Rightarrow c_\mu(A)\overline{q}F$.*

Proof. Let A be fg_μ-closed in (X, μ) and $F \in \mu^c$ be such that $A\overline{q}F$. Then $A \leq 1_X \setminus F \in \mu$. Since A is fg_μ-closed, $c_\mu(A) \leq 1_X \setminus F \Rightarrow c_\mu(A)\overline{q}F$.

Conversely, let for any fuzzy μ-closed set F in X, $A\overline{q}F \Rightarrow c_\mu(A)\overline{q}F$ for a fuzzy set A in X. Let $U \in \mu$ be such that $A \leq U$. Then $1_X \setminus U \leq 1_X \setminus A$ where $1_X \setminus U \in \mu^c$. Therefore, $A\overline{q}(1_X \setminus U)$. By hypothesis, $c_\mu(A)\overline{q}(1_X \setminus U) \Rightarrow c_\mu(A) \leq U$. Hence A is fg_μ-closed set in X. □

Theorem 2.11. *Let (X, μ) be an FGTS and A, B be two fuzzy sets in X with $A \leq B \leq c_\mu(A)$, where A is fg_μ-closed in X. Then B is also fg_μ-closed in X.*

Proof. Let $B \leq U \in \mu$. Then $A \leq B \leq U$. As A is fg_μ-closed, $c_\mu(A) \leq U$. By hypothesis, $B \leq c_\mu(A) \Rightarrow c_\mu(B) \leq c_\mu(c_\mu(A)) = c_\mu(A) \leq U$. Consequently, B is fg_μ-closed in X. □

Theorem 2.12. *In an FGTS (X, μ), $\mu = \Omega$ (the collection of all fuzzy μ-closed sets) iff every fuzzy set in X is fg_μ-closed.*

Proof. Suppose $\mu = \Omega$ and $A \in I^X$ be such that $A \leq U \in \mu$. Then $c_\mu(U) = U$ and so $c_\mu(A) \leq c_\mu(U) = U$. Hence A is fg_μ-closed in X.

Conversely, suppose that every fuzzy set in X is fg_μ-closed in X. Let $U \in \mu$. Then $U \leq U$ and by hypothesis, $c_\mu(U) \leq U$. Therefore, $c_\mu(U) = U$ implies that $U \in \Omega$. Thus $\mu \subseteq \Omega$ ... (i).
Again, let $F \in \Omega$. Then $1_X \setminus F \in \mu$. Also by hypothesis, $1_X \setminus F$ is fg_μ-closed. Therefore, $1_X \setminus F \leq 1_X \setminus F \Rightarrow c_\mu(1_X \setminus F) \leq 1_X \setminus F \Rightarrow 1_X \setminus F \in \mu^c \Rightarrow F \in \mu$. Hence $\Omega \subseteq \mu$ ... (ii).
Combining (i) and (ii), we get $\mu = \Omega$. □

Theorem 2.13. *Let (X, μ) be an FGTS. Then A $(\in I^X)$ is fg_μ-open iff $F \leq i_\mu(A)$ for $F \leq A$ and F is fuzzy μ-closed in X.*

Proof. Let $A (\in I^X)$ be fg_μ-open in X and $F \in \mu^c$ be such that $F \leq A$. Then $1_X \setminus A \leq 1_X \setminus F \in \mu$. By hypothesis, $c_\mu(1_X \setminus A) \leq 1_X \setminus F$. By Result 2.3, $1_X \setminus i_\mu(A) \leq 1_X \setminus F \Rightarrow F \leq i_\mu(A)$.

Conversely, let $F \leq i_\mu(A)$, for $F \leq A$ and $F \in \mu^c$. Let $U \in \mu$ be such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$ where $1_X \setminus U \in \mu^c$. By hypothesis, $1_X \setminus U \leq i_\mu(A)$ $\Rightarrow 1_X \setminus i_\mu(A) \leq U \Rightarrow c_\mu(1_X \setminus A) \leq U \Rightarrow 1_X \setminus A$ is fg_μ-closed in $X \Rightarrow A$ is fg_μ-open in X. □

Theorem 2.14. *let (X, μ) be an FGTS. Then every fuzzy μ-open set in X is fuzzy μ-closed iff every fuzzy set in X is fg_μ-closed in X.*

Proof. Let every fuzzy μ-open set in X be fuzzy μ-closed and $A \in I^X$ be such that $A \leq U \in \mu$. Then by assumption, $U \in \mu^c$ and so $c_\mu(A) \leq c_\mu(U) = U \Rightarrow A$ is fg_μ-closed.

Conversely, let every fuzzy set in X be fg_μ-closed in X and $B \in \mu$. Then $B \leq B$ and so by assumption, $c_\mu(B) \leq B \Rightarrow c_\mu(B) = B \Rightarrow B \in \mu^c$. □

Theorem 2.15. *Let (X, μ) be an FGTS. If A is fg_μ-open and $i_\mu(A) \leq B \leq A$, then B is fg_μ-open.*

Proof. Now $i_\mu(A) \leq B \leq A \Rightarrow 1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus i_\mu(A) = c_\mu(1_X \setminus A)$. As $1_X \setminus A$ is fg_μ-closed, by Theorem 2.11, $1_X \setminus B$ is fg_μ-closed in X and hence B is fg_μ-open in X. □

3. Properties of $f\mu_g$-regular and $f\mu_g$-normal spaces

We now recall the definitions of fuzzy α-regular and fuzzy β-regular spaces from [4] and [5] respectively.

Definition 3.1 ([4])**.** An fts (X, τ) is said to be fuzzy α-regular if for each fuzzy α-open set U in X and each fuzzy point x_α in X with $x_\alpha qU$, there exists a fuzzy α-open set V in X such that $x_\alpha qV \leq \alpha clV \leq U$.

Definition 3.2 ([5])**.** An fts (X, τ) is said to be fuzzy β-regular if for each fuzzy β-open set U in X and each fuzzy point x_α in X with $x_\alpha qU$, there exists a fuzzy β-open set V in X such that $x_\alpha qV \leq \beta clV \leq U$.

Definition 3.3. An FGTS (X, μ) is said to be $f\mu_g$-regular if for each fuzzy point x_α and each fuzzy μ-closed set F of X with $x_\alpha \notin F$, there exist fuzzy μ-open sets U and V such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$.

Remark 3.4. Let (X, μ) be an FGTS. Then every $f\mu_g$-regular space reduces to a fuzzy regular [11] (resp. fuzzy α-regular, fuzzy β-regular) space if one takes μ to be τ (resp., FαO(X), FβO(X)).

Theorem 3.5. *For an FGTS (X, τ), the following are equivalent:*

(a) *X is $f\mu_g$-regular.*

(b) *For each fuzzy point x_α in X and each fuzzy μ-open set U with $x_\alpha qU$, there exists $V \in \mu$ such that $x_\alpha qV \leq c_\mu(V) \leq U$.*

(c) *For each fuzzy μ-closed set F of X, $F = \bigwedge\{c_\mu(V) : F \leq V \in \mu\}$.*

(d) *For any fuzzy set A and any $U \in \mu$ with AqU, there exists $V \in \mu$ such that $AqV \leq c_\mu(V) \leq U$.*

(e) *For any fuzzy set A and any fuzzy μ-closed set F with $A \not\leq F$, there exist $V, W \in \mu$ such that AqV, $F \leq W$ and $V\overline{q}W$.*

(f) *For any fuzzy set A and any fuzzy μ-closed set F with $A \not\leq F$, there exists $U \in \mu$ such that AqU, $c_\mu(U)\overline{q}F$.*

(g) *For any fuzzy point x_α and any fuzzy μ-closed set F with $x_\alpha \notin F$, there exist $U \in \mu$ and an fg_μ-open set V such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$.*

(h) *For any fuzzy set A and any fuzzy μ-closed set F with $A \not\leq F$, there exist $U \in \mu$ and an fg_μ-open set V such that AqU, $F \leq V$ and $U\overline{q}V$.*

(i) *For each fuzzy μ-closed set F of X, $F = \bigwedge\{c_\mu(V) : F \leq V, V$ is fg_μ-open $\}$.*

Proof. (a) $\Rightarrow$ (b) : Let x_α be a fuzzy point in X and $U \in \mu$ be such that $x_\alpha qU$. Then $U(x)+\alpha > 1 \Rightarrow 1-U(x) < \alpha \Rightarrow x_\alpha \notin 1_X \setminus U \in \mu^c$. By (a), there exist fuzzy μ-open sets V and W such that $x_\alpha qV$, $1_X \setminus U \leq W$ and $V\overline{q}W$. Then $V(x)+W(x) \leq 1$, for all $x \in X \Rightarrow V \leq 1_X \setminus W \in \mu^c$. Therefore, $V \leq c_\mu(V) \leq c_\mu(1_X \setminus W) = 1_X \setminus W \leq U$. Then $x_\alpha qV \leq c_\mu(V) \leq U$.

(b) $\Rightarrow$ (c) : Let F be a fuzzy μ-closed set in X. Then $1_X \setminus F \in \mu$. Let $x_\alpha \notin F$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$. Then by (b), there exists $V \in \mu$ such that $x_\alpha qV \leq c_\mu(V) \leq 1_X \setminus F$. Then $F \leq 1_X \setminus c_\mu(V) = U$ (say) $\in \mu$ and $U\overline{q}V$. Indeed, $V \leq c_\mu(V) \Rightarrow U = 1_X \setminus c_\mu(V) \leq 1_X \setminus V \Rightarrow U(x) \leq (1_X \setminus V)(x)$, for all $x \in X \Rightarrow U\overline{q}V$. Then $V \in \mu$ be such that $x_\alpha qV$ and $V\overline{q}U$. Therefore, $x_\alpha \notin c_\mu(U)$ by Result 2.1. Therefore, $\bigwedge\{c_\mu(U) : F \leq U \in \mu\} \leq F$. Obviously, $F \leq \bigwedge\{c_\mu(U) : F \leq U \in \mu\}$. Hence $F = \bigwedge\{c_\mu(U) : F \leq U \in \mu\}$.

(c) $\Rightarrow$ (d) : Let A be any fuzzy set in X and $U \in \mu$ with AqU. Then there exists $x \in X$ such that $A(x) + U(x) > 1$. Let $A(x) = \alpha$. Then $x_\alpha \in A$ and $x_\alpha \notin 1_X \setminus U \in \mu^c$. Then by (c), there exists $W \in \mu$ such that $1_X \setminus U \leq W$ ... (1) and $x_\alpha \notin c_\mu(W)$. Therefore, $c_\mu(W)(x) < \alpha \Rightarrow 1 - c_\mu(W)(x) > 1 - \alpha \Rightarrow x_\alpha qi_\mu(1_X \setminus W)$ where $i_\mu(1_X \setminus W) \in \mu$. Take $i_\mu(1_X \setminus W) = V$. Then $V \in \mu$ such that $x_\alpha qV$ and so $V(x) + \alpha > 1 \Rightarrow V(x) + A(x) > 1 \Rightarrow AqV$. Now $V = i_\mu(1_X \setminus W) \leq 1_X \setminus W$ and so $c_\mu(V) \leq c_\mu(1_X \setminus W) = 1_X \setminus W(\in \mu^c) \leq U$ (by (1).

(d) $\Rightarrow$ (e) : Let A be any fuzzy set in X and $F \in \mu^c$ with $A \not\leq F$. Then there exists $x \in X$ such that $A(x) > F(x)$. Therefore, $1 - A(x) < 1 - F(x) \Rightarrow Aq(1_X \setminus F) \in \mu$. Then by (d), there exists $V \in \mu$ such that $AqV \leq c_\mu(V) \leq 1_X \setminus F$. Now $c_\mu(V) \leq 1_X \setminus F \Rightarrow F \leq 1_X \setminus c_\mu(V) = i_\mu(1_X \setminus V) \in \mu$. Let $i_\mu(1_X \setminus V) = W$. Then $F \leq W \in \mu$. Now $V \leq c_\mu(V) \Rightarrow 1_X \setminus c_\mu(V)(= W) \leq 1_X \setminus V \Rightarrow W\overline{q}V$.

(e) $\Rightarrow$ (a) : Let x_α be a fuzzy point in X and $F \in \mu^c$ with $x_\alpha \notin F$. Then $F(x) < \alpha$. Then x_α is a fuzzy set such that $x_\alpha \not\leq F$. Then by (e), there exist $V, W \in \mu$ such that $x_\alpha qV$, $F \leq W$ and $V\overline{q}W$.

(e) $\Rightarrow$ (f) : Let $A \in I^X$ and $F \in \mu^c$ be such that $A \not\leq F$. By (e), there exist $V, W \in \mu$ such that AqV, $F \leq W$ and $V\overline{q}W$. By Result 2.2, $c_\mu(V)\overline{q}W$ and so $c_\mu(V)\overline{q}F$.

(f) $\Rightarrow$ (e) : Let $A \in I^X$ and $F \in \mu^c$ be such that $A \not\leq F$. By (f), there exists $U \in \mu$ such that AqU and $c_\mu(U)\overline{q}F$. Then $F \leq 1_X \setminus c_\mu(U) = i_\mu(1_X \setminus U) \in \mu$ and $U\overline{q}(1_X \setminus c_\mu(U))$.

(a) $\Rightarrow$ (g) : The proof follows from the fact that every fuzzy μ-open set is fg_μ-open.

(g) $\Rightarrow$ (h) : Let $A \in I^X$ and $F \in \mu^c$ with $A \not\leq F$. Then there exists $x \in X$ such that $A(x) > F(x)$. Let $A(x) = \alpha$. Then $x_\alpha \in A$ and $x_\alpha \notin F$. By (g), there exist $U \in \mu$ and an fg_μ-open set V such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$. Therefore, $U(x) + \alpha > 1 \Rightarrow U(x) + A(x) > 1 \Rightarrow UqA$.

(h) $\Rightarrow$ (a) : Let x_α be a fuzzy point in X and $F \in \mu^c$ with $x_\alpha \notin F$.Then $F(x) < \alpha$. Then x_α is a fuzzy set in X such that $x_\alpha \not\leq F$. Then by (h), there exist $U \in \mu$ and an fg_μ-open set V such that $x_\alpha qU$, $F \leq V$ and $U\overline{q}V$. By Theorem 2.13, $F \leq i_\mu(V) = W$ (say) $\in \mu$. Therefore, $W \leq V$ and so $U\overline{q}W$.

(c) $\Rightarrow$ (i) : We have $F \leq \{c_\mu(V) : F \leq V$ and V is fg_μ-open$\} \leq \{c_\mu(V) : F \leq V$ and V is fuzzy μ-open$\} = F$.

(i) $\Rightarrow$ (a) : Let x_α be any fuzzy point in X and F, a fuzzy μ-closed set in X with $x_\alpha \notin F$. Then by (i), $x_\alpha \notin c_\mu(W)$ for some fg_μ-open set W in X such that $F \leq W$. By Theorem 2.13, $F \leq i_\mu(W)$. Let $V = i_\mu(W)$. Then $F \leq V$. As $x_\alpha \notin c_\mu(W)$, there exists a fuzzy μ-open set M with $x_\alpha q M$, $M \overline{q} W \Rightarrow M \overline{q} V$. Therefore, X is $f\mu_g$-regular. □

Definition 3.6. An FGTS (X, μ) is said to be $f\mu_g$-normal if for any two fuzzy μ-closed sets A and B with $A\overline{q}B$, there exist two fuzzy μ-open sets U and V such that $A \leq U$, $B \leq V$ and $U\overline{q}V$.

Remark 3.7. One can define fuzzy p-normal (resp., fuzzy s-normal, fuzzy α-normal, fuzzy β-normal, fuzzy δ-normal) space by taking fuzzy preopen (resp., fuzzy semiopen, fuzzy α-open, fuzzy β-open, fuzzy δ-open) sets instead of fuzzy open sets and fuzzy preclosed (resp., fuzzy semiclosed, fuzzy α-closed, fuzzy β-closed, fuzzy δ-closed) sets instead of fuzzy closed sets in fuzzy normal space [10]. The above definition gives a unified version of all these definitions if μ takes the role of τ, FPO(X), FSO(X), FαO(X), FβO(X), FδO(X) respectively.

Theorem 3.8. *For an FGTS (X, μ), the following are equivalent :*

(a) *X is $f\mu_g$-normal.*

(b) *For any two fuzzy μ-closed sets A, B with $A\overline{q}B$, there exists a fuzzy μ-open set U such that $A \leq U$, $c_\mu(U)\overline{q}B$.*

(c) *For any two fuzzy μ-closed sets A, B with $A\overline{q}B$, there exist two fg_μ-open sets U and V such that $A \leq U$, $B \leq V$ and $U\overline{q}V$.*

(d) *For each fuzzy μ-closed set A and each fuzzy μ-open set B with $A \leq B$, there exists an fg_μ-open set U such that $A \leq U \leq c_\mu(U) \leq B$.*

(e) *For each fuzzy μ-closed set A and each fg_μ-open set B with $A \leq B$, there exists a fg_μ-open set U such that $A \leq U \leq c_\mu(U) \leq i_\mu(B)$.*

(f) *For each fuzzy μ-closed set A and each fuzzy μ-open set B with $A \leq B$, there exists a fuzzy μ-open set U such that $A \leq U \leq c_\mu(U) \leq B$.*

Proof. (a) $\Rightarrow$ (b) : Let A and B be two fuzzy μ-closed sets with $A\overline{q}B$. By (a), there exist two fuzzy μ-open sets U and V such that $A \leq U$, $B \leq V$ and $U\overline{q}V$. By Result 2.2, $c_\mu(U)\overline{q}V$ and so $c_\mu(U)\overline{q}B$.

(b) $\Rightarrow$ (a) : Let A and B be two fuzzy μ-closed sets with $A\overline{q}B$. By (b), there exists a fuzzy μ-open set U such that $A \leq U$, $c_\mu(U)\overline{q}B$. Then $B \leq 1_X \setminus c_\mu(U) = i_\mu(1_X \setminus U) \in \mu$. Also $U \leq c_\mu(U) \Rightarrow 1_X \setminus U \geq 1_X \setminus c_\mu(U) \Rightarrow U\overline{q}(1_X \setminus c_\mu(U)) \Rightarrow U\overline{q}i_\mu(1_X \setminus U)$.

(a) $\Rightarrow$ (c) : The proof follows from the fact that every fuzzy μ-open set is fg_μ-open in (X, μ).

(c) $\Rightarrow$ (d) : Let $A \in \mu^c$ and $B \in \mu$ with $A \leq B$. Then $A\overline{q}(1_X \setminus B) \in \mu^c$. By (c), there exist two fg_μ-open sets U and V such that $A \leq U$, $1_X \setminus B \leq V$ and $U\overline{q}V \Rightarrow U \leq 1_X \setminus V \leq B$. By Theorem 2.13, $1_X \setminus B \leq i_\mu(V) \Rightarrow c_\mu(1_X \setminus V) = 1_X \setminus i_\mu(V) \leq B$. Thus $A \leq U \leq c_\mu(U) \leq c_\mu(1_X \setminus V) \leq B$. Thus $A \leq U \leq c_\mu(U) \leq B$.

(d) $\Rightarrow$ (a) : Let $A, B \in \mu^c$ with $A\overline{q}B$. Then $A \leq 1_X \setminus B \in \mu$. Then by (d), there exists an fg_μ-open set U such that $A \leq U \leq c_\mu(U) \leq 1_X \setminus B$. By Theorem 2.13, $A \leq i_\mu(U)$. Also $B \leq 1_X \setminus c_\mu(U) = i_\mu(1_X \setminus U)$ and $i_\mu(U)\overline{q}(1_X \setminus c_\mu(U))$.

(b) $\Rightarrow$ (f) : Let $A \in \mu^c$ and $B \in \mu$ be such that $A \leq B$. Then $A\overline{q}(1_X \setminus B) \in \mu^c$. By (b), there exists $U \in \mu$ such that $A \leq U$, $c_\mu(U)\overline{q}(1_X \setminus B) \Rightarrow c_\mu(U) \leq B$. Therefore, $A \leq U \leq c_\mu(U) \leq B$.

(f) $\Rightarrow$ (e) : Let $A \in \mu^c$ and B, an fg_μ-open set in X with $A \leq B$. Then by Theorem 2.13, $A \leq i_\mu(B)$. Let $i_\mu(B) = V$. Then $V \in \mu$. By (f), there exists a fuzzy μ-open set U such that $A \leq U \leq c_\mu(U) \leq V = i_\mu(B)$.

(e) $\Rightarrow$ (c) : Let $A, B \in \mu^c$ be such that $A\overline{q}B$. Then $A \leq 1_X \setminus B \in \mu$. Since every fuzzy μ-open set is fg_μ-open, $1_X \setminus B$ is fg_μ-open in X. By (e), there exists a fg_μ-open set U in X such that $A \leq U \leq c_\mu(U) \leq i_\mu(1_X \setminus B) = 1_X \setminus c_\mu(B)$. Now $B \leq c_\mu(B) \Rightarrow 1_X \setminus B \geq 1_X \setminus c_\mu(B) \geq c_\mu(U) \Rightarrow B \leq 1_X \setminus c_\mu(U) = i_\mu(1_X \setminus U) \in \mu$. Since every fuzzy μ-open set is fg_μ-open, $i_\mu(1_X \setminus U)$ is fg_μ-open in X.
Again, $U\overline{q}(1_X \setminus c_\mu(U)) = i_\mu(1_X \setminus U)$. □

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