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Some geometric properties of generating spaces of semi-norm family

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Abstract. In this paper, definitions of uniformly convex and strictly convex generating spaces of semi-norm family (G.S.S-N.F) are given. The concept of uniform normal structure is introduced. Finally Taylor Foguel's theorem and Browder type fixed point theorem are established in G.S.S-N.F.

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1. Introduction

It is well known that metric and norm structures play a pivotal role in functional analysis. So in order to develop this one has to take care of the suitable generalization of these structures. Historically, the problem of fuzzyfication of the metric structure came first. Different authors introduced ideas of fuzzy-metric space([10], [14], [13], [8]), probabilistic metric spaces [22], quasi metric space, Dislocated fuzzy quasi metric space [7], statistical metric space[22], fuzzy normed linear space([1], [3], [6], [17]), fuzzy soft topological spaces[20], generalized open fuzzy set[21], 2-fuzzy inner product space[4] etc. Chang et.al. ([5], [9], [12]) first introduced a definition of generating spaces of quasi-metric family, which generalizes those of fuzzy metric spaces in the sense of Kaleva & Seikkala [10] and Menger probabilistic metric spaces [22]. They also proved several fixed point theorems in quasi-metric family. J. S. Jung, B. S. Lee and Y. J. Cho, [9] established some fixed point theorems in generating spaces of quasi-metric family. In 2006, Xiao & Zhu [23] introduced a concept of generating spaces of quasi-norm family (G.S.Q-N.F) and studied linear topological structures. They introduced the concept of convergent sequence, Cauchyness, completeness, compactness etc. and established some fixed point theorems specially Schauder-type fixed point theorem in such spaces. In [15], we have established some results in finite dimensional G.S.Q-N.F and derived a G.S.Q-N.F from a generalized B-S fuzzy normed ([1], [2]) linear space. Ideas of bounded linear operators, bounded linear functionals, generating spaces of operators quasi-norm family, dual space etc. are developed in[16]. In [18], we give a constructive proof of Hahn-Banach extension theorem in finite dimensional generating spaces of quasi-norm family. Finally we able to establish Kirk's fixed point theorem[19] for non-expansive mapping on a nonempty, weakly compact, closed and convex subsets in G.S.S-N.F.

The organization of the paper is as follows:

Section 1, comprises some preliminary results.

In section 2, we give the definition of uniformly convex generating space of semi-norm family, normal structure, uniform normal structure and establish Browder's fixed point theorem.

In section 3, the definition of strictly convex generating space of semi-norm family is given and some of its properties are studied. Finally Taylor Foguel's theorem is established in strictly convex generating space of semi-norm family.

Throughout this paper straightforward proofs are omitted.

2. Preliminaries

In this section some preliminary results are given which are related to this paper and used in different portion of this manuscript.

Definition 2.1 ([16]). Let X be a linear space over E(Real or Complex) and θ be the origin of X. Let

$$Q = \{|.|_\alpha : \alpha \in (0,\ 1)\}$$

be a family of mappings from X into $[0,\ \infty)$. $(X,\ Q)$ is called a generating space of quasi-norm family and Q, a quasi-norm family, if the following conditions are satisfied:

(QN1) $|x|_\alpha = 0\ \ \forall \alpha \in (0,\ 1)$ iff $x = \theta$;

(QN2) $|ex|_\alpha = |e||x|_\alpha \quad \forall\ x \in X, \forall \alpha \in (0,\ 1)$ and $\forall\ e \in E$;

(QN3) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that $|x + y|_\alpha \leq |x|_\beta + |y|_\beta$ for all $x,\ y \in X$;

(QN4) for any $x \in X$, $|x|_\alpha$ is non-increasing for $\alpha \in (0,\ 1)$.

(X, Q) is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where

(QN-3u) for any $\alpha \in (0,\ 1]$ there exists $\beta \in (0,\ \alpha]$ such that $|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta$ for any $n \in Z^+$, $x_i \in X (i = 1, 2, ..., n)$;

(QN-3t) for any $\alpha \in (0,\ 1]$ there exists a $\beta \in (0,\ \alpha]$ such that $|x + y|_\alpha \leq |x|_\alpha + |y|_\beta$ for $x,\ y \in X$;

(QN-3e) for any $\alpha \in (0,\ 1]$, it holds that $|x + y|_\alpha \leq |x|_\alpha + |y|_\alpha$ for $x,\ y \in X$.

Definition 2.2 ([16]). Let $(X,\ Q)$ be a generating space of quasi-norm family(G.S.Q-N.F).

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_\alpha = 0$ for each $\alpha \in (0,\ 1)$;

(b) to be a Cauchy sequence if $\lim_{m,\ n\to\infty} |x_n - x_m|_\alpha = 0$ for each $\alpha \in (0,\ 1)$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

Definition 2.3 ([16])**.** Let (X, Q) be a generating space of quasi-norm family:

(a) A subset A of X is said to be bounded if for each $\alpha \in (0,\ 1)$ there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha)\ \ \forall\ x \in\ A$;

(b) A subset A of X is said to be α-level bounded for some $\alpha \in (0,\ 1)$ if there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha)\ \ \forall\ x \in\ A$;

(c) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in\ A$;

(d) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A;

(e) A subset A of X is said to be strongly bounded if there exists a real number $M > 0$ such that $|x|_\alpha \leq\ M\ \ \forall\ x \in\ A\ \ \forall \alpha \in (0,\ 1)$.

Definition 2.4 ([11])**.** Let (X, Q) be a generating space of quasi-norm family:

(a) The closure of a subset A of X is denoted by $\overline{A}$ and is defined by
$\overline{A} = \{x : if\ \exists\ a\ sequence\ \{x_n\}\ in\ A\ such\ that\ \lim_{n\to\infty} x_n = x\}$;

(b) The set of all convex combinations of points of a subset A of X is denoted by $convA$ and is defined by
$convA = \{\lambda\ x + (1-\lambda)\ y\ \ \forall\ x,\ y \in\ A,\ \ \forall \lambda \in [0,\ 1]\}$.

Definition 2.5 ([16])**.** Let $Q_1 = \{|.|_\alpha^1 : \alpha \in (0,\ 1)\}$ and $Q_2 = \{|.|_\alpha^2 : \alpha \in (0,\ 1)\}$ be two quasi-norm families on X_1 and X_2 respectively and $T : (X_1,\ Q_1)\ \rightarrow\ (X_2,\ Q_2)$ be an operator. Then T is said to be continuous at $x \in\ X_1$ if for any sequence $\{x_n\}$ of X_1 with $x_n \rightarrow\ x$ i.e. with $\lim_{n\to\infty} |x_n - x|_\alpha^1 = 0\ \ \forall \alpha \in (0,\ 1)$ implies $T(x_n) \rightarrow\ T(x)$. i.e. $\lim_{n\to\infty} |T(x_n) - T(x)|_\alpha^2 = 0\ \ \forall \alpha \in (0, 1)$. If T is continuous at each point of X_1, then T is said to be continuous on X_1.

Definition 2.6 ([16])**.** Let $T : (X_1,\ Q_1)\ \rightarrow\ (X_2,\ Q_2)$ be an operator. Then T is said to be

(i) bounded if corresponding to each $\alpha \in (0,\ 1)$, $\exists\ M_\alpha > 0$ such that

$$|T(x)|_\alpha^2 \leq\ M_\alpha |x|_{1-\alpha}^1\ \ \forall x \in\ X_1;$$

(ii) $\alpha-$ level bounded for some $\alpha \in (0, 1)$ if $\exists M_\alpha > 0$ such that $|T(x)|_\alpha^2 \leq\ M_\alpha |x|_{1-\alpha}^1$ $\forall x \in\ X_1$.

Definition 2.7 ([19])**.** Let $(X,\ Q)$ be a G.S.Q-N.F and D , H are two strongly bounded subset of X. Set:

(i) $\delta(D) = \bigvee_{\alpha \in (0,\ 1)} [\bigvee\{|x-y|_\alpha,\ \forall x,\ y \in D\}]$;

(ii) $r_u(D) = \bigvee_{\alpha \in (0,\ 1)} [\bigvee\{|u-x|_\alpha,\ \forall x \in D\}],\ (u\ \in H)$;

(iii) $r_H(D) = \bigwedge_{u\in H} \{r_u(D)\}$;

(iv) $C_H(D) = \{u \in H : r_u(D) = r_H(D)\}$.

The number $\delta(D)$ is called the diameter of D, $r_u(D)$ is called the radius of D relative to u, $r_H(D)$ and $C_H(D)$ are called respectively the Chebyshev radius and the Chebyshev center of D relative to H. When $H = D$ the notations $r(D)$ and $C(D)$ are used for $r_H(D)$ and $C_H(D)$ respectively.

Definition 2.8 ([19]). Let (X, Q) be a G.S.Q-N.F and D is a strongly bounded subset of X. A point $u \in D$ is said to be a diametral point if $r_u(D) = \delta(D)$. If u is not a diametral point of D, then it is called a non-diametral point of D.

Definition 2.9 ([19]). Let (X, Q) be a G.S.Q-N.F. A nonempty strongly bounded, convex subset K of X is said to have normal structure if each convex subset S of K with $\delta(S) > 0$ contains a non-diametral point. The space (X, Q) is said to have normal structure if each of its nonempty, strongly bounded, convex subsets has this property.

Definition 2.10 ([19]). Let (X, Q) be a G.S.Q-N.F and $T : X \to X$. The operator T is said to be non-expansive if $|Tx - Ty|_\alpha \le |x - y|_\alpha \forall \alpha \in (0, 1), \forall x, y \in X$.

Definition 2.11 ([19]). Let (X, Q) be a generating space of semi-norm family(G.S.S-N.F):

(a) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

(i) to be weakly convergent to $x \in X$ denoted by $x_n \to^w x$ if $\lim_{n\to\infty} |f(x_n) - f(x)| = 0$ for each $f \in X^*$. In this case x is called the weak limit of the sequence $\{x_n\}$;

(ii) to be a weakly Cauchy sequence if $\lim_{m,\, n\to\infty} |f(x_n) - f(x_m)| = 0$ for each $f \in X^*$;

(b) A subset $B \subset X$ is said to be weakly complete if every weakly Cauchy sequence in B weakly converges in B;

(c) A subset A of X is said to be weakly closed if for any sequence $\{x_n\}$ of points of A with $x_n \to^w x$ implies $x \in A$;

(d) $A \subset X$ is said to be weakly compact if for any sequence $\{x_n\}$ of points of A has a weakly convergent subsequence which is weakly convergent to a point in A.

Theorem 2.12 ([18]). *Let (X_1, Q_1) be a G.S.Q-N.F and (X_2, Q_2) be a generating space of semi-norm family (G.S.S-N.F) satisfying*

(QN6): *if $x(\neq \theta) \in X_2$ then $|x|^2_\alpha > 0 \ \forall \alpha \in (0, 1)$.*

For $T \in B(X_1, X_2)$ and $\alpha \in (0, 1)$ we define

$$|T|^s_\alpha = \bigvee_{x\in X_1,\ |x|^1_{1-\alpha}\le 1} \{|T(x)|^2_\alpha\}$$

Then $(B(X_1, X_2), Q^s)$ is a G.S.S-N.F satisfying (QN6) i.e. a generating space of norm family (G.S.N.F), where $Q^s = \{|.|^s_\alpha : \alpha \in (0, 1)\}$.

Theorem 2.13 ([18]). **(Hahn-Banach)** *Let (X, Q) be a generating space of semi-norm family and $\alpha \in (0, 1)$. If f is an α-level bounded linear operator which is defined on a subspace Z of X, then f has a linear extension $\hat{f}$ from Z to X which is α-level bounded on X and $|f|^s_\alpha = |\hat{f}|^s_\alpha \ \forall \alpha \in (0, 1)$.*

Note 2.14 ([18]). *Let $(X,\ Q)$ be a generating space of semi-norm family and $\alpha \in (0,1)$. Then the space of all α-level bounded linear functional defined on X is called the α-dual space of X and is denoted by X^*_α, which is a normed linear space.*

Theorem 2.15 ([19]). **(Kirk's)** *Let $(X,\ Q)$ be a generating space of semi-norm family and K be a nonempty, weakly compact, convex subset of X. If K has a normal structure then for any non-expansive mapping $T : K \to\ K$ has a fixed point.*

3. Uniformly convex G.S.S-N.F

In this section we give the definition of uniformly convex generating space of semi-norm family, normal structure, uniform normal structure and establish Browder's fixed point theorem.

Definition 3.1. Let (X,Q) be a generating space of semi-norm family. For $0 < \epsilon \leq 2, \alpha \in (0,1)$, α-modulus of convexity of X is a real number

$$\gamma_{\alpha,\ \epsilon} = \bigwedge_{x,\ y \in\ X} \{1 - |\frac{x+y}{2}|_\alpha :\ |x|_\alpha \leq\ 1,\ \ |y|_\alpha \leq\ 1,\ \ |x-y|_\alpha \geq \epsilon\}.$$

Note 3.2. *$\gamma_{\alpha,\ \epsilon}$ is an increasing function with respect to $\epsilon \in (0,\ 2]$.*

Definition 3.3. Let (X,Q) be a generating space of semi-norm family. For $0 < \epsilon \leq 2, \alpha \in (0,1)$, we define

$$S_{\alpha,\ \epsilon}\ =\ \{(x,\ y) :\ x, y \in\ X,\ |x|_\alpha \leq\ 1,\ \ |y|_\alpha \leq\ 1,\ \ |x-y|_\alpha \geq \epsilon\}.$$

Definition 3.4. Let $(X,\ Q)$ be a generating space of semi-norm family. Then $(X,\ Q)$ is said to be a uniformly convex G.S.S-N.F if its α-modulus of convexity $\gamma_{\alpha,\ \epsilon}$ satisfies $\bigwedge_{\alpha \in (0,\ 1)} \gamma_{\alpha,\ \epsilon} > 0\ \ \forall \epsilon \in (0,\ 2]$.

Lemma 3.5. *Let (X,Q) be a generating space of semi-norm family and $\gamma_{\alpha,\epsilon}$ be the α-modulus of convexity of X for each $\alpha \in (0,1)$ and $\epsilon \in (0,2]$. Then*

$$|\tfrac{x+y}{2}|_\alpha \leq\ 1 - \gamma_{\alpha,\ \epsilon} \quad \forall (x,\ y) \in\ S_{\alpha,\ \epsilon}$$

Proof. From definition

$$\gamma_{\alpha,\ \epsilon} = \bigwedge_{x,\ y \in\ X} \{1 - |\frac{x+y}{2}|_\alpha :\ |x|_\alpha \leq\ 1,\ \ |y|_\alpha \leq\ 1,\ \ |x-y|_\alpha \geq \epsilon\}$$

and

$$S_{\alpha,\ \epsilon}\ =\ \{(x,\ y) :\ x, y \in\ X,\ |x|_\alpha \leq\ 1,\ \ |y|_\alpha \leq\ 1,\ \ |x-y|_\alpha \geq \epsilon\}.$$

Hence

$$\gamma_{\alpha,\ \epsilon} = \bigwedge_{(x,\ y) \in\ S_{\alpha,\ \epsilon}} \{1 - |\frac{x+y}{2}|_\alpha\}$$

$$\Rightarrow\ 1 - |\tfrac{x+y}{2}|_\alpha \geq\ \gamma_{\alpha,\ \epsilon} \quad \forall (x,\ y) \in\ S_{\alpha,\ \epsilon}$$

$$\Rightarrow\ |\tfrac{x+y}{2}|_\alpha \leq\ 1 - \gamma_{\alpha,\ \epsilon} \quad \forall (x,\ y) \in\ S_{\alpha,\ \epsilon}.$$

□

Theorem 3.6. *Let $(X,\ Q)$ be a generating space of semi-norm family and K be a nonempty strongly bounded convex subset of X. If X is uniformly convex then K has normal structure.*

Proof. Let K be a convex subset of X and S be a convex subset of K with $\delta\,(S) = \bigvee_{\alpha\in(0,\ 1)} [\bigvee\{|x-y|_\alpha,\ \forall x,\ y\in\ S\}] = d(>0)$ (say) denotes the diameter of S. Then clearly S is nonempty and contains more than one elements. Let $\epsilon \in (0,\ d]$, then there exists $\alpha_0 \in (0,\ 1),\ \ u,\ v(\neq\ u) \in\ S$ such that
$|u-v|_\alpha\ \ \geq \epsilon\quad \forall\alpha\in\ \ (0,\ \alpha_0])$ [Since for any $x\in X$, $|x|_\alpha$ is non-increasing for $\alpha\in(0,\ 1)$]
Take any point $x \in S$.
Then
$|x-u|_\alpha \leq\ d\ \ \forall\alpha\in\ (0,\ 1)$
and
$|x-v|_\alpha \leq\ d\ \ \forall\alpha\in\ (0,\ 1)$;
$\Rightarrow\ |\frac{x-u}{d}|_\alpha\ \leq 1\ \forall\alpha\in\ (0,\ 1)$
and
$|\frac{x-v}{d}|_\alpha\ \leq 1\ \forall\alpha\in\ (0,\ 1)$.
Again $|\frac{x-u}{d}-\frac{x-v}{d}|_\alpha = |\frac{u-v}{d}|_\alpha \geq \frac{\epsilon}{d}\ >\ \frac{\epsilon}{2d}\ >\ 0\ \ \forall\alpha\in\ (0,\ \alpha_0]$.
By the uniform convexity of X and Lemma 2.1, we have
$|\frac{\frac{x-u}{d}+\frac{x-v}{d}}{2}|_\alpha \leq (1-\gamma_{\alpha,\ \frac{\epsilon}{2d}})\ \ \forall\alpha\in\ (0,\ \alpha_0]$;
$\Rightarrow\ |\frac{x-\frac{u+v}{2}}{d}|_\alpha \leq (1-\gamma_{\alpha,\ \frac{\epsilon}{2d}})\ \ \forall\alpha\in\ (0,\ \alpha_0]$;
$\Rightarrow \bigvee_{\alpha\in(0,\ 1)} |x-\dfrac{u+v}{2}|_\alpha\ \leq d\ (1-\bigwedge_{\alpha\in(0,\ 1)}\gamma_{\alpha,\ \frac{\epsilon}{2d}})$;
$\Rightarrow \bigvee_{\alpha\in(0,\ 1)} |x-\dfrac{u+v}{2}|\ <\ d$;
$\Rightarrow \frac{u+v}{2}$ is a non-diametral point of S
$\Rightarrow\ \ X$ has normal structure. □

Theorem 3.7. *Let $(X,\ Q)$ be a generating space of semi-norm family and K be a nonempty strongly bounded convex subset of X such that*

$$\bigwedge_{\alpha\in(0,\ 1)}[\bigvee_{S\in K}\{\frac{\bigwedge_{u\in S}\{\bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u-v|_\gamma,\ \forall v\in S\}\}\}}{\bigvee\{|u-v|_\alpha,\ \forall u,v\in S\}}\}]\ <\ 1$$

where K denotes the collection of all strongly bounded convex subsets of K with $\delta(S)>0$. If X is uniformly convex then K has normal structure.

Proof. Let

$$\bigwedge_{\alpha\in(0,\ 1)}[\bigvee_{S\in K}\{\frac{\bigwedge_{u\in S}\{\bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u-v|_\gamma,\ \forall v\in S\}\}\}}{\bigvee\{|u-v|_\alpha,\ \forall u,\ v\in S\}}\}]\ <\ 1$$

$\Rightarrow \exists\alpha_0\in(0\ ,\ 1)$ such that

$$\bigwedge_{\alpha\in(0,\ 1)}[\bigvee_{S\in K}\{\frac{\bigwedge_{u\in S}\{\bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u-v|_{\gamma},\ \forall v\in S\}\}\}}{\bigvee\{|u-v|_{\alpha_0},\ \forall u,v\in S\}}\}]\ <\ 1$$

$\Rightarrow \bigvee\{|u-v|_{\alpha_0},\ \forall u,\ v\in S\}\ >\ \bigwedge_{u\in S}\{\bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u-v|_{\gamma},\ \forall v\in S\}\}\}$ for all S with $\delta(S)>0$.

$\Rightarrow \bigvee\{|u-v|_{\alpha_0},\ \forall u,\ v\in S\}\ >\ \bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u_S-v|_{\gamma},\ \forall v\in S\}\}$ for some $u_S\in S$ with $\delta(S)>0$

Choose $k_1,\ k_2$ such that,

$\bigvee\{|u-v|_{\alpha_0},\ \forall u,v\in S\}\ >\ k_1\ >\ k_2\ >\ \bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u_S-v|_{\gamma},\ \forall v\in S\}\}$ for some $u_S\in S$ with $\delta(S)>0$.

Now $\bigvee\{|u-v|_{\alpha_0},\ \forall u,\ v\in S\}\ >\ k_1$

$\Rightarrow \delta(S)\ >\ k_1>k_2>\ \bigvee_{\gamma\in(0,\ 1)}\{\bigvee\{|u_S-v|_{\gamma}\quad\forall v\in S\}\}$.

Hence u_S is a non-diametral point of S and K has normal structure. □

Definition 3.8. Let (X,Q) be a generating space of semi-norm family. A nonempty, closed, convex, set K of X is said to have uniform normal structure if there exists a $k\in(0,1)$ such that

$$r(D)\le k\delta(D)$$

for any closed, convex subset D of K. The space (X,Q) is said to have uniform normal structure if each of its nonempty, closed, convex subsets has this property.

Theorem 3.9. *Let $(X,\ Q)$ be a uniformly convex generating space of semi-norm family. Then X has uniform normal structure.*

Proof. Let K be a nonempty, strongly bounded, closed and convex subset of X with $\delta\ (K)=d(>0)$.
Let $\epsilon\in(0,\ d)\cap\ (0,\ 1)$, then there exists $\alpha_0\in(0,\ 1)$, $u,\ v(\ne\ u)\in\ K$ such that $|u-v|_{\alpha}\ >\epsilon\quad\forall\alpha\in\ (0,\ \alpha_0]$.
Take any point $x\in K$.
Then

$|x-u|_{\alpha}\le\ d\ \ \forall\alpha\in\ (0,\ 1)$

and

$|x-v|_{\alpha}\le\ d\ \ \forall\alpha\in\ (0,\ 1)$.

Hence $|\frac{x-u}{d}|_{\alpha}\le\ 1$, $|\frac{x-v}{d}|_{\alpha}\le\ 1$ and $|\frac{x-u}{d}-\frac{x-v}{d}|_{\alpha}>\frac{\epsilon}{d}\ \ \forall\alpha\in\ (0,\ \alpha_0]$.
By Lemma 2.1, we have

$|\frac{x-\frac{u+v}{2}}{d}|_{\alpha}\le(1-\gamma_{\alpha,\ \frac{\epsilon}{d}})\ \ \forall\alpha\in\ (0,\ \alpha_0]$;

$\Rightarrow \bigvee_{\alpha\in(0,\ 1)}|x-\frac{u+v}{2}|_{\alpha}\ \le d\ (1-\bigwedge_{\alpha\in(0,\ \alpha_0)}\gamma_{\alpha,\ \frac{\epsilon}{d}})$.

Since $(X,\ Q)$ is uniformly convex,

$$\bigwedge_{\alpha\in(0,\ 1)} \gamma_{\alpha,\ \epsilon} > 0 \ \ \forall \epsilon \in (0,\ 2]$$

$\Rightarrow \bigvee_{\alpha\in(0,\ 1)} |x - \frac{u+v}{2}|_\alpha \ < \ k\delta(K)$, where $k \in (0,\ 1)$;

$\Rightarrow \bigvee_{y\in\ K} [\bigvee_{\alpha\in(0,\ 1)} |x-k|_\alpha] \ < \ k\delta(K)$.

Now $r_x(K) = \bigvee_{\alpha\in(0,\ 1)} [\bigvee\{|x-y|_\alpha,\ \forall\ y\in K\}] \leq \bigvee_{y\in\ K} [\bigvee_{\alpha\in(0,\ 1)} |x-k|_\alpha] \ < \ k\delta(K)$.

$\Rightarrow \ r(K) < \ k\delta(K)$.

Hence X has uniform normal structure. □

Theorem 3.10. *Let (X, Q) be a generating space of semi-norm family and $K \subset X$. If K has uniform normal structure then it has normal structure.*

Proof. Let $D \subset \ K$ be any nonempty, strongly bounded, closed and convex subset of K with $\delta\ (D) = d(> 0)$. We have to prove D has non-diametral point. If possible let D has non-diametral point. Since K has uniform normal structure, there exists a $k \in (0,\ 1)$ such that

$$r(D) \leq \ k\delta(D)................................(1)$$

Since D contains only diametral points

$r_x(D) = \delta(D) \quad \forall\ x \in \ D$;

$\Rightarrow \ r(D) = \delta(D)$ which contradicts with (1).

Hence K has normal structure. □

Theorem 3.11. *Let K be a nonempty, strongly bounded, weakly compact, convex subset of a uniformly convex generating space of semi-norm family $(X,\ Q)$. Then every non-expansive mapping $T:\ K \rightarrow\ K$ has a fixed point.*

Proof. By Theorem 3.5, K has normal structure and by Theorem 2.16, K has a unique fixed point. □

4. Strictly convex G.S.S-N.F

In this section we give the definition of strictly convex generating space of semi-norm family and study some properties of it.

Definition 4.1. Let $(X,\ Q)$ be a generating space of semi-norm family. Then $(X,\ Q)$ is said to be a strictly convex G.S.S-N.F if for $\alpha \in (0,\ 1)$ and for $x,\ y \in\ X$,

$$\left.\begin{array}{l} |x|_\alpha \leq\ 1 \\ |y|_\alpha \leq\ 1 \\ |x-y|_\alpha > 0 \end{array}\right\} \Rightarrow\ |\frac{x+y}{2}|_\alpha <\ 1.$$

Example 4.2. Let X be an inner product space and define

$$|x|_\alpha = \left\{ \begin{array}{l} \frac{(0.7-\alpha)}{\alpha}||x|| =< x, x >^{\frac{1}{2}} \quad for \ \ \alpha \in (0, 0.7) \\ 0 \quad for \ \ \alpha \in [0.7, 1). \end{array}\right.$$

Then $Q = \{|.|_\alpha : \alpha \in (0,1)\}$ is is a semi-norm family and $(X,\ Q)$ is a strictly convex G.S.S-N.F.

Proof. Since X is an inner product space
$||\frac{x+y}{2}||^2 = 2(||\frac{x}{2}||^2 + ||\frac{y}{2}||^2) - ||\frac{x-y}{2}||^2$;
$\Rightarrow\ |\frac{x+y}{2}|_\alpha^2 = 2(|\frac{x}{2}|_\alpha^2 + |\frac{y}{2}|_\alpha^2) - |\frac{x-y}{2}|_\alpha^2\ \ \forall\alpha \in\ (0,\ 1)$.
Hence for $\alpha \in (0,\ 1)$ and for $x,\ y \in\ X$,

$$\left.\begin{array}{l}|x|_\alpha \leq\ 1\\ |y|_\alpha \leq\ 1\\ |x-y|_\alpha > 0\end{array}\right\} \Rightarrow\ |\frac{x+y}{2}|_\alpha <\ 1.$$

So $(X,\ Q)$ is a strictly convex G.S.S-N.F. □

Theorem 4.3. *If $(X,\ Q)$ is an uniformly convex generating space of semi-norm family, then it is strictly convex.*

Proof. Let $\alpha \in (0,\ 1)$, $x,\ y \in\ X$, $|x|_\alpha \leq\ 1$, $|y|_\alpha \leq\ 1$, and $|x-y|_\alpha = \epsilon\ > 0$. Then clearly $\epsilon \in\ (0,\ 2]$ and $(x,\ y) \in\ S_{\alpha,\ \epsilon}$. By Lemma 2.1,
$|\frac{x+y}{2}|_\alpha \leq\ \ 1 - \gamma_{\alpha,\ \epsilon}$.
Since $(X,\ Q)$ is an uniformly convex generating space of semi-norm family,
$\gamma_{\alpha,\ \epsilon} > 0\ \Rightarrow\ |\frac{x+y}{2}|_\alpha <\ 1$.
Hence the Theorem. □

Theorem 4.4. *A generating space of semi-norm family $(X,\ Q)$ is strictly convex iff $\gamma_{\alpha,\ 2} = 1\ \ \forall\alpha \in\ (0,\ 1)$.*

Proof. Let $\gamma_{\alpha,\ 2} = 1\ \ \forall\alpha \in\ (0,\ 1)$. We have to prove $(X,\ Q)$ is strictly convex i.e. for $\alpha \in (0,\ 1)$ and for $x,\ y \in\ X$,

$$\left.\begin{array}{l}|x|_\alpha \leq\ 1\\ |y|_\alpha \leq\ 1\\ |x-y|_\alpha > 0\end{array}\right\} \Rightarrow\ |\frac{x+y}{2}|_\alpha <\ 1.$$

If possible let there exists $\alpha_0 \in\ (0,\ 1)$, $x,\ y \in\ X$ such that
$|x|_{\alpha_0} =\ 1,\ \ |y|_{\alpha_0} =\ 1,\ \ |x-y|_{\alpha_0} > 0$ but $|\frac{x+y}{2}|_{\alpha_0} =\ 1$. By Lemma 2.1,
$|\frac{x-y}{2}|_{\alpha_0} = |\frac{x+(-y)}{2}|_{\alpha_0} \leq\ \ 1 - \gamma_{\alpha_0,\ 2} = 0$
$\Rightarrow\ |x-y|_\alpha = 0$, which contradicts with our assumption.

Conversely let, $(X,\ Q)$ be strictly convex. Let $\alpha \in\ (0,\ 1)$, $x,\ y \in\ X$ such that $|x|_\alpha =\ 1,\ \ |y|_\alpha =\ 1$ and $|x-y|_\alpha = 2$. From definition

$$\gamma_{\alpha,\ 2} = \bigwedge_{x,\ y\in\ X} \{1 - |\frac{x+y}{2}|_\alpha :\ |x|_\alpha \leq\ 1,\ \ |y|_\alpha \leq\ 1,\ \ |x-y|_\alpha \geq\ 2\}.$$

Now we claim that $|x|_\alpha \leq\ 1,\ |y|_\alpha \leq\ 1$ and $|x-y|_\alpha = 2$ implies $|x+y|_\alpha = 0$, otherwise by the strictly convex property of $(X,\ Q)$,
$1 = |\frac{x-y}{2}|_\alpha = |\frac{x+(-y)}{2}|_\alpha < 1$, which is a contradiction. Hence $\gamma_{\alpha,2} = 1 \forall\alpha \in (0,1)$. □

Definition 4.5. A normed linear space $(X,\ ||.||)$ is strictly convex if for $x,\ y \in\ X$,

$$\left.\begin{array}{l}||x|| \leq\ 1\\ ||y|| \leq\ 1\\ ||x-y|| > 0\end{array}\right\} \Rightarrow\ ||\frac{x+y}{2}||\ <\ 1.$$

Theorem 4.6. (Taylor Foguel) *Let $(X,\ Q)$ be a generating space of semi-norm family and $\alpha \in (0,\ 1)$. Then every α-level bounded linear functional defined on a subspace of X has a unique norm-preserving linear extension to X iff X^*_α is strictly convex.*

Proof. Let X^*_α be strictly convex. Let Y be a subspace of X and $g \in Y^*_\alpha$. If possible let $f_1,\ f_2 \in X^*_\alpha$ be two distinct linear extension of g. Then $f_1 + f_2 \in X^*_\alpha$ and $(\frac{f_1+f_2}{2})(x) = g(x)\ \ \forall\, x \in Y$. Hence $|g|^s_\alpha \leq |\frac{f_1+f_2}{2}|^s_\alpha$. On the other hand
$|\frac{f_1+f_2}{2}|^s_\alpha \leq \frac{|f_1|^s_\alpha+|f_2|^s_\alpha}{2} = |g|^s_\alpha$.
Thus $|\frac{f_1+f_2}{2}|^s_\alpha = |g|^s_\alpha$.
Since X^*_α is a strictly convex normed linear space,

$$\left.\begin{array}{l} |f_1|^s_\alpha = |g|^s_\alpha \\ |f_2|^s_\alpha = |g|^s_\alpha \\ |f_1 - f_2|^s_\alpha > 0 \end{array}\right\} \Rightarrow |\frac{f_1+f_2}{2}|^s_\alpha < |g|^s_\alpha,$$

which is a contradiction.

Conversely let, X admits unique norm-preserving linear extension for every α-level bounded linear functional defined on a subspace of X. If possible let there exists $\alpha \in (0,\ 1)$ and $f_1,\ f_2 \in X^*_\alpha$ such that
$f_1 \neq f_2$ and $|f_1|^s_\alpha = |f_2|^s_\alpha = |\frac{f_1+f_2}{2}|^s_\alpha = 1$.
Let $Y = \{x \in X\ :\ f_1(x) = f_2(x)\}$, then Y is a subspace of X. Since $f_1 \neq f_2$, there exists $a \in X$ such that $f_1(a) = 1 \neq f_2(a)$.
Since $|\frac{f_1+f_2}{2}|^s_\alpha = 1$, there exists a sequence $\{x_n\}$ in X such that $|x_n|_{1-\alpha} = 1$ and $\lim\limits_{n\to\infty} |f_1(x_n) + f_2(x_n)| = 2$ Since $|f_1(x_n)| \leq 1,\ \ |f_2(x_n)| \leq 1\ \ \ \forall\, n \in N$, we have $\lim\limits_{n\to\infty} |f_1(x_n)| = 1$ and $\lim\limits_{n\to\infty} |f_2(x_n)| = 1$.
Let $k_n = \frac{(f_1(x_n)-f_2(x_n))}{(1-f_2(a))}\quad \forall\, n \in N$.
Then $\lim\limits_{n\to\infty} k_n = 0$ and we have
$1 = |x_n|_{1-\alpha} \leq |x_n - k_n a|_{1-\alpha} + |a k_n|_{1-\alpha}$
$\Rightarrow \lim\limits_{n\to\infty} |x_n - k_n a|_{1-\alpha} \geq 1$. So there exists a $n_0 \in N$ such that
$|x_n - k_n a|_{1-\alpha} > 0 \quad \forall\, n \geq n_0$.
$y_n = \frac{(x_n - k_n a)}{(|x_n - k_n a|_{1-\alpha})} \quad \forall\, n \geq n_0$.
Then $f_1(y_n) = \frac{(f_1(x_n) - k_n)}{(|x_n - k_n a|_{1-\alpha})}$
$= \frac{(f_2(x_n) - f_2(a) f_1(x_n))}{(1-f_2(a))(|x_n - k_n a|_{1-\alpha})}$
$= \frac{(f_2(x_n) - k_n f_2(a))}{(|x_n - k_n a|_{1-\alpha})} = f_2(y_n)$.
Thus $y_n \in Y$ and $|y_n|_{1-\alpha} = 1\ \ \forall\, n \geq n_0$. Also, since $\lim\limits_{n\to\infty} k_n = 0$, $\lim\limits_{n\to\infty} |f_1(x_n)| = 1$ and $\lim\limits_{n\to\infty} |f_2(x_n)| = 1$, we see that $\lim\limits_{n\to\infty} |f_1(y_n)| = 1$ and $\lim\limits_{n\to\infty} |f_2(y_n)| = 1$. Let $g \in Y^*_\alpha$. Then $|g|^s_\alpha = 1$ and g has two distinct norm-preserving linear extension from Y to X. Which is a contradiction. □

5. Conclusion

In this paper, definitions of uniformly convex and strictly convex generating spaces of semi-norm family (G.S.S-N.F) are given. The concept of uniform normal structure is introduced. Using the relation between strictly convex G.S.S-N.F and α dual space we have established Taylor Foguel's theorem in this space. Finally, as an application of Kirk's theorem, Browder type fixed point theorem is derived in G.S.S-N.F. Since the work has been done in this field are very few, we think that there is a large scope of developing more results of functional analysis in this context.

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