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Multi-fuzzy rough sets and relations

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Abstract. Pawlak's Rough set theory was originally proposed as a general mathematical tool for dealing with uncertainty in modeling imperfect knowledge. The purpose of this paper is to introduce the concept of multi-fuzzy rough sets by combining the multi-fuzzy set and rough set models. Some operations such as Complement, Union, Intersection etc. are defined for multi-fuzzy rough sets and De Morgan's laws are proved. Finally, the concept of multi-fuzzy rough relations and composition of multi-fuzzy rough relations are introduced and some properties of such relations are studied.

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1. Introduction

The concept of a set is fundamental for the whole of mathematics. Theory of fuzzy sets, theory of rough sets, theory of multisets, theory of multi-fuzzy sets are some of the popular generalizations of classical set theory. Among these Zadeh's [19] fuzzy sets are extensively studied by a number of researchers all over the world and within the last forty five years, the theory of fuzzy sets has been applied to almost all the branches of mathematics. Many extensions and generalizations of Zadeh's [19] fuzzy set theory were developed, a few among those are intuitionistic fuzzy sets, L-fuzzy set, rough fuzzy sets, fuzzy rough sets and multisets ([1],[2],[3],[4],[5], [18]).

Rough set theory is a new mathematical approach to imperfect knowledge. It is a new tool to deal with partial information. Rough set theory was initiated by Pawlak ([7],[8],[9]) based on equivalence relations for dealing with vagueness and granularity in information. Many generalization of Pawlak rough sets were proposed and all of them deal with approximations of concepts in terms of granules. This theory deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, metrology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields. Rough set theory overlaps with many other theories. By using an equivalence relation on U, one can introduce lower and upper approximations in fuzzy set theory to obtain an extended notion called rough fuzzy set. Alternatively, a fuzzy similarity relation can be used to replace an equivalence relation. The result is a deviation of rough set theory called fuzzy rough sets. Theory of multi-fuzzy sets [13] is a generalization of theories of fuzzy sets, L-fuzzy sets and intuitionistic fuzzy sets. The notion of multi-fuzzy sets provides a new method to represent some problems, which are difficult to explain in other extensions of fuzzy set theory. A detailed account of studies on multi-fuzzy sets can be seen in ([10],[11],[12],[14],[15],[16],[17]).

In this paper we combine the concepts of multi-fuzzy sets and rough sets to obtain a new hybrid intelligent structure called multi-fuzzy rough sets. The significance of introducing hybrid set structures is that the computational techniques based on any one of these structures alone will not always yield the best results but a fusion of two or more of them can often give better results. This is the motivation for introducing multi-fuzzy rough sets, which is a combination of multi-fuzzy sets and rough sets. In section 2, basic definitions and notions regarding multi-fuzzy sets and fuzzy rough sets are collected. Multi-fuzzy rough sets are introduced in section 3 and various identities regarding them including the De Morgan laws are proved. Multi-fuzzy rough relations, their compositions and various properties are studied in section 4.

2. Preliminaries

Definition 2.1 ([1])**.** Let X be a non empty set and L be a partially ordered set. A mapping A: X→ L is refered as an L-fuzzy set. The family of all L-fuzzy sets on X will be denoted by L^X.

Definition 2.2 ([13])**.** Let X be a nonempty set, N the set of all natural numbers and $\{L_i$: i ∈ N$\}$ a family of complete lattices. A multi-fuzzy set A in X is a set of ordered sequences,A = $\{< x, \mu_1(x), \mu_2(x)... \mu_i(x), ...> : x \in X\}$ where $\mu_i \in {L_i}^X$, for i ∈ N.The family of all multi-fuzzy sets in a non empty set X with value domains $\{L_i$:I ∈ N$\}$ is denoted by $MF({L_i}^X)$.

Definition 2.3 ([13])**.** Let {Li: i ∈ N} be a family of complete lattices;

$$A = \{< x, {\mu_1}^A(x), {\mu_2}^A(x)...{\mu_i}^A(x)... >: x \in X, {\mu_i}^A \in {L_i}^X, i \in N\},$$

$$B = \{< x, {\mu_1}^B(x), {\mu_2}^B(x), ..., {\mu_i}^B(x), ... >: x \in X, {\mu_i}^B \in {L_i}^X, i \in N\}$$

be multi-fuzzy sets in a nonempty set X. The following relations and operations are defined:

(a) A ⊆ B if and only if ${\mu_i}^A(x) \le {\mu_i}^B(x)$,∀x ∈ X and for all i ∈ N;

(b) A = B if and only if ${\mu_i}^A(x) = {\mu_i}^B(x)$,∀ x ∈ X and for all i ∈ N;

(c) A∪B = $\{<x, {\mu_1}^A(x) \vee {\mu_1}^B(x), {\mu_2}^A(x) \vee {\mu_2}^B(x)... {\mu_i}^A(x) \vee {\mu_i}^B(x) ...>: x \in X\}$;

(d) A ∩ B = $\{<x, {\mu_1}^A(x) \wedge {\mu_1}^B(x), {\mu_2}^A(x) \wedge {\mu_2}^B(x)... {\mu_i}^A(x) \wedge {\mu_i}^B(x) ...>: x \in X\}$.

In general if L_i's are complete lattices, then for a family $\{A_\lambda\}$ of multi-fuzzy sets we have

$E = \bigcup_\lambda A_\lambda$ iff ${\mu_i}^E(x) = \sup_\lambda \mu_i^{A_\lambda}(x)$ ∀x∈X and i∈N

$F = \bigcap_\lambda A_\lambda$ iff ${\mu_i}^F(x) = \inf_\lambda \mu_i^{A_\lambda}(x)$ ∀x∈X and i∈N

In particular if $\{L_i\}$ are all complemented lattices the complement of a multi-fuzzy set A^c is defined by $\mu_i^{A^c}(x) = 1 - \mu_i^A(x)$; $1 - \mu_i^A(x)$ is the complement of ${\mu_i}^A(x)$ in L_i. The family of all multi-fuzzy sets in a non empty set X with value domains $\{L_i$:i∈N$\}$ is denoted by $MF({L_i}^X)$.

Definition 2.4 ([6])**.** Let U be a non empty set. Let B be complete sub algebra of the Boolean algebra P (U) of power set of U. The pair (U, B) is called a rough universe.

Definition 2.5 ([6])**.** Let V= (U, B) be a given fixed rough universe, let R be the relation defined by: A = $(A_L, A_U) \in$ R if and only if A_L as well as $A_U \in$ B and A_L is a subset of A_U. The elements of R are called rough sets and elements of B are called exact sets. Let A = (A_L, A_U) and let B = (B_L, B_U) be any two rough sets. Then
$A \cup B = (A_L \cup B_L, A_U \cup B_U)$
$A \cap B = (A_L \cap B_L, A_U \cap B_U)$ A ⊂ B if and only if A ∩ B = A
Thus A ⊂ B iff $A_L \subset B_L$, and $A_U \subset B_U$

Definition 2.6 ([6])**.** Let U be a set and B a Boolean sub algebra of the Boolean algebra of all subsets of U. Let L be a lattice. Let X be a rough set .Then
$X = (X_L, X_U) \in B^2$ with $X_L \subset X_U$.
A fuzzy rough set A= (A_L, A_U) in X is characterized by a pair of maps
$\mu_{AL} : X_L \to L$ and $\mu_{AU}:X_U \to L$ with the property that $\mu_{AL}(x) \leq \mu_{AU}(x)$ for all x∈ X_L.
For any two fuzzy rough sets A = (A_L, A_U) and B= (B_L, B_U) in X we define
(1) A = B iff $\mu_{AL}(x) = \mu_{BL}(x)$ for all x∈ X_L
and $\mu_{AU}(x) = \mu_{BU}(x)$ for all x∈ X_U
(2) A ⊂ B iff $\mu_{AL}(x) \leq \mu_{BL}(x)$ for all x∈ X_L
and $\mu_{AU}(x) \leq \mu_{BU}(x)$ for all x∈X_U
(3) C = A ∪ B iff $\mu_{CL}(x) = \max[\mu_{AL}(x), \mu_{BL}(x)]$ for all x∈X_L
and $\mu_{CU}(x) = \max[\mu_{AU}(x), \mu_{BU}(x)]$ for all x∈ X_U
(4) D = A ∩ B iff $\mu_{DL}(x) = \min[\mu_{AL}(x), \mu_{BL}(x)]$ for all x∈ X_L
and $\mu_{DU}(x) = \min[\mu_{AU}(x), \mu_{BU}(x)]$ for all x∈ X_U
More generally, if L is a complete lattice, then for any index set I, if $\{A_i$: i∈I$\}$ is a family of fuzzy rough sets we have
$E = \bigcup_{i\in I} A_i$ iff $\mu_{EL}(x) = \sup_{i\in I} \mu_{ALi}(x)$ for all x∈ X_L and $\mu_{EU}(x) = \sup_{i\in I} \mu_{AUi}(x)$ for all x∈ X_U similarly,
$F = \bigcap_{i\in I} A_i$ iff $\mu_{FL}(x) = \inf_{i\in I} \mu_{ALi}(x)$ for all x∈ X_L and $\mu_{FU}(x) = \inf_{i\in I} \mu_{AUi}(x)$ for all x∈ X_U.
We define the complement A^c of A by the ordered pair $({A_L}^c, {A_U}^c)$ of membership functions where $\mu_{AL^c}(x) = 1 - \mu_{AL}(x)$ for all x∈ X_L and $\mu_{AU^c}(x) = 1 - \mu_{AU}(x)$ for all x∈ X_U.

3. Multi-fuzzy rough sets

Definition 3.1. Let U be a set and B be Boolean sub algebra of the Boolean algebra of all subsets of U. Let {L_i :i∈N} be a collection of lattices. Let X be a rough set. Then X= (X_L,X_U) ∈ B^2 with $X_L \subset X_U$
A multi fuzzy rough set A in X is a pair A = (A_L, A_U) where A_L and A_U are multi fuzzy subsets of X. A is characterized by a pair of maps,

$$\mu_i^{AL} : X_L \to L_i, i \in N$$

$$\mu_i^{AU} : X_U \to L_i, i \in N$$

with the property that $\mu_i^{AL}(x) \leq \mu_i^{AU}(x)$ for all x ∈ X_L.
For any two multi fuzzy rough sets A = (A_L, A_U) and B= (B_L, B_U) in X we define
(1) A = B iff $\mu_i^{AL}(x) = \mu_i^{BL}$ (x) for all x∈ $X_L$and , i ∈ N
and μ_i^{AU} (x) = μ_i^{BL}(x) for all x∈ X_U, i ∈ N
(2) A ⊂ B iff $\mu_i^{AL}(x) \leq \mu_i^{BL}$ for all x∈ X_L, i ∈ N
and $\mu_i^{AU}(x) \leq \mu_i^{BU}$ for all x∈ X_U, i ∈ N
(3) C = A ∪ B iff $\mu_i^{CL}(x) = \max[\mu_i^{AL}(x), \mu_i^{BL}(x)]$ for each x∈ X_L, i ∈ N
and $\mu_i^{CU}(x) = \max[\mu_i^{AU}(x), \mu_i^{BU}(x)]$ for each x∈ X_U, i ∈ N
(4) D = A∩ B iff $\mu_i^{DL}(x) = \min[\mu_i^{AL}(x), \mu_i^{BL}(x)]$ for each x∈ X_L, i ∈ N
and $\mu_i^{DU}(x) = \min[\mu_i^{AU}(x), \mu_i^{BU}(x)]$ for each x∈ X_U, i ∈ N
More generally, if L is a complete lattice, then for any index set I, if {A_i :i∈I} is a family of fuzzy rough sets we have
$E = \bigcup_{i\in I} A_i$ iff $\mu_i^{EL}(x) = \sup_{i\in I} \mu_i^{AL}(x)$ for all x∈ X_L, i ∈ N and $\mu_i^{EU}(x) = \sup_{i\in I} \mu_i^{AU}(x)$ for all x∈ X_U, i ∈ N
Similarly,
$F = \bigcap_{i\in I} A_i$ iff $\mu_i^{FL}(x) = \inf_{i\in I} \mu_i^{AL}(x)$ for all x∈ X_L,i ∈ N and $\mu_i^{FU}(x) = \inf_{i\in I} \mu_i^{AU}(x)$ for all x∈ X_U, i ∈ N.
We define the complement A^c of A by the ordered pair (${A_L}^c$, ${A_U}^c$) of membership functions where $\mu_i^{AL^c}$ (x)=1-$\mu_i^{AL}(x)$ for all x ∈ X_L , i ∈ N and $\mu_i^{AU^c}(x) = 1 - \mu_i^{AU}(x)$ for all x ∈ X_U , i ∈ N.

Lemma 3.2. *If A, B, C are multi fuzzy rough sets, then*
(1) $A \cap A = A, \quad A \cup A = A$
(2) $A \cup B = B \cup A, \quad A \cap B = B \cap A$
(3) $(A \cap B) \cap C = A \cap (B \cap C), \quad (A \cup B) \cup C = A \cup (B \cup C),$
(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \quad (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Proof. : We will show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,
Case 1: $\mu_i^{CL}(x) \leq (\mu_i^{AL}(x), \mu_i^{BL}(x)) \qquad \forall\ i \in N\ , \forall x \in X_L$
Consider membership value of L.H.S

$$\begin{aligned} \mu_i^{[(A\bigcap B)\bigcup C]L}(x) &= \max(\mu_i^{(A\bigcap B)L}(x), \mu_i^{CL}(x)) \\ &= \max\{\min\{\mu_i^{AL}(x), \mu_i^{BL}(x)\}, \mu_i^{CL}(x)\} \end{aligned}$$

$$(3.1) \qquad = \min\{\mu_i^{AL}(x), \mu_i^{BL}(x)\} \forall i \ldots\ldots$$

Consider R.H.S $= \min\{\max(\mu_i^{AL}(x), \mu_i^{CL}(x)), \max(\mu_i^{BL}(x), \mu_i^{CL}(x)\}$
(3.2)
$= \min\{\mu_i^{AL}(x), \mu_i^{BL}(x)\}$ ....
From (3.1) and (3.2) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
Case 2: $\mu_i^{CL}(x) \geq (\mu_i^{AL}(x), \mu_i^{BL}(x)) \quad \forall\ i \in N\ , \forall x \in X_L$

$$\begin{aligned} \mu_i^{[(A \cap B) \cup C]L}(x) &= \max[\mu_i^{(A \cap B)L}(x), \mu_i^{CL}(x)] \\ &= \max\{\min\{\mu_i^{AL}(x), \mu_i^{BL}(x)\}, \mu_i^{CL}(x)\} \end{aligned}$$

(3.3)
$= \mu_i^{CL}(x)$ ....
Consider R.H.S

$$\begin{aligned} &= \min\{\max(\mu_i^{AL}(x), \mu_i^{CL}(x)), \max(\mu_i^{BL}(x), \mu_i^{CL}(x)\} \\ &= \min\{\mu_i^{CL}(x), \mu_i^{CL}(x)\} \end{aligned}$$

(3.4)
$= \mu_i^{CL}(x)$ ....
From (3.3) and (3.4) , L.H.S = R.H.S
Case 3 : $\mu_i^{AL}(x) \leq \mu_i^{CL}(x) \leq \mu_i^{BL}(x)]\ i \in N\ , \forall x \in X_L$

$$\begin{aligned} \mu_i^{[(A \cap B) \cup C]L}(x) &= \max[\mu_i^{(A \cap B)L}(x), \mu_i^{CL}(x)] \\ &= \max\{\min\{\mu_i^{AL}(x), \mu_i^{BL}(x)\}, \mu_i^{CL}(x)\} \\ &= \min\{\mu_i^{CL}(x), \mu_i^{CL}(x)\} \end{aligned}$$

(3.5)
$= \mu_i^{CL}(x)$ ....
Consider R.H.S

$$\begin{aligned} &= \min\{\max(\mu_i^{AL}(x), \mu_i^{CL}(x)), \max(\mu_i^{BL}(x), \mu_i^{CL}(x)\} \\ &= \min\{\mu_i^{CL}(x), \mu_i^{BL}(x)\} \end{aligned}$$

(3.6)
$= \mu_i^{CL}(x)$ ....
From (3.5) and (3.6), L.H.S = R.H.S
Case 4 : $\mu_i^{BL}(x) \leq \mu_i^{CL}(x) \leq \mu_i^{AL}(x)]\ i \in N\ , \forall x \in X_L$
Proof of Case 4 is similar to Case 3.
Similarly we can do for upper approximation.
Thus $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ □

Theorem 3.3. *(De Morgan's Law) Let A and B be two multi-fuzzy rough sets where the value domains $\{L_i\}$ are complemented lattices then*
(1) $(A \cup B)^c = A^c \cap B^c$
(2) $(A \cap B)^c = A^c \cup B^c$

Proof. (1) Let A and B be multi-fuzzy rough sets with ordered pairs of membership functions (μ_i^{AL}, μ_i^{AU}) and (μ_i^{BL}, μ_i^{BU})respectively. Then

$$\mu_i^{((A \bigcup B)U)^c}(x) = 1 - \mu_i^{(A \bigcup B)U}(x) = 1 - \max(\mu_i^{AU}(x), \mu_i^{BU}(x)) i \in N, \forall x \in X_U$$

$$\mu_i^{(A^c \bigcap B^c)U}(x) = \min(\mu_i^{A^c U}(x), \mu_i^{B^c U}(x) = \min\{1 - \mu_i^{AU}(x), 1 - \mu_i^{BU}(x)\}$$

Therefore

$$\mu_i^{((A \bigcup B)U)^c}(x) = \mu_i^{(A^c \bigcap B^c)U}(x) i \in N, \forall x \in X_U$$

Similarly

$$\mu_i^{((A \bigcup B)L)^c}(x) = \mu_i^{(A^c \bigcap B^c)L}(x) i \in N, \forall x \in X_L$$

And this completes the proof of (1) similarly (2) □

Remark 3.4. *If* $\{L_i\}^{,s}$ *are complete complemented lattices then for any collection* $\{A_{\ \lambda}\}$ *of multi-fuzzy rough sets*
(1) $(\bigcup_\lambda A_\lambda)^c = \bigcap_\lambda (A_\lambda)^c$
(2) $(\bigcap_\lambda A_\lambda)^c = \bigcup_\lambda (A_\lambda)^c$

4. Multi-fuzzy rough relations

Definition 4.1. A multi-fuzzy relation is a sub multiset of the Cartesian product. A multi-fuzzy relation R: V to W is a multi fuzzy set of the form $R = \{< (x, y), \mu_1(x, y), \mu_2(x, y), ...\mu_k(x, y).. >: x \in V$ and $y \in W\}$ here $\mu_i \in L_i^{V \times W}$.

Definition 4.2. Let X = (X_L, X_U) and Y = (Y_L, Y_U) be two rough sets, multi-fuzzy rough relation R = $(R_L,\ R_U)$ from X to Y is a multi-fuzzy rough set in X × Y characterized by the two multi-fuzzy relations

$$R_L : X_L \to Y_L$$

$$R_U : X_U \to Y_U$$

with $R_L \subseteq R_U$, a multi-fuzzy rough relation from X to Y will be denoted by R(X → Y).

Example 4.3. Let $U = \{x, y, z\}$, $B = P(U)$, $X = (X_L, X_U), Y = (Y_L, Y_U)$ are rough sets in U defined as $X_L = \{x\}, X_U = \{x, y\}, Y_L = \{z\}$ and $Y_U = \{y, z\}$. Take $i = 2$ with $L_1 = L_2 = [0, 1]$ and $\mu_1, \mu_2 : X\text{x}Y \to [0, 1]$.A multi-fuzzy rough relation $R = (R_L,\ R_U)$ from X to Y may be defined as follows:
$R_L : X_L \to Y_L, R_U : X_U \to Y_U$ with $R_L = \{((x, z), 0.3, 0.2)\}$ and
$R_U = \{((x, y), 0.3, 0.1), ((x, z), 0.4, 0.3), ((y, y), 0.5, 0.8), ((y, z), 0.4, 0.2)\}$

Definition 4.4. If A = $(\text{A}_L,\ \text{A}_U)$ is a multi-fuzzy rough set, $R = (R_L, R_U)$ is a multi-fuzzy rough relation from X to Y, the composition of R with A is a multi-fuzzy rough set in Y denoted by B = R ∘A , and is defined by

$$(R \circ A)_L(y) = \bigvee_{x \in X_L} (A_i^L(X) \wedge \text{R}_\text{i}^L(x, y)) \quad for\ any\ y \in Y_L$$

$$(R \circ A)_U(y) = \bigvee_{x \in X_U} (A_i^U(X) \wedge \text{R}_\text{i}^U(x, y)) \quad for\ any\ y \in Y_U$$

Definition 4.5. Let R (X → Y) and Q (Y → Z) be two multi-fuzzy relation, the composition R ∘Q is a multi-fuzzy relation from X to Z, defined by the membership function

$$(R \circ Q)_L(x, z) = \bigvee_{y \in Y_L} (\mathrm{R}_L(x, y) \wedge Q_L(y, z)) \quad \forall x \in X_L, z \in Z_L$$

$$(R \circ Q)_U(x, z) = \bigvee_{y \in Y_U} (\mathrm{R}_U(x, y) \wedge Q_U(y, z)) \quad \forall x \in X_U, z \in Z_U$$

Properties of Multi-fuzzy Rough Relations

The composition of multi-fuzzy rough relation has following important properties:

Theorem 4.6. *Let R (U → V), S (V → W), Q (V → W), P (W → Z) be four multi-fuzzy rough relations, then*

(1) $(R \circ S) \circ P = R \circ (S \circ P)$

(2) $R \circ (S \cup Q) = (R \circ S) \cup (R \circ Q)$, $(S \cup Q) \circ P = (S \circ P) \cup (Q \circ P)$,

(3) $R \circ (S \cap Q) \subseteq (R \circ S) \cap (R \circ Q)$, $(S \cap Q) \circ P \sqsubseteq (S \circ P) \cap (Q \circ P)$,

(4) $S \subseteq Q \Longrightarrow R \circ S \subseteq R \circ Q$ *and* $S \circ P \subseteq Q \circ P$

(5) $(R \circ S)^T = S^T \circ R$, $(R^T)^T = R$

Where $R^T (Y \longrightarrow X)$ *is called the reverse of* $R(Y \longrightarrow X)$, *defined by*

$$R_L^T(y, x) = R_L(y, x), R_U^T(y, x) = R_U(y, x)$$

(6) $(R \circ S)_\lambda = R_\lambda \circ S_\lambda$, $0 \leq \lambda \leq 1$. *Where* R_λ *is called the cut relation of R, defined by*

$$(R_\lambda)_L = \{(x, y)/R_L(x, y) \geq \lambda, x \in X_L, y \in Y_L$$

$$(R_\lambda)_U = \{(x, y)/R_U(x, y) \geq \lambda, x \in X_U, y \in Y_U$$

Proof. 1) For any u ∈ U_L, z ∈ Z_L,

$$((R \circ S) \circ P_L(u, z) = \bigvee_{w \in W_L} [(R \circ S)_L(u, w) \wedge P_L(w, z)]$$

$$\begin{aligned}
&= \bigvee_{w \in W_L} [(\bigvee_{v \in V_L} (R_L(u, v) \wedge S_L(v, w))) \wedge P_L(w, z)] \\
&= \bigvee_{w \in W_L} \bigvee_{v \in V_L} [(R_L(u, v) \wedge S_L(v, w) \wedge P_L(w, z)] \\
&= \bigvee_{v \in V_L} \bigvee_{w \in W_L} [(R_L(u, v) \wedge S_L(v, w) \wedge P_L(w, z)] \\
&= \bigvee_{v \in V_L} [(R_L(u, v) \wedge \left(\bigvee_{w \in W_L} S_L(v, w) \right) \wedge P_L(w, z)] \\
&= \bigvee_{v \in V_L} [(R_L(u, v) \wedge (S \circ P)_L(v, z)] \\
&= (R \circ (S \circ P))_L(u, z)]
\end{aligned}$$

Therefore $((R \circ S) \circ P)_L = (R \circ (S \circ P))_L$

Similarly $((R \circ S) \circ P)_U = (R \circ (S\ P))_U$.

2) For any u ∈U_L, w ∈ W_L

(R ∘(S ∩ Q)) $_L$(u, w)

$$= [R_L(u,v) \wedge (S \cup Q)_L(v,w)]$$
$$= \bigvee_{v \in V_L} [(R_L(u,v) \wedge (S_L(v,w) \vee Q_L(v,w))]$$
$$= \bigvee_{v \in V_L} [(R_L(u,v) \wedge S_L(v,w)) \vee (R_L(u,v) \wedge Q_L(v,w))]$$
$$= \left\{ \bigvee_{v \in V_L} [R_L(u,v) \wedge S_L(v,w)] \right\} \vee \left\{ \bigvee_{v \in V_L} [R_L(u,v) \wedge Q_L(v,w)] \right\}$$

= (R ∘S) $_L$ (u, w) ∨ (R ∘Q)$_L$ (u, w) = (R ∘S) ∪ (R ∘Q))$_L$ (u, w)
Then R ∘(S ∩ Q)) $_L$=((R ∘S) ∪ (R ∘Q))$_L$.
Similarly R ∘(S ∩ Q)) $_U$= ((R ∘S) ∪ (R ∘Q))$_U$
Therefore, R ∘(S ∪ Q) = (R ∘S) ∪ (R ∘Q),
Similarly 3), 4), 5) can be proved.

6) $\forall (u,w) \in ((R \circ S)_\lambda)_L \Rightarrow (R \circ S)_L(u,w) \geq \lambda$

$$i.e \quad \bigvee_{v \in V_L} (R_L(u,v) \wedge (S_L(v,w)) \geq \lambda \Rightarrow \exists\, v_1 \in V_L \quad \text{such that}$$

$R_L(u,v_1) \wedge S_L(v_1,w)) \geq \lambda \Rightarrow R_L(u,v_1) \geq \lambda$ *and* $S_L(v_1,w) \geq \lambda \Rightarrow (u,v_1) \in (R_\lambda)_L$
and $(v_1,w) \in (R_\lambda)_L \Rightarrow (u,w) \in (R_\lambda \circ S_\lambda)_L$
Then ((R ∘ S) $_\lambda$) $_L$⊆ (R $_\lambda$∘S $_\lambda$)$_L$.
Similarly ((R ∘ S) $_\lambda$) $_U$⊆ (R $_\lambda$∘S $_\lambda$)$_U$
So, (R ∘ S) $_\lambda$= R $_\lambda$∘S $_\lambda$ □

5. Conclusion

In this paper, the hybrid intelligent structure multi fuzzy rough sets are obtained by fusing the concepts of multi fuzzy sets [10] and fuzzy rough sets [6]. Besides giving the basic definitions, some identities including De Morgan type laws are proved. Further, Relations and Compositions in this context are also investigated. The introduced hybrid structure has the nice properties of both constituents and hence can be applied to many application problems in a more effective manner than that can be achieved by any one of the structures alone.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] W. D Blizard, Multiset theory, Notre Dame J. Formal Logic 30(1) (1989) 36–66.
[3] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (1990) 191–208.
[4] J. A Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[5] S. Miyamoto, Fuzzy multisets and their generalizations, Fuzzy Sets and Systems 10 (1998) 157–167.
[6] S. Nanda and S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems 45 (1992) 157–160.
[7] Z. Pawlak, Rough sets, International Journal of Information and Computer Sciences 11 (1982) 341–356.
[8] Z. Pawlak, Rough sets and fuzzy sets, Fuzzy Sets and Systems 17 (1985) 99–102.

[9] Z. Pawlak, Rough sets, Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers Dordrecht 1991.
[10] T. V Ramakrishnan and Sabu Sebastian, A study on multi-fuzzy sets, Int. J. Appl. Math. 23(4) (2010) 713–721.
[11] Sabu Sebastian, A study on multi-fuzziness, Ph. D. Dissertation, Kannur University. 2011.
[12] Sabu Sebastian and T. V Ramakrishnan, Multi-fuzzy sets, Int. Math. Forum. 5(50) (2010) 2471–2476.
[13] Sabu Sebastian and T. V Ramakrishnan, Multi-fuzzy sets: An extension of fuzzy sets, Fuzzy Inf. Eng. 3(1) (2011) 35–43.
[14] Sabu Sebastian and T. V. Ramakrishnan, Multi-fuzzy subgroups, Int. J. Contemp. Math. Sci. 6(8) (2011) 365–372.
[15] Sabu Sebastian and T. V. Ramakrishnan, Multi-fuzzy topology, Int. J. Appl. Math. 24(1) (2011) 117–129.
[16] Sabu Sebastian and T. V. Ramakrishnan, Multi-fuzzy extensions of functions, Adv. Adapt. Data Anal. 3(3) (2011) 339–350.
[17] Sabu Sebastian and T. V. Ramakrishnan, Multi-fuzzy extension of crisp functions using bridge functions, Ann. Fuzzy Math. Inform. 2(1) (2011) 1–8.
[18] SHU Lan, Fuzzy rough relation and its properties, Journal of System Science and Systems Engineering 11 (2002) 367–370.
[19] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–352.

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