Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 5, (May 2014), pp. 773–789
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

D-proximity structures and D-ideals

A. Kandil, S. A. El-Sheikh, M. M. Yakout, Shawqi A. Hazza

Received 7 April 2012; Accepted 14 July 2013

Abstract. In this paper, we present a new approach to double (intuitionistic) proximity(D-Proximity, for short) structures based on the recognition that many of the entities important in the theory of double (intuitionistic) ideals(D-ideal, for short). So we given a characterization of the basic double proximity using double ideals. Also, we introduce the concept of f- double proximities and we show that for different choice of "f" one can obtain many of the the known types of double proximities. Also, characterizations of some types of these double proximities-(f_0, l_0)- have obtained.

2010 AMS Classification: 06D72, 54A40

Keywords: Ideals, Basic proximity, Proximity space, D-ideals, Basic D-proximity, D-proximity space, f- D-proximities, D-topological space, Intuitionistic (Flou) topological spaces.

Corresponding Author: Shawqi Ahmed Hazza (shawqialalimi@yahoo.com)

1. Introduction

Ideals in topological spaces were introduced by Kuratowski [16], Vaidyanathaswamy [20] and Jankovic and Hamlett[10]. In the crisp, various classes of generalized proximities have been extensively studied by many authors including Lodato[17], [18]. In[9], the authors were introduced a new approach to construct generalized proximity structures based on the concept of ideal and an EF-Proximity structure. Kandil et. al [11] presented a new approach to proximity structures using the theory of ideals. After Atanassov [1, 2, 3] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Çoker [5] introduced the notion of intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. The notion of intuitionistic sets which is a classical version of an intuitionistic fuzzy sets was first given by Çoker in [4]. He studied topology on intuitionistic sets in [6]. The authors in [15] introduced the concept of flou set and studied the basic properties of flou topological spaces. In this paper, we follow the suggestion of Rodabaugh [8] that double set is more appropriate name than intuitionistic (Flou) set, and therefore, adopted the term double set for the intuitionistic(Flou) set and double topology for the intuitionistic (Flou) topology. Also, The authors in [12] obtained a new double topology form the old by using a double ideal. In this paper, we present a new approach to double proximity structures based on the recognition that many of the entities important in the theory of double ideals, and study some of its properties. The concepts of a basic double proximity on a double set and a basic proximal neighborhood of a double set with respect to a basic double proximity are obtained. Also, we introduce the concept of f- double proximities and we show that for different choice of "f" one can obtain many types of double proximities.

2. Preliminaries

Definition 2.1 ([15])**.** Let X be a nonempty set:

(1) A double set $\underline{A}$ is an ordered pair $\underline{A} = (A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.

(2) $D(X) = \{(A_1, A_2) : (A_1, A_2) \in P(X) \times P(X),\ A_1 \subseteq A_2\}$ is the family of all double sets on X.

(3) Let $x \in X$. Then the double sets $x_{0.5} = (\phi, \{x\})$ and $x_1 = (\{x\}, \{x\})$ are said to be double points in X.
$X_p = \{x_t : x \in X,\ t \in \{0.5, 1\}\}$ is the set of all double points of X.

(4) $x_1 \in \underline{A}$ iff $x \in A_1$, and $x_{0.5} \in \underline{A}$ iff $x \in A_2$.

(5) Let $\eta_1, \eta_2 \subseteq P(X)$. Then the double product of η_1 and η_2 is denoted by $\eta_1 \hat{\times} \eta_2$ and is defined by $\eta_1 \hat{\times} \eta_2 = \{(A_1, A_2) : (A_1, A_2) \in \eta_1 \times \eta_2,\ A_1 \subseteq A_2\}$.

(6) The double set $\underline{X} = (X, X)$ is called the universal double set.

(7) The double set $\underline{\phi} = (\phi, \phi)$ is called the empty double set.

(8) The double set $\underline{A} = (A_1, A_2)$ is said to be finite double set if A_2 is finite set.

(9) The double set $\underline{A} = (A_1, A_2)$ is said to be countable double set if A_2 is countable set.

Note that a double set in the sense of Çoker [4] is of the form $\underline{A} = (A_1, A_2) \in P(X) \times P(X)$, where $A_1 \cap A_2 = \phi$. But $\underline{A} = (A_1, A_2) \in P(X) \times P(X)$ is a double set in the sense of Kandil et. al [15], where $A_1 \subseteq A_2$. Then $\underline{A} = (A_1, A_2)$ is a double set in the sense of Çoker if and only if $\underline{A} = (A_1, A_2^c)$ is a double set in the sense of Kandil. And one can see that a one to one correspondence mapping between the two types. On the other hand, Kandil's notion simplify the concepts, specially in the case of intuitionistic fuzzy points or double fuzzy point see [19].

Definition 2.2 ([15])**.** Let $\underline{A} = (A_1, A_2)$, $\underline{B} = (B_1, B_2) \in D(X)$. Then

(1) $\underline{A} = \underline{B} \Leftrightarrow A_i = B_i,\ i = 1, 2$.

(2) $\underline{A} \subseteq \underline{B} \Leftrightarrow A_i \subseteq B_i,\ i = 1, 2$.

(3) $\underline{A} \cap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$ and $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.

(4) If $\{\underline{A}_\alpha : \alpha \in \Lambda\} \subseteq D(X)$ such that $\underline{A}_\alpha = (A_{1_\alpha}, A_{2_\alpha})$, then $\bigcup_{\alpha\in\Lambda} \underline{A}_\alpha = (\bigcup_{\alpha\in\Lambda} A_{1_\alpha}, \bigcup_{\alpha\in\Lambda} A_{2_\alpha})$ and $\bigcap_{\alpha\in\Lambda} \underline{A}_\alpha = (\bigcap_{\alpha\in\Lambda} A_{1_\alpha}, \bigcap_{\alpha\in\Lambda} A_{2_\alpha})$.

(5) $\underline{A}^c = (A_2^c, A_1^c)$, where $\underline{A}^c$ is the complement of $\underline{A}$.

(6) $\underline{A} \backslash \underline{B} = \underline{A} \cap \underline{B}^c$.

Remark 2.3 ([13])**.** We have here three major deviations from the ordinary case:

(1) $\underline{A} \cup \underline{A}^c$ need not to be the universal double set.
(2) $\underline{A} \cap \underline{A}^c$ need not to be the empty double set.
(3) $x_t \in \underline{A}$ does not implies $x_t \notin \underline{A}^c$, in general.

Definition 2.4 ([13]). Two double sets $\underline{A}$ and $\underline{B}$ are said to be quasi-coincident, denoted by $\underline{A}q\underline{B}$, if and only if $A_1 \cap B_2 \neq \phi$ or $A_2 \cap B_1 \neq \phi$. $\underline{A}$ is not quasi-coincident with $\underline{B}$, denoted by $\underline{A}\bar{q}\underline{B}$, if and only if $A_1 \cap B_2 = \phi$ and $A_2 \cap B_1 = \phi$.

Theorem 2.5 ([13]). *Let $\underline{A}, \underline{B}, \underline{C} \in D(X)$ and $x_t, y_r \in X_p$. Then:*
(1) $\underline{A}q\underline{B} \Rightarrow \underline{A} \cap \underline{B} \neq \underline{\phi}$,
(2) $\underline{A}q\underline{B} \Leftrightarrow \exists x_t \in \underline{A}$ *such that* $x_t q\underline{B}$,
(3) $\underline{A}\bar{q}\underline{B} \Leftrightarrow \underline{A} \subseteq \underline{B}^c$,
(4) $x_t\bar{q}\underline{A} \Leftrightarrow x_t \in \underline{A}^c$,
(5) $\underline{A} \subseteq \underline{B} \Leftrightarrow x_t \in \underline{A}$ *implies* $x_t \in \underline{B} \Leftrightarrow x_t q\underline{A}$ *implies* $x_t q\underline{B}$,
(6) $\underline{A}\bar{q}\underline{A}^c$.
(7) $\underline{A} = \bigcup\{x_t : x_t \in \underline{A}\} = \bigcup\{x_t : x_t\bar{q}\underline{A}^c\}$.

Definition 2.6 ([15]). Let X be a nonempty set. Then, $\eta \subseteq D(X)$ is called a double topology on X if the following axioms are satisfied:
(i) $\underline{\phi}$, $\underline{X} \in \eta$,
(ii) If $\underline{A}$, $\underline{B} \in \eta$, then $\underline{A} \cap \underline{B} \in \eta$, and
(iii) If $\{\underline{A}_\alpha : \alpha \in \Lambda\} \subseteq \eta$, then $\cup_{\alpha\in\Lambda}\underline{A}_\alpha \in \eta$.

Definition 2.7 ([13]). Let X be a nonempty set. A nonempty collection $\underline{\mathcal{I}} \subseteq D(X)$ is said to be a double ideal(D-ideal, for short) on X, if it satisfies the following two conditions:
(i) $\underline{A} \in \underline{\mathcal{I}}$ and $\underline{B} \subseteq \underline{A} \Rightarrow \underline{B} \in \underline{\mathcal{I}}$ (hereditary),
(ii) $\underline{A} \in \underline{\mathcal{I}}$ and $\underline{B} \in \underline{\mathcal{I}} \Rightarrow \underline{A} \cup \underline{B} \in \underline{\mathcal{I}}$ (finite additivity).
The set of all D-ideals on X is denoted by $DI(X)$.

One of the important D-ideals is $\underline{\mathcal{I}}_{\underline{A}}(= \{\underline{B} : \underline{B} \in D(X), \underline{B} \subseteq \underline{A}\})$.

Definition 2.8 ([14]). A mapping $c : D(X) \to D(X)$ is said to be a D-$\check{C}ech$ closure operator if it satisfies the following axioms:
(i) $c(\underline{\phi}) = \underline{\phi}$,
(ii) $\underline{A} \subseteq c(\underline{A})\ \ \forall\ \underline{A} \in D(X)$,
(iii) $c(\underline{A} \cup \underline{B}) = c(\underline{A}) \cup c(\underline{B})\ \ \forall\ \underline{A}, \underline{B} \in D(X)$.
If c satisfies the following additional axiom, then c is called a D-closure operator, and sometimes called a D-Kuratowski's closure operator.
(iv) $c(c(\underline{A})) = c(\underline{A})\ \forall\ \underline{A} \in D(X)$.

Definition 2.9 ([11]). Let δ be a binary relation on the power set $P(X)$ of a nonempty set X. For all $A \in P(X)$, we define

$$\delta[A] = \{B : B \in P(X), B\bar{\delta}A\}.$$

Definition 2.10 ([11]). A binary relation δ on the power set $P(X)$ of a nonempty set X is said to be a basic proximity on X if it satisfies the following conditions: For any $A, B, C \in P(X)$
PI_1 : $A \in \delta[B] \Rightarrow B \in \delta[A]$,
PI_2 : $A \in \delta[C]$ and $B \in \delta[C] \Leftrightarrow A \cup B \in \delta[C]$,

PI_3 : $\phi \in \delta[A]$, for all $A \in P(X)$, and
PI_4 : $A \in \delta[B] \Rightarrow A \cap B = \phi$.
δ is said to be an EF - proximity on X if it is a basic proximity on X and it satisfies the following condition:
PI_5 : $A \in \delta[B] \Rightarrow \exists H \in P(X)$ such that $A \in \delta[H]$ and $H^c \in \delta[B]$.
δ is said to be separated proximity on X if it is an EF - proximity on X and it satisfies the following condition:
PI_6 : $x \neq y \Rightarrow \{x\} \in \delta[\{y\})]$.
For all $x \in X$, $x \in \delta[A]$ stands for $\{x\} \in \delta[A]$ and $\delta[x]$ stands for $\delta[\{x\}]$.

Lemma 2.11 ([11]). *For every subsets A and B of a basic proximity space (X, δ). If $A \in \delta[B]$ and $E \subseteq B$, then $A \in \delta[E]$.*

Proposition 2.12 ([11]). *Let (X, δ) be a basic proximity space. Then*

$$\delta[A] \text{ is an ideal on } X,\ \forall\, A \in P(X).$$

Lemma 2.13 ([11]). *Let (X, δ) be a basic proximity space. Then the two simplest ideals on X which generated by δ are $\delta[\phi] = P(X)$ and $\delta[X] = \{\phi\}$.*

Theorem 2.14 ([11]). *A binary relation δ on the power set $P(X)$ of a nonempty set X is a basic proximity on X if and only if it satisfies the following conditions:*
I_1 : $A \in \delta[B] \Rightarrow B \in \delta[A]$,
I_2 : $\delta[A]$ *is an ideal on* X $\forall\, A \in P(X)$, *and*
I_3 : $\delta[A] \subseteq \mathcal{I}_{A^c}$, *where* $\mathcal{I}_{A^c} = \{B : B \in P(X), B \subseteq A^c\}$.

Corollary 2.15 ([11]). *Let $\delta \in m(X)$. Then δ is an EF - Proximity iff it is a g_0 - Proximity.*

Corollary 2.16 ([11]). *Let $\delta \in m(X)$. Then δ is an RH - Proximity iff it is an h_0 - Proximity.*

3. Some properties of a basic D-proximities and D-ideals

Definition 3.1. Let X be a nonempty set and let δ be a binary relation on $D(X)$. For any $\underline{A} \in D(X)$ we define

$$\delta[\underline{A}] = \{\underline{B} : \underline{B} \in D(X), \underline{A}\bar{\delta}\underline{B}\} \text{ (here and henceforth also, } \bar{\delta} \text{ means non- } \delta).$$

Definition 3.2. Let X be a nonempty set and let δ be a binary relation on $D(X)$. For any $\underline{A}, \underline{B}, \underline{C} \in D(X)$, consider the following axioms:
DP_1: $\underline{A} \in \delta[\underline{B}] \Rightarrow \underline{B} \in \delta[\underline{A}]$,
DP_2: $\underline{A} \in \delta[\underline{C}]$ and $\underline{B} \in \delta[\underline{C}] \Leftrightarrow \underline{A} \cup \underline{B} \in \delta[\underline{C}]$,
$D\acute{P}_2$: $\underline{A} \in \delta[\underline{C}]$, $\underline{B} \in \delta[\underline{C}] \Leftrightarrow \underline{A} \cup \underline{B} \in \delta[\underline{C}]$, and $\underline{C} \in \delta[\underline{A}]$, $\underline{C} \in \delta[\underline{B}] \Leftrightarrow \underline{C} \in \delta[\underline{A} \cup \underline{B}]$,
DP_3: $\underline{\phi} \in \delta[\underline{A}]$ $\forall$ $\underline{A} \in D(X)$,
DP_4: $\underline{A} \in \delta[\underline{B}] \Rightarrow \underline{A}\bar{q}\underline{B}$, and
DP_5: $\underline{A} \in \delta[\underline{B}] \Rightarrow \exists \underline{H} \in D(X)$ such that $\underline{A} \in \delta[\underline{H}]$ and $\underline{H^c} \in \delta[\underline{B}]$ (here and henceforth also, $\underline{E}^c = \underline{X} \backslash \underline{E}$),
DP_6: $x_t\bar{q}y_r \Rightarrow x_t \in \delta[y_r]$,
DP_7: $\underline{A} \notin \delta[\underline{B}]$ and $b_t \notin \delta[\underline{C}]$ $\forall$ $b_t q B \Rightarrow \underline{A} \notin \delta[\underline{C}]$,

$D\acute{P}_7$: $x_t \notin \delta[\underline{B}]$ and $b_r \notin \delta[\underline{C}]\ \forall\ b_r qB \Rightarrow x_t \notin \delta[\underline{C}]$.

Then δ is said to be :

(a) A basic D-proximity on X, if it satisfies DP_1, DP_2, DP_3 and DP_4.

(b) An EF - D-proximity on X, if it is a basic D-proximity on X and satisfies DP_5.

(c) A separated D-proximity on X if it is an EF - D-proximity on X and it satisfies DP_6.

(d) An LE - D-proximity on X, if it satisfies $D\acute{P}_2$, DP_3, DP_4 and DP_7.

(e) An LO - D-proximity on X, if it is an LE - D-proximity on X and satisfies DP_1.

(f) An S - D-proximity on X, if it is a basic D-proximity on X and satisfies DP_6, $D\acute{P}_7$.

If δ is a basic D-proximity (resp. EF - D-proximity, separated D-proximity, LE - D-proximity, LO - D-proximity, S - double proximity) on X, then the pair (X, δ) is called a basic D-proximity (resp. EF - D-proximity, separated D-proximity, LE - D-proximity, LO - D-proximity, S - D-proximity) space.

We denote $\underline{m}(X)$ for the set of all basic D-proximities on X and $x_t \in \delta[\underline{A}]$ for $\underline{x}_t \in \delta[\underline{A}]$.

Definition 3.3. A binary relation δ on the set $D(X)$ is said to be RH-D-proximity on X if it satisfies the following conditions:

DR_1: $\underline{A} \in \delta[\underline{B}] \Rightarrow \underline{B} \in \delta[\underline{A}]$,

DR_2: $\underline{A} \in \delta[\underline{C}]$ and $\underline{B} \in \delta[\underline{C}] \Leftrightarrow \underline{A} \cup \underline{B} \in \delta[\underline{C}]$,

DR_3: $\underline{\phi} \in \delta[\underline{X}]$,

DR_4: $\underline{A} \in \delta[\underline{A}] \Rightarrow \underline{A} = \underline{\phi}$, and

DR_5: $x_t \in \delta[\underline{B}] \Rightarrow \exists \underline{H} \in D(X)$ such that $x_t \in \delta[\underline{H}]$ and $\underline{H}^c \in \delta[\underline{B}]$.

Lemma 3.4. *Let $\delta \in \underline{m}(X)$ and let $\underline{A}, \underline{B} \in D(X)$. Then If $\underline{A}\delta\underline{B}$, $\underline{A} \subseteq \underline{C}$ and $\underline{B} \subseteq \underline{D}$, then $\underline{C}\delta\underline{D}$.*

Lemma 3.5. *Let (X, δ) be a basic D-proximity space. For every $\underline{A},\underline{B} \in D(X)$. Then,*

(i) *If $\underline{A}\delta\underline{B}$, $\underline{A} \subseteq \underline{C}$, implies $\underline{B}\delta\underline{C}$.*

(ii) *If $\underline{A}\delta\underline{B}$, $\underline{B} \subseteq \underline{C}$, implies $\underline{A}\delta\underline{C}$.*

Theorem 3.6. *Let $\delta \in \underline{m}(X)$ and let $\underline{A}, \underline{B} \in D(X)$. Then*

(i) *If $\underline{A} \in \delta[\underline{B}]$ and $\underline{H} \subseteq \underline{B}$, then $\underline{A} \in \delta[\underline{H}]$.*

(ii) *$\underline{A} \subseteq \underline{B} \Rightarrow \delta[\underline{B}] \subseteq \delta[\underline{A}]$.*

(iii) *If $\underline{A} \in \delta[\underline{B}]$, then $a_t \in \delta[\underline{B}]\ \forall\ a_t \in \underline{A}$.*

Proof. (i) Let $\underline{A} \in \delta[\underline{B}]$ and $\underline{H} \subseteq \underline{B}$. Assume that $\underline{A} \notin \delta[\underline{H}]$. Then $\underline{H}\delta\underline{A}$, but $\underline{A} \subseteq \underline{A}$ and $\underline{H} \subseteq \underline{B}$ then(by Lemma 3.4) $\underline{A}\delta\underline{B}$ i.e. $A \notin \delta[\underline{B}]$ a contradiction.

(ii) it obvious from (i).

(iii) Let $\underline{A} \in \delta[\underline{B}]$ and assume that $\exists\ a_t \in \underline{A}$ such that $a_t \notin \delta[\underline{B}]$. Then $a_t\delta B$, but $\underline{a}_t \subseteq \underline{A}$ and $\underline{B} \subseteq \underline{B} \Rightarrow \underline{A}\delta\underline{B}$(by Lemma 3.4), which contradicts with $\underline{A} \in \delta[\underline{B}]$. □

Theorem 3.7. *Let $\delta \in \underline{m}(X)$. Then, $\delta[\underline{A}]$ is a D-ideal on X, $\forall\ \underline{A} \in D(X)$.*

Proof. Since $\underline{\phi} \in \delta[\underline{A}]$(by PI_3), then $\delta[\underline{A}]$ is a nonempty collection. Let $\underline{H} \in \delta[\underline{A}]$ and $\underline{M} \subseteq \underline{H}$. Then $\underline{A} \in \delta[\underline{H}]$ and $\underline{M} \subseteq \underline{H} \Rightarrow \underline{M} \in \delta[\underline{A}]$(by Theorem 3.6 (i), DP_1).

Now, let $\underline{H} \in \delta[\underline{A}]$ and $\underline{M} \in \delta[\underline{A}]$ then $\underline{H} \cup \underline{M} \in \delta[\underline{A}]$(by DP_2). Hence $\delta[\underline{B}]$ is a D-ideal on X. □

Remark 3.8. Note that if $\delta \in \underline{m}(X)$, then the two simplest D-ideals on X which generated by δ are $\delta[\underline{\phi}] = D(X)$ and $\delta[\underline{X}] = \{\underline{\phi}\}$.

Theorem 3.9. *A binary relation δ on $D(X)$ is said to be a basic D-proximity on X if and only if it satisfies the following conditions:*
DI_1 : $\underline{A} \in \delta[\underline{B}] \Rightarrow \underline{B} \in \delta[\underline{A}]$,
DI_2 : $\delta[\underline{A}]$ *is a D-ideal on* X $\forall$ $\underline{A} \in D(X)$, *and*
DI_3 : $\delta[\underline{A}] \subseteq \underline{\mathcal{I}}_{\underline{A}^c}$, *where* $\underline{\mathcal{I}}_{\underline{A}^c} = \{\underline{B} : \underline{B} \in D(X), \underline{B} \subseteq \underline{A}^c\}$.

Proof. Suppose that δ is a basic D-proximity on X. Then DP_1 equivalent to DI_1, and DI_2 holds (by Theorem 3.7). For DI_3, let $\underline{B} \in \delta[\underline{A}] \Rightarrow \underline{A}\bar{q}\underline{B}$(by DP_4) which implies $\underline{B} \subseteq \underline{A}^c$ (by Theorem 2.5), so $\underline{B} \in \underline{\mathcal{I}}_{\underline{A}^c}$. Hence $\delta[\underline{A}] \subseteq \underline{\mathcal{I}}_{\underline{A}^c}$.
Conversely, suppose that DI_1, DI_2 and DI_3 are hold. Then DI_1 equivalent to DP_1. Since $\delta[\underline{A}]$ is a D-ideal for all $\underline{A} \in D(X)$ then DP_2 and DP_3 are hold. Now, let $\underline{B} \in \delta[\underline{A}] \Rightarrow \underline{B} \subseteq \underline{A}^c$ (byDI_3)and so $\underline{A}\bar{q}\underline{B}$. Hence DP_4 holds. Consequently, δ is a basic D-proximity on X. □

Example 3.10. Let X be a nonempty set. If we define a binary relation δ on $D(X)$ such that $\underline{A}\delta\underline{B} \Leftrightarrow \underline{A}q\underline{B}$, then $\delta \in \underline{m}(X)$.

Example 3.11. Let X be a nonempty set. If we define a binary relation δ on $D(X)$ such that $\underline{A}\delta\underline{B} \Leftrightarrow \underline{A} \neq \underline{\phi}$ and $\underline{B} \neq \underline{\phi}$, then $\delta \in \underline{m}(X)$.

Theorem 3.12. *Let $\delta \in \underline{m}(X)$. For any $\underline{A}$, $\underline{B} \in D(X)$, then*
(i) $\delta[\underline{A} \cup \underline{B}] = \delta[\underline{A}] \cap \delta[\underline{B}] \subseteq \delta[\underline{A} \cap \underline{B}]$, *and*
(ii) *If* $\underline{H} \in \delta[\underline{A}]$ *and* $\underline{M} \in \delta[\underline{B}] \Rightarrow \underline{H} \cap \underline{M} \in \delta[\underline{A} \cup \underline{B}]$.

Proof. (i) Since $\underline{A}$, $\underline{B} \subseteq \underline{A} \cup \underline{B}$,then $\delta[\underline{A} \cup \underline{B}] \subseteq \delta[\underline{A}], \delta[\underline{B}]$ (by Theorem 3.6 (ii)), and so $\delta[\underline{A} \cup \underline{B}] \subseteq \delta[\underline{A}] \cap \delta[\underline{B}]$. Next, let $\underline{H} \notin \delta[\underline{A} \cup \underline{B}] \Rightarrow \underline{A} \cup \underline{B} \notin \delta[\underline{H}]$ which implies that $\underline{A} \notin \delta[\underline{H}]$ or $\underline{B} \notin \delta[\underline{H}]$ (byDP_2). So $\underline{H} \notin \delta[\underline{A}]$ or $\underline{H} \notin \delta[\underline{B}]$ which implies that $\underline{H} \notin \delta[\underline{A}] \cap \delta[\underline{B}]$. Therefore, $\delta[\underline{A} \cup \underline{B}] = \delta[\underline{A}] \cap \delta[\underline{B}]$. Now, let $\underline{H} \in \delta[\underline{A}] \cap \delta[\underline{B}]$. Then $\underline{A}$, $\underline{B} \in \delta[\underline{H}]$. Since $\underline{A} \cap \underline{B} \subseteq \underline{A}, \underline{B}$, hence $\underline{A} \cap \underline{B} \in \delta[\underline{H}]$ (by DI_2) and so $\underline{H} \in \delta[\underline{A} \cap \underline{B}]$, therefore, $\delta[\underline{A}] \cap \delta[\underline{B}] \subseteq \delta[\underline{A} \cap \underline{B}]$.
(ii) Let $\underline{M} \in \delta[\underline{A}]$ and $\underline{H} \in \delta[\underline{B}]$. Since $\underline{M} \cap \underline{H} \subseteq \underline{M}, \underline{H}$ then $\underline{M} \cap \underline{H} \in \delta[\underline{A}]$ and $\underline{M} \cap \underline{H} \in \delta[\underline{B}] \Rightarrow M \cap \underline{H} \in \delta[\underline{A}] \cap \delta[\underline{B}] = \delta[\underline{A} \cup \underline{B}]$. □

Definition 3.13. A double set $\underline{B}$ of a basic D-proximity space (X, δ) is said to be a δ- neighborhood(δ-nbd, in short) of a double set $\underline{A}$ iff $\underline{B}^c\bar{\delta}\underline{A}$. The set of all δ- nbd of a double set $\underline{A}$ is denoted by $N(\delta, \underline{A})$, i.e. $N(\delta, \underline{A}) = \{\underline{B} : \underline{B} \in D(X), \underline{B}^c \in \delta[\underline{A}]\}$. When there is no ambiguity we will write $N_\delta(\underline{A})$ for $N(\delta, \underline{A})$.

Definition 3.14. Let δ_1, δ_2 be two basic D-proximities on a nonempty set X, we define

$$\delta_1 < \delta_2 \text{ iff } \underline{A}\delta_2\underline{B} \Rightarrow \underline{A}\delta_1\underline{B}.$$

The above expression refers to that δ_2 is a finer than δ_1, or δ_1 is a coarser than δ_2.

Proposition 3.15. *Let δ_1, $\delta_2 \in \underline{m}(X)$. then $\delta_1 < \delta_2 \Leftrightarrow \delta_1[\underline{A}] \subseteq \delta_2[\underline{A}]$ $\forall\underline{A} \in D(X)$.*

Proof. Straightforward. □

Definition 3.16. Let $\delta_1, \delta_2 \in \underline{m}(X)$ and $\underline{A}, \underline{B} \in D(X)$, we define

$$\underline{A}(\delta_1 \cup \delta_2)\underline{B} \Leftrightarrow \underline{A}\delta_1\underline{B} \text{ or } \underline{A}\delta_2\underline{B}.$$

Theorem 3.17. *Let $\delta, \delta_1, \delta_2 \in \underline{m}(X)$ and let $\underline{A}, \underline{B} \in D(X)$. Then*

1. $N_\delta(\underline{\phi}) = D(X)$,
2. $\underline{A} \subseteq \underline{B} \Rightarrow N_\delta(\underline{B}) \subseteq N_\delta(\underline{A})$,
3. $\underline{X} \in N_\delta(\underline{A})\ \ \forall\, \underline{A} \in D(X)$,
4. $\underline{B} \in N_\delta(\underline{A}) \Rightarrow \underline{A} \subseteq \underline{B}$,
5. $\underline{A} \in N_\delta(\underline{B}) \Leftrightarrow \underline{B}^c \in N_\delta(\underline{A}^c)$,
6. $N_\delta(\underline{A} \cup \underline{B}) = N_\delta(\underline{A}) \cap N_\delta(\underline{B})$,
7. $\underline{A} \in N_\delta(\underline{H})$ *and* $\underline{B} \in N_\delta(\underline{M}) \Rightarrow \underline{A} \cup \underline{B} \in N_\delta(\underline{H} \cup \underline{M})$,
8. $N_{\delta_1 \cup \delta_2}(\underline{A}) = N_{\delta_1}(\underline{A}) \cap N_{\delta_2}(\underline{A})$,
9. *If* $\delta_1 < \delta_2$, *then* $N_{\delta_1}(\underline{A}) \subseteq N_{\delta_2}(\underline{A})\ \ \forall\, \underline{A} \in D(X)$.

Proof. Straightforward. □

Theorem 3.18. *Let (X, δ) be a basic D-proximity space. If $\underline{A}\delta\underline{B}$, then $\underline{A} \cap \underline{H} \neq \underline{\phi}\ \ \forall\, \underline{H} \in N_\delta(\underline{B})$.*

Proof. Let $\underline{A}\delta\underline{B}$ and assume that there exists $\underline{H} \in N_\delta(\underline{A})$ such that $\underline{B} \cap \underline{H} = \underline{\phi}$. Then $\underline{B}\bar{q}\underline{H} \Rightarrow \underline{B} \subseteq \underline{H}^c \Rightarrow \underline{A} \in \delta[\underline{B}]$(by DP_1, Theorem 3.6 (i)) a contradiction. □

Theorem 3.19. *Let (X, δ) be a basic D-proximity space then the operator $c_\delta : D(X) \to D(X)$ given by*

$$c_\delta(\underline{A}) = \cap\{\underline{B} : \underline{B} \in N_\delta(\underline{A})\}, \text{ for all } \underline{A} \in D(X)$$

is a D-Čech closure operator.

Proof. Clearly $c_\delta(\underline{\phi}) = \underline{\phi}$. Let $x_t \in \underline{A}$ and assume that $x_t \notin c_\delta(\underline{A})$, then there exists $\underline{M} \in D(X)$ such that $x_t \notin \underline{M} \in N_\delta(\underline{A})$, since $\underline{M} \in N_\delta(\underline{A})$, then $\underline{A} \subseteq \underline{M}$(by Theorem 3.17) a contradicts with $x_t \in \underline{A}$. Hence, $\underline{A} \subseteq c_\delta(\underline{A})$ for any $\underline{A} \in D(X)$. Now, let $\underline{A} \subseteq \underline{B}$ and let $x_t \in c_\delta(\underline{A})$, then $x_t \in \underline{H}\ \forall\, \underline{H} \in N_\delta(\underline{A}) \Rightarrow x_t \in \underline{H}\ \forall\, \underline{H} \in N_\delta(\underline{B})$ (as $N_\delta(\underline{B}) \subseteq N_\delta(\underline{A})) \Rightarrow x_t \in c_\delta(\underline{B})$. Hence $c_\delta(\underline{A}) \subseteq c_\delta(\underline{B})$. Since $\underline{A}, \underline{B} \subseteq \underline{A} \cup \underline{B}$, then $c_\delta(\underline{A}), c_\delta(\underline{B}) \subseteq c_\delta(\underline{A} \cup \underline{B}) \Rightarrow c_\delta(\underline{A}) \cup c_\delta(\underline{A}) \subseteq c_\delta(\underline{A} \cup \underline{B})$. Next, let $x_t \notin c_\delta(\underline{A}) \cup c_\delta(\underline{B})$, then there exists $\underline{H}, \underline{M} \in D(X)$ such that $x_t \notin \underline{H} \in N_\delta(\underline{A})$ and $x_t \notin \underline{M} \in N_\delta(\underline{B}) \Rightarrow x_t \notin (\underline{H} \cup \underline{M}) \in N_\delta(\underline{A} \cup \underline{B})$ (by Theorem 3.17) and so $x_t \notin c_\delta(\underline{A} \cup \underline{B})$. Hence, $c_\delta(\underline{A} \cup \underline{B}) \subseteq c_\delta(\underline{A}) \cup c_\delta(\underline{B})$. Consequently, c_δ is a D-*Čech* closure operator. □

Theorem 3.20. *Let (X, δ) be an EF - D-proximity space then the operator c_δ is a D-closure operator and the collection $\eta_\delta = \{\underline{A} \in D(X) : c_\delta(\underline{A}^c) = \underline{A}^c\}$ is a double topology on X.*

Proof. Straightforward. □

Lemma 3.21. *Let $\delta_1, \delta_2 \in \underline{m}(X)$. If $\delta_1 < \delta_2$, then $c_{\delta_2}(\underline{A}) \subseteq c_{\delta_1}(\underline{A})\ \ \forall\, \underline{A} \in D(X)$.*

Proof. Let $x_t \in c_{\delta_2}(\underline{A})$, then $x_t \in \underline{B} \forall\, \underline{B} \in N_{\delta_2}(\underline{A}) \Rightarrow x_t \in \underline{B} \forall\, \underline{B} \in N_{\delta_1}(\underline{A})$ (by Theorem 3.17) and so $x_t \in \cap\{\underline{B} : \underline{B} \in N_{\delta_1}(\underline{A})\} = c_{\delta_1}(\underline{A})$. Hence, $c_{\delta_2}(\underline{A}) \subseteq c_{\delta_1}(\underline{A}) \forall\, \underline{A} \in D(X)$. □

Definition 3.22. Let $\delta \in \underline{m}(X)$ and $\underline{A} \in D(X)$ we define

$$CN_\delta(\underline{A}) = \{\underline{B} : \underline{B} \in D(X), \underline{B} \notin N_\delta(\underline{A})\}.$$

Lemma 3.23. *Let $\delta \in \underline{m}(X)$, $\underline{A} \in D(X)$ and let $\underline{\mathcal{I}} \in DI(X)$. Then $N_\delta(\underline{A}) \cap \underline{\mathcal{I}} = \underline{\phi} \Leftrightarrow \underline{\mathcal{I}} \subseteq CN_\delta(\underline{A})$.*

Theorem 3.24. *Let $\delta \in \underline{m}(X)$, $\underline{A} \in D(X)$ and let $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2 \in DI(X)$. Then $\underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2 \subseteq CN_\delta(\underline{A}) \Rightarrow \underline{\mathcal{I}}_1 \subseteq CN_\delta(\underline{A})$ or $\underline{\mathcal{I}}_2 \subseteq CN_\delta(\underline{A})$.*

Proof. Assume that $\underline{\mathcal{I}}_1 \nsubseteq CN_\delta(\underline{A})$ and $\underline{\mathcal{I}}_2 \nsubseteq CN_\delta(\underline{A})$. Then there exists $\underline{H}_1 \in \underline{\mathcal{I}}_1 \setminus CN_\delta(\underline{A})$ and $\underline{H}_2 \in \underline{\mathcal{I}}_2 \setminus CN_\delta(\underline{A})$. Since $\underline{H}_1 \cap \underline{H}_2 \subseteq \underline{H}_1, \underline{H}_2$ and $\underline{\mathcal{I}}_1, \underline{\mathcal{I}}_2$ are a D-ideals on X, then $\underline{H}_1 \cap \underline{H}_2 \in \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2 \subseteq CN_\delta(\underline{A}) \Rightarrow \underline{H}_1 \cap \underline{H}_2 \in CN_\delta(\underline{A})$ which implies that $\underline{H}_1 \cap \underline{H}_2 \notin N_\delta(\underline{A}) \Rightarrow (\underline{H}_1^c \cup \underline{H}_2^c) \notin \delta[\underline{A}] \Rightarrow \underline{H}_1^c \notin \delta[\underline{A}]$ or $\underline{H}_2^c \notin \delta[\underline{A}]$ $(byDP_2)$, thus $\underline{H}_1^c \notin N_\delta(\underline{A})$ or $\underline{H}_2^c \notin N_\delta(\underline{A})$. Hence $\underline{H}_1 \in CN_\delta(\underline{A})$ or $\underline{H}_2 \in CN_\delta(\underline{A})$, a contradiction. □

Theorem 3.25. *Let $\underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2$ and $\underline{\mathcal{J}}$ are D-ideals on a nonempty set X. Then if $\underline{\mathcal{J}} \subseteq \underline{\mathcal{I}}_1 \cup \underline{\mathcal{I}}_2$, then $\underline{\mathcal{J}} \subseteq \underline{\mathcal{I}}_1$ or $\underline{\mathcal{J}} \subseteq \underline{\mathcal{I}}_2$.*

Proof. Assume that $\underline{\mathcal{J}} \nsubseteq \underline{\mathcal{I}}_1$ and $\underline{\mathcal{J}} \nsubseteq \underline{\mathcal{I}}_2$. Then there exists $\underline{A} \in \underline{\mathcal{J}} \setminus \underline{\mathcal{I}}_1$ and $\underline{B} \in \underline{\mathcal{J}} \setminus \underline{\mathcal{I}}_2$, so $\underline{A} \cup \underline{B} \in \underline{\mathcal{J}} \subseteq \underline{\mathcal{I}}_1 \cup \underline{\mathcal{I}}_2$. Therefore $\underline{A} \cup \underline{B} \in \underline{\mathcal{I}}_1$ or $\underline{A} \cup \underline{B} \in \underline{\mathcal{I}}_2$ which implies $\underline{A} \in \underline{\mathcal{I}}_1$ or $\underline{B} \in \underline{\mathcal{I}}_2$, a contradiction. □

4. f - D-proximities

In this section we introduce the concept of g- double proximities(f - D-proximities, for short) and we show that for different choice of "f" one can obtain many of the the known types of D-proximities. Also, characterizations of some types of these D-proximities-(f_0, l_0)- have obtained.

Definition 4.1. A mapping $f : \underline{m}(X) \times DI(X) \to DI(X)$ is said to be a D-ideal operator on X if $\forall\, \delta \in \underline{m}(X)$ and $\forall\, \underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2 \in DI(X)$, we have

$$f(\delta, \underline{\mathcal{I}}_1) \subseteq f(\delta, \underline{\mathcal{I}}_2) \text{ whenever } \underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2.$$

Definition 4.2. Let f be a D-ideal operator on X. Then a basic D-proximity δ on X is said to be a $f-$ D-proximity iff $\delta[\underline{A}] \subseteq f(\delta, \delta[\underline{A}])$, $\forall\, \underline{A} \in D(X)$. The family of all f - D-proximities will denoted by $\mathcal{D}_f$.

Definition 4.3. A D-ideal operator f is said to be:
in class M_1 if $f(\delta, \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2) = f(\delta, \underline{\mathcal{I}}_1) \cap f(\delta, \underline{\mathcal{I}}_2)$ $\forall\, \delta \in \underline{m}(X)$ and $\forall\, \underline{\mathcal{I}}_1$, $\underline{\mathcal{I}}_2 \in DI(X)$.
in class M_2 if $f(\delta, \bigcap_{\alpha\in\Lambda} \underline{\mathcal{I}}_\alpha) = \bigcap_{\alpha\in\Lambda} f(\delta, \underline{\mathcal{I}}_\alpha)$ $\forall\, \delta \in \underline{m}(X)$ and $\forall\, \underline{\mathcal{I}}_\alpha \in DI(X)$.
in class C if $f(\delta_1, \underline{\mathcal{I}}) = f(\delta_2, \underline{\mathcal{I}})$ with $c_{\delta_1} = c_{\delta_2}$ $\forall\, \delta_1, \delta_2 \in \underline{m}(X)$ and $\forall\, \underline{\mathcal{I}} \in DI(X)$.
in class I if $f(\delta_1, \underline{\mathcal{I}}) \subseteq f(\delta_2, \underline{\mathcal{I}})$ whenever $\delta_1 < \delta_2$ $\forall\, \underline{\mathcal{I}} \in DI(X)$.
in class N if $f(\delta, \underline{\mathcal{I}}) \subseteq f(\delta, f(\delta, \underline{\mathcal{I}}))$, $\forall\, \delta \in \mathcal{D}_f$, $\forall\, \underline{\mathcal{I}} \in DI(X)$.

Definition 4.4. For a set X, for all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$ we define
$id(\delta, \underline{\mathcal{I}}) = \underline{\mathcal{I}}$,
$f_0(\delta, \underline{\mathcal{I}}) = \{\underline{A} : \underline{A} \in D(X), N_\delta(\underline{A}) \cap \underline{\mathcal{I}} \neq \underline{\phi}\}$,
$f_1(\delta, \underline{\mathcal{I}}) = \{\underline{A} : \underline{A} \in D(X), c_\delta(\underline{A}) \in \underline{\mathcal{I}}\}$,
$f_2(\delta, \underline{\mathcal{I}}) = \{\underline{A} : \underline{A} \in D(X), \underline{x}_t \in \delta[\underline{A}] \cup \underline{\mathcal{I}},\ \forall\, x_t \in X_p\}$,
$l_0(\delta, \underline{\mathcal{I}}) = \{\underline{A} : \underline{A} \in D(X), N_\delta(\underline{a}_t) \cap \underline{\mathcal{I}} \neq \underline{\phi}\ \forall\, a_t \in \underline{A}\}$,

$l_1(\delta, \underline{\mathcal{I}}) = \{\underline{A} : \underline{A} \in D(X), c_\delta(\underline{A}) \in \delta[x_t]$ *with* $\underline{\mathcal{I}} \subseteq \delta[x_t]\}$. When there is no ambiguity we will write f_i for $f_i(\delta, \underline{\mathcal{I}})$ and l_i for $l_i(\delta, \underline{\mathcal{I}})$.

Theorem 4.5. *Let $\delta \in \underline{m}(X)$ and $\underline{\mathcal{I}}$ be an arbitrary element in $DI(X)$. Then f is a D-ideal operator on X for all $f \in \{id, f_0, f_1, f_2, l_0, l_1\}$.*

Proof. We prove the cases f_0 and f_2, the other cases are similar. Suppose that $\delta \in \underline{m}(X)$ and $\mathcal{I} \in DI(X)$. Now, since $N_\delta(\underline{\phi}) \cap \underline{\mathcal{I}} = \underline{\mathcal{I}} \neq \underline{\phi}$, then $\underline{\phi} \in f_0$. If $\underline{A} \in f_0$, $\underline{B} \subseteq \underline{A}$ then $N_\delta(\underline{A}) \cap \underline{\mathcal{I}} \neq \underline{\phi} \Rightarrow N_\delta(\underline{B}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$ (by Theorem 3.17), hence $\underline{B} \in f_0$. If $\underline{A}$, $\underline{B} \in f_0$, then $N_\delta(\underline{A}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$ and $N_\delta(\underline{B}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$. So $\exists \underline{H}$, $\underline{M} \in \underline{\mathcal{I}}$ such that $\underline{H} \in N_\delta(\underline{A})$ and $\underline{M} \in N_\delta(\underline{B})$ which implies that $\underline{H} \cup \underline{M} \in N_\delta(\underline{A} \cup \underline{B})$ (by Theorem 3.17) and so $\underline{H} \cup \underline{M} \in N_\delta(\underline{A} \cup \underline{B}) \cap \underline{\mathcal{I}}$, consequently, $N_\delta(\underline{A} \cup \underline{B}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$. Hence $\underline{A} \cup \underline{B} \in f_0$. Therefore, $f_0 \in DI(X)$. Now, let $\underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2$ and let $\underline{H} \in f_0(\delta, \underline{\mathcal{I}}_1)$. Then $N_\delta(\underline{H}) \cap \underline{\mathcal{I}}_1 \neq \underline{\phi} \Rightarrow N_\delta(\underline{H}) \cap \underline{\mathcal{I}}_2 \neq \underline{\phi}$. So $\underline{H} \in f_0(\delta, \underline{\mathcal{I}}_2)$. Hence f_0 is a D-ideal operator on X.
Next, since $\delta[\underline{\phi}] = D(X)$, then $x_t \in \delta[\underline{\phi}] \cup \underline{\mathcal{I}}\ \forall\ x_t \in X_p \Rightarrow \underline{\phi} \in f_2$. If $\underline{A} \in f_2$, $\underline{B} \subseteq \underline{A}$ then $x_t \in \delta[\underline{A}] \cup \underline{\mathcal{I}}\ \forall\ x_t \in X_p \Rightarrow x_t \in \delta[\underline{B}] \cup \underline{\mathcal{I}}\ \forall\ x_t \in X_p$(by Theorem 3.6 (ii)), and so $\underline{B} \in f_2$. If $\underline{A}$, $\underline{B} \in f_2$, then $x_t \in (\delta[\underline{A}] \cup \underline{\mathcal{I}}) \cap (\delta[\underline{B}] \cup \underline{\mathcal{I}})\ \forall\ x_t \in X_p \Rightarrow x_t \in (\delta[\underline{A}] \cap \delta[\underline{B}]) \cup \underline{\mathcal{I}}\ \forall\ x_t \in X_p \Rightarrow x_t \in \delta[\underline{A} \cup \underline{B}] \cup \underline{\mathcal{I}}\ \forall\ x_t \in X_p$ (by Theorem 3.12 (i)), and so, $\underline{A} \cup \underline{B} \in f_2$. Hence $f_2 \in DI(X)$. Also, it is clear that if, $\underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2$, then $f_2(\delta, \underline{\mathcal{I}}_1) \subseteq f_2(\delta, \underline{\mathcal{I}}_2)$. Consequently, f_2 is a D-ideal operator on X. □

Theorem 4.6. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$, we have $id, f_1, f_2 \in M_2 \subseteq M_1$ and $f_0, l_0 \in M_1$.*

Proof. Clearly $M_2 \subseteq M_1$. Also, trivially, $id, f_1, f_2 \in M_2$. Now, let $\underline{A} \in f_0(\delta, \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2)$. Then, $N_\delta(\underline{A}) \cap (\underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2) \neq \underline{\phi} \Rightarrow N_\delta(\underline{A}) \cap \underline{\mathcal{I}}_1 \neq \underline{\phi}$ and $N_\delta(\underline{A}) \cap \underline{\mathcal{I}}_2 \neq \underline{\phi} \Rightarrow \underline{A} \in f_0(\delta, \underline{\mathcal{I}}_1) \cap f_0(\delta, \underline{\mathcal{I}}_2)$. Hence $f_0(\delta, \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2) \subseteq f_0(\delta, \underline{\mathcal{I}}_1) \cap f_0(\delta, \underline{\mathcal{I}}_2)$. Now, let $A \in f_0(\delta, \underline{\mathcal{I}}_1) \cap f_0(\delta, \underline{\mathcal{I}}_2)$. Then $N_\delta(\underline{A}) \cap \underline{\mathcal{I}}_1 \neq \underline{\phi}$ and $N_\delta(\underline{A}) \cap \underline{\mathcal{I}}_2 \neq \underline{\phi}$ which implies $N_\delta(\underline{A}) \cap (\underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2) \neq \underline{\phi}$(by Lemma 3.23, Theorem 3.24). So $\underline{A} \in f_0(\delta, \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2)$. Hence $f_0(\delta, \underline{\mathcal{I}}_1) \cap f_0(\delta, \underline{\mathcal{I}}_2) \subseteq f_0(\delta, \underline{\mathcal{I}}_1 \cap \underline{\mathcal{I}}_2)$. Therefore, $f_0 \in M_1$. Similarly $l_0 \in M_1$. □

Theorem 4.7. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$ we have $f \in C\ \ \forall\ f \in \{id, f_1, , l_1\}$.*

Proof. It follows from Definition 4.4. □

Theorem 4.8. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$, we have $f \in I\ \ \forall\ f \in \{id, f_0, f_1, f_2, l_0, l_1\}$.*

Proof. It follows from Theorem 3.17 and Lemma 3.21. □

Theorem 4.9. *Let $\delta \in \underline{m}(X)$. Then the following statements are equivalent:*

(1) *δ is an EF - D-Proximity on X,*
(2) $\underline{A} \in \delta[\underline{B}] \Rightarrow N_\delta(\underline{A}) \cap \delta[\underline{B}] \neq \underline{\phi}$,
(3) $N_\delta(\underline{A}) \cap \delta[\underline{B}] = \underline{\phi} \Rightarrow \underline{A} \notin \delta[\underline{B}]$,
(4) *δ is a g_0 - D-Proximity,*
(5) $\underline{A} \in N_\delta(\underline{B}) \Rightarrow \exists \underline{H} \in N_\delta(\underline{B})$ *such that* $\underline{A} \in N_\delta(\underline{H})$.

Proof. (1) $\Rightarrow$ (2): let $\underline{A} \in \delta[\underline{B}]$. Then $\exists \underline{H} \in D(X)$ such that $\underline{A} \in \delta[\underline{H}]$ and $\underline{H}^c \in \delta[\underline{B}]$. It follows that $\underline{H} \in \delta[\underline{A}]$ and $\underline{H}^c \in \delta[\underline{B}]$. Hence $\underline{H}^c \in N_\delta(\underline{A}) \cap \delta[\underline{B}]$, and so $N_\delta(\underline{A}) \cap \delta[\underline{B}] \neq \underline{\phi}$.

(2) $\Leftrightarrow$ (3): it is obvious.

(2) $\Rightarrow$ (4): let $\underline{H} \in \delta[\underline{A}]$. Then, $N_\delta(\underline{H}) \cap \delta[\underline{A}] \neq \underline{\phi} \Rightarrow \underline{H} \in f_0(\delta, \delta[\underline{A}])$. Hence δ is an f_0 -D-Proximity.

(4)$\Rightarrow$ (2): let $\underline{A} \in \delta[\underline{B}]$. Then, $\underline{A} \in f_0(\delta, \delta[\underline{B}]) \Rightarrow N_\delta(\underline{A}) \cap \delta[\underline{B}] \neq \underline{\phi}$.

(2) $\Rightarrow$ (5): let $\underline{A} \in N_\delta(\underline{B})$. Then $\underline{A}^c \in \delta[\underline{B}] \Rightarrow N_\delta(\underline{A}^c) \cap \delta[\underline{B}] \neq \underline{\phi} \Rightarrow \exists \underline{M} \in D(X)$ such that $\underline{M} \in \delta[\underline{B}]$ and $\underline{M} \in N_\delta(\underline{A}^c)$. Hence $\underline{M}^c \in N_\delta(\underline{B})$ and $\underline{A} \in N_\delta(\underline{M}^c)$, putting $\underline{H} = \underline{M}^c$. So (5) holds.

(5) $\Rightarrow$ (1): let $\underline{A} \in \delta[\underline{B}]$. Then $\underline{A}^c \in N_\delta(\underline{B}) \Rightarrow \exists \underline{H} \in D(X)$ such that $\underline{H} \in N_\delta(\underline{B})$ and $\underline{A}^c \in N_\delta(\underline{H}) \Rightarrow \underline{H}^c \in \delta[\underline{B}]$ and $\underline{A} \in \delta[\underline{H}]$. Hence δ is an EF -D-Proximity. □

Corollary 4.10. *Let $\delta \in \underline{m}(X)$. Then, δ is an EF - D-Proximity $\Leftrightarrow$ it is f_0 - D-Proximity.*

Theorem 4.11. *Let $\delta \in \underline{m}(X)$. Then, $\delta \in \mathcal{D}_{f_1} \Rightarrow c_\delta$ is a D-closure operator.*

Proof. Straightforward. □

Theorem 4.12. *Let $\delta \in \underline{m}(X)$. Then δ is a f_1 - D-Proximity if and only if $\forall\ \underline{B} \in \delta[\underline{A}] \Rightarrow c_\delta(\underline{B}) \in \delta[\underline{A}]$.*

Proof. Suppose that δ is a f_1 - D-Proximity and let $\underline{B} \in \delta[\underline{A}]$. Then, $\underline{B} \in f_1(\delta, \delta[\underline{A}])$ $\Rightarrow c_\delta(\underline{B}) \in \delta[\underline{A}]$.
Conversely, let $\underline{B} \in \delta[\underline{A}]$. Then, $c_\delta(\underline{B}) \in \delta[\underline{A}] \Rightarrow \underline{B} \in f_1(\delta, \delta[\underline{A}])$. So, $\delta[\underline{A}] \subseteq f_1(\delta, \delta[\underline{A}])$, $\forall\ \underline{A} \in D(X)$. Hence δ is a f_1 - D-Proximity. □

Theorem 4.13. *Let $\delta \in \underline{m}(X)$. Then δ is an LO - D-Proximity $\Leftrightarrow$ it is f_1 - D-Proximity.*

Proof. Straightforward. □

Theorem 4.14. *Let $\delta \in \underline{m}(X)$, $\underline{\mathcal{I}} \in DI(X)$. Then, $f(\delta, \underline{\mathcal{I}}) \subseteq \underline{\mathcal{I}}\ \forall\ f \in \{id, f_0, f_1\}$.*

Proof. Straightforward. □

Theorem 4.15. *Let $\delta \in \underline{m}(X)$. Then*
$\delta \in \mathcal{D}_{f_2} \Leftrightarrow \underline{A} \in \delta[\underline{B}] \Rightarrow (\underline{A} \in \delta[x_t]$ *or* $\underline{B} \in \delta[x_t])\ \ \forall\ x_t \in Xp$.

Proof. Suppose that δ is a f_2 - D-Proximity and let $\underline{A} \in \delta[\underline{B}]$. Then, $\underline{A} \in f_2(\delta, \delta[\underline{B}])$ $\Rightarrow \underline{x}_t \in \delta[\underline{A}] \cup \delta[\underline{B}]\ \forall\ x_t \in X_p$ and so, $\underline{A} \in \delta[x_t]$ or $\underline{B} \in \delta[x_t]\ \ \forall\ x_t \in X_p$.
Conversely, let $\underline{H} \in \delta[\underline{A}]$. Then, $\underline{H} \in \delta[x_t]$ or $\underline{A} \in \delta[x_t]\ \ \forall\ x_t \in X_p \Rightarrow x_t \in \delta[\underline{H}] \cup \delta[\underline{A}]\ \forall\ x_t \in X_p$. Hence $\underline{H} \in f_2(\delta, \delta[\underline{A}])\ \forall\ \underline{A} \in D(X))$, and consequently, δ is a f_2 - D-Proximity. □

Theorem 4.16. *Let $\delta \in \underline{m}(X)$. Then the following statements are equivalent:*

(1) $x_t \in \delta[\underline{A}] \Rightarrow \exists \underline{H} \in D(X)$ *such that* $x_t \in \delta[\underline{H}]$ *and* $\underline{H}^c \in \delta[\underline{A}]$,

(2) $x_t \in \delta[\underline{A}] \Rightarrow N_\delta(x_t) \cap \delta[\underline{A}] \neq \underline{\phi}$,

(3) $N_\delta(x_t) \cap \delta[\underline{A}] = \underline{\phi} \Rightarrow x_t \notin \delta[\underline{A}]$,

(4) *δ is an l_0 - D-Proximity,*

(5) $\underline{A} \in N_\delta(x_t) \Rightarrow \exists \underline{B} \in N_\delta(x_t)$ *such that* $\underline{A} \in N_\delta(\underline{B})$.

Proof. (1) $\Rightarrow$ (2): let $x_t \in \delta[\underline{A}]$. Then, $\exists \underline{H} \in D(X)$ such that $x_t \in \delta[\underline{H}]$ and $\underline{H}^c \in \delta[\underline{A}]$(by (1)). It follows that $\underline{H}^c \in \delta[\underline{A}]$ and $\underline{H}^c \in N_\delta(x_t)$. Hence $N_\delta(_t) \cap \delta[\underline{A}] \neq \underline{\phi}$.

(2) $\Leftrightarrow$ (3) it is obvious.

(2) $\Rightarrow$ (4): let $\underline{B} \in \delta[\underline{A}]$. Then, $b_t \in \delta[\underline{A}]\ \ \forall\, b_t \in \underline{B}$(by Theorem 3.6 (iii)). By (2), $N_\delta(b_t) \cap \delta[\underline{A}] \neq \underline{\phi}$, $\ \forall\, b_t \in \underline{B}$ which implies that $\underline{B} \in l_0(\delta, \delta[\underline{A}])$. Hence δ is an l_0 - D-Proximity.

(4) $\Rightarrow$ (2): it is obvious.

(2) $\Rightarrow$ (5): let $\underline{A} \in N_\delta(x_t)$. Then, $x_t \in \delta[\underline{A}^c]$, (by (2)) $N_\delta(x_t) \cap \delta[\underline{A}^c] \neq \underline{\phi}$. It follows that $\exists \underline{B} \in N_\delta(x_t), \underline{B} \in \delta[\underline{A}^c]$. So, $\underline{A}^c \in \delta[\underline{B}]$ and $\underline{B} \in N_\delta(x_t)$, this lead to $\underline{A} \in N_\delta(\underline{B})$.

(5) $\Rightarrow$ (1): let $x_t \in \delta[\underline{A}]$. Then, $\underline{A}^c \in N_\delta(x_t)$. By (5), $\exists \underline{H} \in N_\delta(x_t)$ such that $\underline{A}^c \in N_\delta(\underline{H})$. It follows that $\underline{A} \in \delta[\underline{H}]$ and $\underline{H}^c \in \delta[x_t]$. So, $x_t \in \delta[\underline{H}^c], \underline{H} \in \delta[\underline{A}]$. □

Corollary 4.17. *Let $\delta \in \underline{m}(X)$. Then, δ is an RH - D-Proximity $\Leftrightarrow$ it is l_0 - D-Proximity.*

Theorem 4.18. *Let $\delta \in \underline{m}(X)$. Then*

(1) *If $\delta \in \mathcal{D}_{l_1}$, then $x_t \in \delta[\underline{A}] \Rightarrow x_t \in \delta[c_\delta(\underline{A})]$.*

(2) *$\forall\, \delta \in \mathcal{D}_{l_1}$, c_δ is a D-closure operator.*

Proof. (1) Suppose that δ is an l_1 -D-proximity and let $x_t \in \delta[\underline{A}]$. Then $\underline{A} \in \delta[x_t] \subseteq l_1(\delta, \delta[x_t]) \Rightarrow \underline{A} \in l_1(\delta, \delta[x_t]) \Rightarrow c_\delta(\underline{A}) \in \delta[y_r]$ with $\delta[x_t] \subseteq \delta[y_r]$. But $\delta[x_t] \subseteq \delta[x_t]$, hence $c_\delta(\underline{A}) \in \delta[x_t] \Rightarrow x_t \in \delta[c_\delta(\underline{A})]$.

(2) It's obvious. □

Lemma 4.19. *Let δ be an S -D-Proximity. If $\underline{A} \in \delta[x_t]$, then $c_\delta(\underline{A}) \in \delta[x_t]$.*

Theorem 4.20. *Let $\delta \in \underline{m}(X)$. Then δ is an S - D-Proximity $\Leftrightarrow$ it is an l_1 - D-Proximity.*

Proof. Straightforward. □

Theorem 4.21. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$, we have $f \in N\ \ \forall\, f \in \{id, f_0, f_2, l_0\}$.*

Proof. Clearly, $id \in N$. Suppose that $\delta \in \mathcal{D}_{f_0}$, $\underline{A} \in f_0(\delta, \underline{\mathcal{I}})$. Then, $N_\delta(\underline{A}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$, so there exists $\underline{M} \in N_\delta(\underline{A})$, $\underline{M} \in \underline{\mathcal{I}}$. Since $\delta \in \mathcal{D}_{f_0}$, then (by Theorem 4.9) there exists $\underline{H} \in N_\delta(\underline{A})$ such that $\underline{M} \in N_\delta(\underline{H}) \Rightarrow N_\delta(\underline{H}) \cap \underline{\mathcal{I}} \neq \underline{\phi}$. So, $\underline{H} \in f_0(\delta, \underline{\mathcal{I}})$. But, $\underline{H} \in N_\delta(\underline{A})$, then $N_\delta(\underline{A}) \cap f_0(\delta, \underline{\mathcal{I}}) \neq \underline{\phi}$. Hence $\underline{A} \in f_0(\delta, f_0(\delta, \underline{\mathcal{I}}))$. Consequently, $f_0(\delta, \underline{\mathcal{I}}) \subseteq f_0(\delta, f_0(\delta, \underline{\mathcal{I}}))$. So, $f_0 \in N$. Similarly with f_2, l_0. □

Theorem 4.22. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$, then $f_0(\delta, \underline{\mathcal{I}}) = \bigcup_{\underline{A}^c \in \underline{\mathcal{I}}} \delta[\underline{A}]$.*

Proof. Straightforward. □

Theorem 4.23. *For all $\delta \in \underline{m}(X)$ and for all $\underline{\mathcal{I}} \in DI(X)$, then*
$\mathcal{D}_{f_2} \subseteq \mathcal{D}_{f_1}$, $\mathcal{D}_{l_1} \subseteq \mathcal{D}_{f_1}$ and $\mathcal{D}_{l_0} \subseteq \mathcal{D}_{f_0}$.

Proof. Straightforward. □

5. The relation between basic proximity and basic D-proximity

In this section, we investigate the relationship among basic proximity and basic double proximity.

Definition 5.1. Let δ be any arbitrary relation on the power set of a nonempty set X. We define

$$\xi_\delta[\underline{A}] = \{\underline{B} \in D(X) : B_2 \in \delta[A_2]\},\ \underline{A} \in D(X).$$

Theorem 5.2. *Let (X, δ) be a basic proximity space. Then ξ_δ is a basic D-proximity on X which is called the basic D-proximity induced by the basic proximity δ on X.*

Proof. Let $\underline{A} \in D(X)$.

DI_1: Let $\underline{B} \in \xi_\delta[\underline{A}]$. Then $B_2 \in \delta[A_2] \Rightarrow A_2 \in \delta[B_2]$ [by Theorem 2.14 (I_1)] $\Rightarrow \underline{A} \in \xi_\delta[\underline{B}]$. Hence DI_1 holds.

DI_2: We shall show that $\xi_\delta[\underline{A}]$ is a D-ideal on X. Since $\underline{A} \in D(X)$. Then $A_2 \in P(X) \Rightarrow \phi \in \delta[A_2]$ [by Theorem 2.14 (I_2)], and so $\underline{\phi} \in \xi_\delta[\underline{A}]$, i.e. $\xi_\delta[\underline{A}]$ is a nonempty. Let $\underline{B} \in \xi_\delta[\underline{A}]$ and $\underline{H} \subseteq \underline{B}$. Then $B_2 \in \delta[A_2]$ and $H_2 \subseteq B_2 \Rightarrow H_2 \in \delta[A_2]$ and so $\underline{H} \in \xi_\delta[\underline{A}]$. Now, let $\underline{B}, \underline{H} \in \xi_\delta[\underline{A}]$. Then $B_2, H_2 \in \delta[A_2] \Rightarrow B_2 \cup H_2 \in \delta[A_2]$ and so $\underline{B} \cup \underline{A} \in \xi_\delta[\underline{A}]$. Hence $\xi_\delta[\underline{A}]$ is a D-ideal on X for all $\underline{A} \in D(X)$, i.e. DI_2 holds.

For the condition DI_3, let $\underline{B} \in \xi_\delta[\underline{A}]$. Then, $B_2 \in \delta[A_2] \Rightarrow B_2 \in \mathcal{I}_{A_2^c}$, which implies that $B_2 \cap A_2 = \phi \Rightarrow A_1 \cap B_2 = \phi$ and $A_2 \cap B_1 = \phi \Rightarrow \underline{A}\overline{q}\underline{B} \Leftrightarrow \underline{B} \subseteq \underline{A}^c$ and so $\underline{B} \in \underline{\mathcal{I}}_{\underline{A}^c}$. Hence $\xi_\delta[\underline{A}] \subseteq \underline{\mathcal{I}}_{\underline{A}^c}$ i.e. DI_3 holds. Consequently, By Theorem 3.9, ξ_δ is a basic D- proximity on X. □

Example 5.3. Let X be a nonempty set and let $A\delta B \Leftrightarrow A \cap B \neq \phi$. Then δ is a basic proximity on X. The basic D-proximity which induced by δ is ξ_δ such that $\underline{A}\xi_\delta\underline{B} \Leftrightarrow A_2 \cap B_2 \neq \phi$.

Theorem 5.4. *Let (X, δ) be a basic proximity space. Then δ is an EF-proximity iff ξ_δ is an EF-D-proximity.*

Proof. Let δ be an EF-proximity and let $\underline{A} \in \xi_\delta[\underline{B}]$. Then $A_2 \in \delta[B_2] \Rightarrow N_\delta(A_2) \cap \delta[B_2] \neq \phi$. Then there exists $H \in P(X)$ such that $H \in N_\delta(A_2)$ and $H \in \delta[B_2] \Rightarrow H^c \in \delta[A_2], H \in \delta[B_2] \Rightarrow \underline{H}^c = (H^c, H^c) \in \xi_\delta[\underline{A}]$ and $\underline{H} = (H, H) \in \xi_\delta[\underline{B}] \Rightarrow \underline{H} \in N_{\xi_\delta}(\underline{A}), \underline{H} \in \xi_\delta[\underline{B}] \Rightarrow N_{\xi_\delta}(\underline{A}) \cap \xi_\delta[\underline{B}] \neq \underline{\phi}$. Hence, by Theorem 4.9, ξ_δ is an EF-D-proximity.

Conversely, Let ξ_δ is an EF-D-proximity and let $A \in \delta[B]$. Then there exists $A_1, B_1 \in P(X)$ such that $(A_1, A) \in \xi_\delta[(B_1, B)]$. So, by hypothesis, $\exists (H_1, H_2) \in D(X)$ such that $(H_1, H_2) \in \xi_\delta[(A_1, A)]$ and $(H_2^c, H_1^c) \in \xi_\delta[(B_1, B)] \Rightarrow H_2 \in \delta[A]$ and $H_1^c \in \delta[B]$. Since $\delta[B]$ is an ideal and $H_2^c \subseteq H_1^c$, then $H_2^c \in \delta[B]$. Hence δ is an EF-proximity. □

Corollary 5.5. *Let (X, δ) be a basic proximity space. Then δ is an g_0-proximity iff ξ_δ is an f_0-D-proximity.*

Theorem 5.6. *Let δ be any relation on the power set of a nonempty set X. Then δ is an RH- proximity iff ξ_δ is an RH-D-proximity.*

Proof. Let δ be an RH-proximity. Let $\underline{A} \in \xi_\delta[\underline{B}]$, then $A_2 \in \delta[B_2] \Leftrightarrow B_2 \in \delta[A_2] \Leftrightarrow \underline{B} \in \xi_\delta[\underline{A}]$. Hence DR_1 holds. Similarly, DR_2, DR_3 and DR_4 are hold. Now, let $x_t \in \xi_\delta[\underline{A}]$. Then $\{x\} \in \delta[A_2] \Rightarrow \exists H \in P(X)$ such that $H \in \delta[\{x\}]$ and $H^c \in \delta[A_2] \Rightarrow (H, H) \in \xi_\delta[x_t]$ and $(H^c, H^c) \in \xi_\delta[\underline{A}]$,i.e. there exists $\underline{H} = (H, H) \in D(X)$ such that $\underline{H} \in \xi_\delta[x_t]$ and $\underline{H}^c \in \xi_\delta[\underline{A}]$. Hence DR_5 holds. Consequently, ξ_δ is an RH-D-proximity.
The sufficiency of the Theorem is similar. $\square$

Corollary 5.7. *Let (X, δ) be a basic proximity space. Then δ is an h_0-proximity iff ξ_δ is an l_0-D-proximity.*

Theorem 5.8. *Let (X, δ) be a basic proximity space. Then ξ_δ is a separated-D-proximity $\Rightarrow$ δ is a separated proximity.*

Proof. Straightforward. $\square$

Definition 5.9. Let ξ be any arbitrary relation on $D(X)$. We define
$$\delta_\xi[A] = \{B \in P(X) : (B, B) \in \xi[(A, A)]\},\ A \in P(X).$$

Theorem 5.10. *Let (X, ξ) be a basic D-proximity space. Then δ_ξ is a basic proximity on X which is called the basic proximity induced by the basic D-proximity ξ on X.*

Proof. Let $A \in P(X)$.

I_1: Let $B \in \delta_\xi[A]$. Then $(B, B) \in \xi[(A, A)] \Rightarrow (A, A) \in \xi[(B, B)]$ and so $B \in \delta_\xi[B]$. Hence I_1 holds.

I_2: We shall show that $\delta_\xi[A]$ is an ideal on X. Since $\underline{\phi} \in \xi[(A, A)]\ \forall A \in P(X)$, then $\phi \in \delta_\xi[A]$ i.e. $\delta_\xi[A]$ is a nonempty. Let $B \in \delta_\xi[A]$ and $C \subseteq B$. Then $(B, B) \in \xi[(A, A)]$ and $(C, C) \subseteq (B, B) \Rightarrow (C, C) \in \xi[(A, A)] \Rightarrow C \in \delta_\xi[A]$. Now, let $B, H \in \delta_\xi[A]$. Then $(B, B), (H, H) \in \xi[(A, A)] \Rightarrow (B \cup H, B \cup H) \in \xi[(A, A)] \Rightarrow B \cup H \in \delta_\xi[A]$. Hence $\delta_\xi[A]$ is an ideal on X i.e. I_2 holds.

I_3: Let $B \in \delta_\xi[A]$. Then $(B, B) \in \xi[(A, A)] \Rightarrow B \subseteq A^c \Rightarrow B \in \mathcal{I}_{A^c}$. Hence $\delta_\xi[A] \subseteq \mathcal{I}_{A^c}$ i.e. I_3 holds. Consequently, δ_ξ is a basic proximity on X. $\square$

Example 5.11. Let X be a nonempty set and let $\underline{A}\xi\underline{B} \Leftrightarrow \underline{A}q\underline{B}$. Then the basic proximity which induced by ξ is δ_ξ such that $A\delta_\xi B \Leftrightarrow A \cap B \neq \phi$.

Theorem 5.12. *Let (X, δ) be a basic proximity space and $\underline{A} = (A_1, A_2), \underline{B} = (B_1, B_2) \in D(X)$. Then*
$$B_1 \in N_\delta(A_2) \Leftrightarrow \underline{B} \in N_{\xi_\delta}(\underline{A}).$$

Proof. Straightforward. $\square$

6. Continuity in basic D-proximity spaces

In this section, we introduced the concept of continuity in basic double proximity spaces.

Definition 6.1. Let (X, ξ_1) and (Y, ξ_2) be two basic D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$ be a map. Then f is called a basic D- proximally continuous (BDP-continuous, for short) map if $\underline{A}\xi_1\underline{B}$ implies $f(\underline{A})\xi_2 f(\underline{B})$.
Equivalently, if $f(\underline{A}) \in \xi_2[f(\underline{B})]$ implies $\underline{A} \in \xi_1[\underline{B}]$, $\forall \underline{A}, \underline{B} \in D(X)$.

If (X, ξ_1) and (Y, ξ_2) be two EF-D-proximity spaces, then f is called double proximally continuous (DP-continuous, for short).

Theorem 6.2. *Let (X, ξ_1) and (Y, ξ_2) be two basic D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$ be a map. Then*
f is a BDP-continuous if and only if $\underline{C} \in \xi_2[\underline{D}] \Rightarrow f^{-1}(\underline{C}) \in \xi_1[f^{-1}(\underline{D})]$ $\forall \underline{C}, \underline{D} \in D(Y)$.

Proof. Let $\underline{C}, \underline{D} \in D(Y)$ such that $\underline{C} \in \xi_2[\underline{D}]$. Since $ff^{-1}(\underline{D}) \subseteq \underline{D}$, then, by Theorem 3.6 (i), $\underline{C} \in \xi_2[ff^{-1}(\underline{D})]$. Since $\xi_2[ff^{-1}(\underline{D})]$ is a D-ideal on Y and $ff^{-1}(\underline{C}) \subseteq \underline{C}$, then $ff^{-1}(\underline{C}) \in \xi_2[ff^{-1}(\underline{D})]$. We claim that $f^{-1}(\underline{C}) \in \xi_1[f^{-1}(\underline{D})]$. In fact, if $f^{-1}(\underline{C}) \notin \xi_1[f^{-1}(\underline{D})]$, then $f^{-1}(\underline{C})\xi_1 f^{-1}(\underline{D}) \Rightarrow ff^{-1}(\underline{C})\xi_2 ff^{-1}(\underline{D})$ (by continuity of f) $\Rightarrow ff^{-1}(\underline{C}) \notin \xi_2[ff^{-1}(\underline{D})]$, a contradiction. Hence $f^{-1}(\underline{C}) \in \xi_1[f^{-1}(\underline{D})]$.
Conversely, let $\underline{A}, \underline{B} \in D(X)$ such that $f(\underline{A}) \in \xi_2[f(\underline{B})]$. Then , by hypothesis, $f^{-1}(f(A)) \in \xi_1[f^{-1}(f(\underline{B}))]$. Since $\xi_1[f^{-1}(f(\underline{B}))]$ is a D-ideal on X and $\underline{A} \subseteq f^{-1}f(\underline{A})$, then $\underline{A} \in \xi_1[f^{-1}(f(\underline{B}))] \Rightarrow f^{-1}(f(\underline{B})) \in \xi_1[\underline{A}]$, but $\underline{B} \subseteq f^{-1}(f(\underline{B}))$, then $\underline{B} \in \xi_1[\underline{A}]$. Hence f is a BDP-continuous. □

Theorem 6.3. *Let (X, ξ_1) and (Y, ξ_2) and (Z, ξ_3) be a basic D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$, $g : (Y, \xi_2) \to (Z, \xi_3)$ be two maps. Then*
If f and g are BDP-continuous maps, then gof is BDP-continuous map.

Proof. Straightforward. □

Theorem 6.4. *Let (X, ξ_1) and (Y, ξ_2) be two a basic D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$ be a map. Then*
If f is BDP-continuous, then $f : (X, \delta_{\xi_1}) \to (Y, \delta_{\xi_2})$ is BP-continuous.

Proof. Let $A, B \in P(X)$ and $A\delta_{\xi_1}B$. Then, $(A, A)\xi_1(B, B) \Rightarrow f(A, A)\xi_2 f(B, B)$(by continuity of f) $\Rightarrow (f(A), f(A))\xi_2(f(B), f(B)) \Rightarrow f(A)\delta_{\xi_2}f(B)$. Hence f is a BP-continuous. □

Theorem 6.5. *Let (X, δ_1) and (Y, δ_2) be two a basic proximity spaces and $f : (X, \delta_1) \to (Y, \delta_2)$ be a map. Then*
$f : (X, \delta_1) \to (Y, \delta_2)$ is BP-continuous $\Leftrightarrow f : (X, \xi_{\delta_1}) \to (Y, \xi_{\delta_2})$ is BDP-continuous.

Proof. Let $f : (X, \delta_1) \to (Y, \delta_2)$ be a BP-continuous, $\underline{C}, \underline{D} \in D(Y)$ and let $\underline{C} \in \xi_{\delta_2}[\underline{D}]$. Then $C_2 \in \delta_2[D_2]$
$\Rightarrow f^{-1}(C_2) \in \delta_1[f^{-1}(D_2)]$
$\Rightarrow (f^{-1}(C_1), f^{-1}(C_2)) \in \xi_{\delta_1}[(f^{-1}(D_1), f^{-1}(D_2))]$
$\Rightarrow f^{-1}(\underline{C}) \in \xi_{\delta_1}[f^{-1}(\underline{D})]$
Hence $f : (X, \xi_{\delta_1}) \to (Y, \xi_{\delta_2})$ is BDP-continuous.
Conversely, Let $f : (X, \xi_{\delta_1}) \to (Y, \xi_{\delta_2})$ be a BDP-continuous, $A, B \in P(Y)$ and $A \in \delta_2[B]$. Then $(A_1, A) \in \xi_{\delta_2}[(B_1, B)]$, for some $A_1, B_1 \in P(Y) \Rightarrow (f^{-1}(A_1), f^{-1}(A)) \in \xi_{\delta_1}[(f^{-1}(B_1), f^{-1}(B))] \Rightarrow f^{-1}(A) \in \delta_1[f^{-1}(B)]$. Hence $f : (X, \delta_1) \to (Y, \delta_2)$ is BP-continuous. □

Lemma 6.6. *Let (X, ξ_1) and (Y, ξ_2) be two EF-D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$ be a DP-continuous. Then*

$$\underline{V} \in N_{\xi_2}(\underline{H}) \Rightarrow c_{\xi_1}(f^{-1}(\underline{V}^c)) \subseteq f^{-1}(\underline{H}^c)\ \forall \underline{V}, \underline{H} \in D(Y).$$

Proof. Let $\underline{V}, \underline{H} \in D(Y)$ such that $\underline{V} \in N_{\xi_2}(\underline{H})$. Then $\underline{V}^c \in \xi_2[\underline{H}]$. It follows that, by Theorem 6.2, $f^{-1}(\underline{V}^c) \in \xi_1[f^{-1}(\underline{H})] \Rightarrow [f^{-1}(\underline{V}^c)]^c \in N_{\xi_1}(f^{-1}(\underline{H}))$. But, $[f^{-1}(\underline{V}^c)]^c = f^{-1}(\underline{V})$, then $f^{-1}(\underline{V}) \in N_{\xi_1}(f^{-1}(\underline{H})) \Rightarrow [f^{-1}(\underline{H})]^c \in N_{\xi_1}([f^{-1}(\underline{V})]^c)$ (by Theorem 3.17 (5)) $\Rightarrow f^{-1}(\underline{H}^c) \in N_{\xi_1}(f^{-1}(\underline{V}^c))$. Since, $c_{\xi_1}(f^{-1}(\underline{V}^c)) = \cap\{\underline{M} \in D(X) : \underline{M} \in N_{\xi_1}(f^{-1}(\underline{V}^c))\}$, then $c_{\xi_1}(f^{-1}(\underline{V}^c)) \subseteq f^{-1}(\underline{H}^c)$. □

Theorem 6.7. *Let (X, ξ_1) and (Y, ξ_2) be two EF-D-proximity spaces and $f : (X, \xi_1) \to (Y, \xi_2)$ be a map. Then f is a DP-continuous $\Rightarrow f : (X, \eta_{\xi_1}) \to (X, \eta_{\xi_2})$ is a D-continuous with respect to the double topologies η_{ξ_1} and η_{ξ_2}.*

Proof. Let $\underline{V} \in \eta_{\xi_2}$. Then $c_{\xi_2}(\underline{V}^c) = \underline{V}^c$, we shall show that $f^{-1}(\underline{V}) \in \eta_{\xi_1} \Leftrightarrow c_{\xi_1}(f^{-1}(\underline{V}^c)) = f^{-1}(\underline{V}^c)$. Clear that, $f^{-1}(\underline{V}^c) \subseteq c_{\xi_1}(f^{-1}(\underline{V}^c))$. Now, by Lemma 6.6, we have
$c_{\xi_1}(f^{-1}(\underline{V}^c)) \subseteq \cap\{f^{-1}(\underline{H}^c) : \underline{V} \in N_{\xi_2}(\underline{H})\}$
$= \cap\{f^{-1}(\underline{H}^c) : \underline{H}^c \in N_{\xi_2}(\underline{V}^c)\}$
$= f^{-1}[\cap\{\underline{H}^c \in D(Y) : \underline{H}^c \in N_{\xi_2}(\underline{V}^c)\}]$
$= f^{-1}(c_{\xi_2}(\underline{V}^c))$
$= f^{-1}(\underline{V}^c)$.
Hence $c_{\xi_1}(f^{-1}(\underline{V}^c)) \subseteq f^{-1}(\underline{V}^c)$. Consequently, $c_{\xi_1}(f^{-1}(\underline{V}^c)) = f^{-1}(\underline{V}^c)$ and $f^{-1}(\underline{V}) \in \eta_{\xi_1}$. Therefore, f is a D-continuous. □

7. Categorical point of view

In this section, we are going to find a categorical relationship between basic proximity spaces and basic D-proximity spaces.
Let CBP be the category of all basic proximity spaces and BP-continuous maps. Also, let CBDP be the category of all basic D-proximity spaces and BDP-continuous maps.

Definition 7.1 ([7])**.** A category $\mathcal{C}$ consists of a collection $\{C_\alpha : \alpha \in \Lambda\}$ of elements called **objects** and a collection $\{f_i : i \in I\}$ of elements called **mappings**.

Definition 7.2 ([7])**.** Let $\mathcal{C}$ and $\mathcal{F}$ be a categories and let F be a function which maps the objects of $\mathcal{C}$ into the objects of $\mathcal{F}$ and, in addition, assigns to each map $f \in \mathcal{C}$ a map $F(f) \in \mathcal{F}$. The map F is called a functor from $\mathcal{C}$ to $\mathcal{F}$ if it satisfies the following conditions:
For any $C, C_1, C_2, f, f_1, f_2 \in \mathcal{C}$

(1) $f : C_1 \to C_2 \Rightarrow F(f) : F(C_1) \to F(C_2)$.
(2) $F(i_C) = i_{F(C)}$.
(3) If $f_2.f_1$ is defined, then $F(f_2.f_1) = F(f_2).F(f_1)$.

Theorem 7.3. *Let $F : \underline{CBP} \to \underline{CBDP}$ defined by*
$$F(X, \delta) = (X, \xi_\delta) \text{ and } F(f) = f,$$
where, $\underline{A}\xi_\delta\underline{B} \Leftrightarrow A_2\delta B_2\ \forall \underline{A}, \underline{B} \in D(X)$. Then F is a functor.

Proof. Clearly, by Theorem 5.2, $F(X, \delta)$ is a basic D-proximity space. Now, we shall show that if $f : (X, \delta_1) \to (Y, \delta_2)$ is a BP-continuous, then $f : (X, \xi_{\delta_1}) \to (Y, \xi_{\delta_2})$ is a BDP-continuous. Let $\underline{A}, \underline{B} \in D(Y)$ such that $\underline{A} \in \xi_{\delta_2}[\underline{B}]$. Then $A_2 \in \delta_2[B_2] \Rightarrow f^{-1}(A_2) \in \delta_1[f^{-1}(B_2)]$. Since $f^{-1}(\underline{A}) = (f^{-1}(A_1), f^{-1}(A_2))$ and

$f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2))$, then $f^{-1}(\underline{A}) \in \xi_{\delta_1}[f^{-1}(\underline{B})]$. Hence $f : (X, \xi_{\delta_1}) \to (Y, \xi_{\delta_2})$ is a BDP-continuous. Therefore, F is a functor. □

Theorem 7.4. *Let $G : \underline{CBDP} \to \underline{CBP}$ defined by*

$$G(X, \xi) = (X, \delta_\xi) \text{ and } G(f) = f,$$

where, $A\delta_\xi B \Leftrightarrow (A, A)\xi(B, B)\ \forall A, B \in P(X)$. Then G is a functor.

Proof. Clearly, by Theorem 5.10, $G(X, \xi)$ is a basic proximity space. Now, we shall show that if $f : (X, \xi_1) \to (Y, \xi_2)$ is a BDP-continuous, then $f : (X, \delta_{\xi_1}) \to (Y, \delta_{\xi_2})$ is a BP-continuous. Let $A, B \in P(Y)$ such that $A \in \delta_{\xi_2}[B]$. Then $(A, A) \in \xi_2[(B, B)] \Rightarrow$ (by Theorem 6.2) $f^{-1}(A, A) \in \xi_1[f^{-1}(B, B)]$. Since $f^{-1}(A, A) = (f^{-1}(A), f^{-1}(A))$ and $f^{-1}(B, B) = (f^{-1}(B), f^{-1}(B))$, then $f^{-1}(A) \in \delta_{\xi_1}[f^{-1}(B)]$. Hence $f : (X, \delta_{\xi_1}) \to (Y, \delta_{\xi_2})$ is a BP-continuous. Therefore, G is a functor. □

Acknowledgements. The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editors-in-chief and managing editors for their important comments which helped to improve the presentation of the paper.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986) 87–96.
[2] K. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33(1) (1989) 37–45.
[3] K. Atanassov, New operators defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61(2) (1993) 131–142.
[4] D. Çoker, A note on intuitionistic sets and intuitionistic points, Turkish J. Math. 20(3) (1996) 343–351.
[5] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997) 81–99.
[6] D. Çoker, An introduction to intuitionistic topological spaces, BUSEFAL. 81 (2000) 51–66.
[7] Eilenberg and Maclane, Categories and Functors, Trans. Amer. Math. Soc. 58 (1945) 231–294.
[8] J. G. Garcia and S. E. Rodabaugh, Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies, Fuzzy Sets and Systems 156(3) (2005) 445–484.
[9] R. A. Hosny and O. A. E. Tantawy, New proximities from old via ideals, Acta Math. Hungar. 110(1-2) (2006) 37–50.
[10] D. S. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97(4) (1990) 295–310.
[11] A. Kandil, S. A. El-Sheikh, M. M. Yakout and Shawqi A. Hazza, Proximity structures and ideals, accepted.
[12] A. Kandil, O. A. E. Tantawy and M. Abdelhakem, Compatibility relation and new operator on flou topological spaces with flou ideals, Bull. Fac. of Sci., Zagazig Univ. 32 (2010).
[13] A. Kandil, O. A. E. Tantawy and M. Abdelhakem, Flou topological spaces via flou ideals, Int. J. Appl. Math. 23(5) (2010) 873–885.
[14] A. Kandil, O. A. E. Tantawy and M. Abdelhakem, New double proximities from old via double ideal, submitted.
[15] A. Kandil, O. A. E. Tantawy and M. Wafaie, On flou (INTUITIONISTIC) topological spaces, J. Fuzzy Math. 15(2) (2007).
[16] K. Kuratowski, Topology, New York: Academic Press, vol. I. 1966.
[17] M. W. Lodato, On topologically induced general proximity relations, I, Proc. Amer. Math. Soc. 15 (1964) 417–433.
[18] M. W. Lodato, On topologically induced general proximity relations, II, Pacific J. Math. 17 (1966) 133–135.

[19] S. A. Saleh, On interval valued fuzzy topological spaces, Ph. D. thesis in Math., Fac. of Edu., Ain Shams Univ., Egypt (2009).
[20] R. Vaidyanathaswamy, The localisation theory in set topology, Proc. Indian Acad. Sci., Sect. A. 20, (1944) 51–61.

A. Kandil (dr.ali-kandil@yahoo.com)
Department of Mathematics, Faculty of Science, Helwan University, Helwan, Egypt

S. A. El-Sheikh (sobhyelsheikh@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt

M. M. Yakout (mmyakout@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt

Shawqi A. Hazza (shawqialalimi@yahoo.com)
Department of Mathematics, Faculty of Education, Taiz University, Taiz, Yemen