Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 5, (May 2014), pp. 755–763
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy prime ideals in ternary semirings

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Received 13 July 2013; Revised 18 August 2013; Accepted 15 September 2013

Abstract. In this paper we introduce the notions of Fuzzy Prime ideals and Fuzzy m-systems in ternary semirings. We have shown that the image of fuzzy prime ideal will contain only two elements in ternary semirings. We have shown that μ is a fuzzy prime ideal if and only if $1 - \mu$ is a fuzzy m-system.

2010 AMS Classification: 08A72, 16Y60, 20M12, 17A40

Keywords: Fuzzy ideals, Fuzzy Prime ideal, Fuzzy m-system.

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1. Introduction

The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [13] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [2] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [2, 3, 4, 5, 6, 7, 8, 9]. The theory of fuzzy sets was first studied by Zadeh [15] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Kavikumar et al.[10] and [11] studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings. Ronnason Chinram et al. [14] studied L-fuzzy ideals in ternary semirings. Recently A.H. Mohammed[1] studied Anti fuzzy k-ideals of ternary semirings. In this paper we introduce the notions of Fuzzy prime ideals and Fuzzy m-systems in ternary semirings. We have shown that the image of fuzzy prime ideal will contain only two elements in ternary semirings. In fact we have shown that the image of a fuzzy prime ideals in any algebraic structure will contain only two elements. We have shown that μ is a fuzzy prime ideal if and only if $1 - \mu$ is a fuzzy m-system.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. A nonempty set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring if $(S,+)$ is a commutative semigroup, $(S,\cdot)$ is a semigroup and multiplicative distributes over addition both from the left and the right, i.e., $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a,b,c \in S$.

Definition 2.2. A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S,+)$ is a commutative semigroup satisfying the following conditions:
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$ and
(iv) $ab(c+d) = abc + abd$ for all $a,b,c,d,e \in S$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider Z_0^-, the set of all non-positive integers under usual addition and multiplication, we see that Z_0^- is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, Z_0^- is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.3. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0+x=x=x+0$ and $0xy=x0y=xy0=0$ for all $x,y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Definition 2.4. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1,t_2,t_3 \in T$.

Definition 2.5. An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1,s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.6. A proper ideal P of a ternary semiring S is called a prime ideal of S, if $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three ideals A,B,C of S.

Let X be a non-empty set. A map $\mu : X \to [0,1]$ is called a fuzzy set in X.

Definition 2.7. Let f ,g and h be any three fuzzy subsets of a ternary semiring S. Then $f\cap g,\ f\cup g,\ f+g,\ f\circ g\circ h$ are fuzzy subsets of S defined by

$$(f\cap g)(x) = min\{f(x),\ g(x)\}$$
$$(f\cup g)(x) = max\{f(x),\ g(x)\}$$
$$(f+g)(x) = \begin{cases} sup\{min\{f(y),g(z)\}\} & if \quad x=y+z \\ 0 & otherwise \end{cases}$$

$$(f \circ g \circ h)(x) = \begin{cases} sup\{min\{f(u), g(v), h(w)\}\} & if \quad x = uvw, \\ 0 \qquad otherwise \end{cases}$$

Definition 2.8. Let X be a nonempty set and let μ be a fuzzy subset of X. Let $0 \leq t \leq 1$. Then the set $\mu_t = \{x \in X \ / \ \mu(x) \geq t\}$ is called a level set of X with respect to μ.

Definition 2.9. For any $x \in S$ and $t \in [0, 1]$, we define the fuzzy point x_t as

$$x_t(y) = \begin{cases} t & if \quad y = x \\ 0 & if \quad y \neq x. \end{cases}$$

If x_t is a fuzzy point and μ is any fuzzy subset of S and $x_t \subseteq \mu$, then we write $x_t \in \mu$. Note that $x_t \in \mu$, if and only if $x \in \mu_t$, where μ_t is a level subset of μ. For any fuzzy subset f of S, it is obvious that $f = \bigcup\limits_{a_t \in f} a_t$.

Definition 2.10. Let μ be a fuzzy set of a ternary semiring S. Then μ is called a fuzzy ternary subsemiring of S if
1. $\mu(x + y) \geq \mu(x) \wedge \mu(y)$
2. $\mu(xyz) \geq \mu(x) \wedge \mu(y) \wedge \mu(z)$ for all $x, y, z \in S$.

Definition 2.11. A fuzzy set μ of a ternary semiring S is called a fuzzy ideal of S if

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$
(ii) $\mu(xyz) \geq \mu(x)$
(iii) $\mu(xyz) \geq \mu(z)$ and
(iv) $\mu(xyz) \geq \mu(y)$ for all $x, y, z \in S$.

A fuzzy subset μ with conditions (i) and (ii) is called a fuzzy right ideal of S. If μ satisfies (i)and(iii), then it is called a fuzzy left ideal of S. Also if μ satisfies (i)and(iv), then it is called a fuzzy lateral ideal of S. It is clear that μ is a fuzzy ideal of a ternary semiring S if and only if $\mu(xyz) \geq max\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in S$.

Theorem 2.12 ([10])**.** *Let μ be a fuzzy set of a ternary semiring S. Then μ is a fuzzy right (resp. left, lateral) ideal of S if and only if $\mu_t \neq \phi$ is a right (resp. left, lateral) ideal of S for all $t \in [0, 1]$.*

Theorem 2.13 ([14])**.** *Let μ be a fuzzy set of a ternary semiring S and $\mu_0 = \{x \in S \ /\mu(x) = \mu(0)\}$. If μ is a fuzzy right (resp. left, lateral) ideal of S then μ_0 is a ternary right (resp. left, lateral) ideal of S.*

Proposition 2.14 ([8])**.** *Let S be a ternary semiring and $a \in S$. Then the principal ideal generated by a is given by $\langle a \rangle = \{\sum p_i q_i a + \sum a r_j s_j + \sum u_k a v_k + \sum p'_l q'_l a r'_l s'_l + na / p_i, q_i, r_j, s_j, u_k, v_k, p'_l, q'_l, r'_l, s'_l \in S; n \in Z_0^+\}$.*

Definition 2.15. A subset M of a ternary semiring S is called an m-system if for every $a, b, c \in M$, there exist $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$ and $c_1 \in \langle c \rangle$ such that $a_1 b_1 c_1 \in M$.

Note: Dutta and Kar [6] defined m-system in Ternary semirings as follows: A non-empty subset A of S is called an m-system if for each $a, b, c \in A$ there exist elements x_1, x_2, x_3, x_4 of S such that $a x_1 b x_2 c \in A$ or $a x_1 x_2 b x_3 x_4 c \in A$ or $a x_1 x_2 b x_3 c x_4 \in A$

or $x_1ax_2bx_3x_4c \in A$.
If A is a m-system defined by Kar [6] then it is clearly a m-system defined as in definition 2.15. If M is a m-system defined as in definition 2.15 such that M does not contain zero, then using Zorn's lemma we can obtain a prime ideal P such that $P \cap M = \phi$. Then $S \setminus P = A$ is an m-system defined by Kar [6] containing M. Thus m-system by Kar [6] implies m-system defined by us in definition 2.15 and if M defined by us then there exists a m-system A defined by Kar [6] containing M. Hence M is the largest m-system defined as in definition 2.15 then clearly M is a m-system defined by Kar [6].
The following two Lemmas can be easily verified.

Lemma 2.16. *Let f_1, f_2 and f_3 be any three fuzzy subsets of S. If $f_1 \le f$, $f_2 \le g$ and $f_3 \le h$, then $f_1 \circ f_2 \circ f_3 \le f \circ g \circ h$ for any fuzzy subsets f, g and h.*

Lemma 2.17. *Let a_r, b_s and c_t be any three fuzzy points of S such that $a_r \in f$, $b_s \in g$ and $c_t \in h$, where f, g and h are any fuzzy subset of S. Then $a_r \circ b_s \circ c_t \in f \circ g \circ h$.*

3. Fuzzy prime ideals

Throughout this paper S denotes a ternary semiring with zero.

Definition 3.1. A fuzzy ideal μ of a ternary semiring S is said to be a fuzzy prime ideal of S if

(i) μ is not a constant function and
(ii) for any fuzzy ideals f, g, h in S if $f \circ g \circ h \subseteq \mu$, then $f \subseteq \mu$ or $g \subseteq \mu$ or $h \subseteq \mu$.

Note: $f \subseteq \mu$ means $f(x) \le \mu(x)$ for all $x \in S$.

Example 3.2. Consider the ternary semiring $S = Z_6^- = \{0, -1, -2, -3, -4, -5\}$ with the usual addition and ternary multiplication,
we have

+	0	-1	-2	-3	-4	-5
0	0	-1	-2	-3	-4	-5
-1	-1	-2	-3	-4	-5	0
-2	-2	-3	-4	-5	0	-1
-3	-3	-4	-5	0	-1	-2
-4	-4	-5	0	-1	-2	-3
-5	-5	0	-1	-2	-3	-4

·	0	-1	-2	-3	-4	-5
0	0	0	0	0	0	0
-1	0	1	2	3	4	5
-2	0	2	4	0	2	4
-3	0	3	0	3	0	3
-4	0	4	2	0	4	2
-5	0	5	4	3	2	1

·	0	1	2	3	4	5
0	0	0	0	0	0	0
-1	0	-1	-2	-3	-4	-5
-2	0	-2	-4	0	-2	-4
-3	0	-3	0	-3	0	-3
-4	0	-4	-2	0	-4	-2
-5	0	-5	-4	-3	-2	-1

Let a fuzzy subset $\mu : Z_6^- \to [0, 1]$ be defined by $\mu(0) = 1$, $\mu(-1) = 0.3$, $\mu(-2) = 1$, $\mu(-3) = 0.3$, $\mu(-4) = 1$ and $\mu(-5) = 0.3$. Then μ is a fuzzy prime ideal of S.

Theorem 3.3. *Let S be a ternary semiring and let μ be a fuzzy prime ideal of S. Then $|Im(\mu)| = 2$; that is μ is two-valued.*

Proof. Since μ is not constant, $|Im(\mu)| \geq 2$. Suppose that $|Im(\mu)| \geq 3$. Let $\mu(0) = s$ and $k = g.l.b.\{\mu(x) : x \in S\}$. Then there exists $t, m \in Im(\mu)$ such that $t < m < s$ and $t \geq k$. Let $t_1 = \frac{1}{2}(t + m)$, $t_2 = \frac{1}{2}(t_1 + m)$. Clearly $t < t_1 < t_2 < m$. Let f, g and h be fuzzy subsets of S such that $f(x) = t_1$ and $g(x) = t_2$ for all $x \in S$ and $h(x) = k$ if $x \notin \mu_m$ and $h(x) = s$ if $x \in \mu_m$. Clearly f and g are fuzzy ideals of S. We now show that h is a fuzzy ideal of S. Let $x, y, z \in S$. If $x, y \in \mu_m$ then $x + y \in \mu_m$ and $h(x + y) = s = min(h(x), h(y))$. If $x \in \mu_m$ and $y \notin \mu_m$, then $h(y) = k$ and $h(x + y) \geq k = min(h(x), h(y))$. If $x \notin \mu_m$ and $y \notin \mu_m$ then $h(x) = h(y) = k$ and thus $h(x + y) \geq k = min(h(x), h(y))$. Hence $h(x+y) \geq min(h(x), h(y))$ for all $x, y \in S$. Now if $x \in \mu_m$, then $xyz, yxz, yzx \in \mu_m$ since μ_m is an ideal. Thus in this case $h(xyz) = h(yxz) = h(yzx) = s = h(x)$. If $x \notin \mu_m$ then $h(xyz) \geq k = h(x)$, $h(yxz) \geq k = h(x)$ and $h(yzx) \geq k = h(x)$. Consequently h is a fuzzy ideal of S. We now claim that $f \circ g \circ h \subseteq \mu$. Let $x \in S$. Consider the following cases,

(i) $x = 0$. Then $f \circ g \circ h(x) = \sup\limits_{x=bcd} \{min(f(b), g(c), h(d))\} \leq t_1 < s = \mu(0)$.

(ii) $x \neq 0$, $x \in \mu_m$. Then $\mu(x) \geq m$, and $f \circ g \circ h(x) = \sup\limits_{x=bcd} \{min(f(b), g(c), h(d))\}$ $\leq t_1 < m \leq \mu(x)$.

(iii) $x \neq 0$, $x \notin \mu_m$. Then for any $b, c, d \in S$ such that $x = bcd$, $b \notin \mu_m$, $c \notin \mu_m$ and $d \notin \mu_m$. Thus $h(d) = k$. Hence $f \circ g \circ h(x) = \sup\limits_{x=bcd} \{min(f(b), g(c), h(d))\} =$ $k \leq \mu(x)$.

Thus in any case, $f \circ g \circ h(x) \leq \mu(x)$. Hence $f \circ g \circ h \subseteq \mu$. Now there exists $u \in S$ such that $\mu(u) = t$. Then $f(u) = t_1 > \mu(u)$ and $g(u) = t_2 > \mu(u)$. Hence $f \nsubseteq \mu$ and $g \nsubseteq \mu$. Also there exists $x \in S$ such that $\mu(x) = m$. Then $x \in \mu_m$ and thus $h(x) = s > m = \mu(x)$. Hence $h \nsubseteq \mu$. Thus $f \nsubseteq \mu$, $g \nsubseteq \mu$ and $h \nsubseteq \mu$. This shows that μ is not a fuzzy prime ideal of S, which is a contradiction of the hypothesis. Hence $|Im(\mu)| = 2$. □

Remark 3.4. The image of a fuzzy prime ideal with usual definition in any algebraic structure like rings, near-rings, semirings, ternary semirings, n-nary semirings will contain only two elements. Since between any two real numbers we can find any finite number of real numbers, using the same method in the previous theorem, the image of fuzzy prime ideal will contain only two elements in any algebraic structure.

Theorem 3.5. *Let μ be a fuzzy prime ideal of S. Then $\mu(0) = 1$.*

Proof. Since μ is a fuzzy prime ideal, by Theorem 3.3, $|Im(\mu)| = 2$. Let $Im(\mu) = \{t, s\}$ and $t < s$. Let $t_1 = \frac{1}{2}(t + s)$, $t_2 = \frac{1}{2}(t_1 + s)$. Clearly $t < t_1 < t_2 < s$. Then $\mu(0) = s$. Suppose that $s \neq 1$. Let $s < m \leq 1$. Let f, g and h be fuzzy subsets of S such that $f(x) = t_1$, $g(x) = t_2$ for all $x \in S$ and $h(x) = t$ if $x \notin \mu_0$ and $h(x) = m$ if $x \in \mu_0$, where $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$. Clearly f and g are fuzzy ideals of S. Since μ_0 is an ideal of S, h is a fuzzy ideal of S. It can be easily checked that $f \circ g \circ h \subseteq \mu$. Since $\mu(0) = s < m = h(0)$, $h \nsubseteq \mu$. Also there exists $x \in S$ such that $\mu(x) = t < t_1 = f(x)$ and $\mu(x) = t < t_2 = g(x)$. Thus $f \nsubseteq \mu$ and $g \nsubseteq \mu$. Hence $f \nsubseteq \mu$, $g \nsubseteq \mu$ and $h \nsubseteq \mu$. This is a contradiction to the hypothesis that μ is a fuzzy prime ideal of S. Hence $\mu(0) = 1$. □

Theorem 3.6. *If μ is a fuzzy prime ideal of S. Define $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$. Then μ_0 is a prime ideal in S.*

Proof. Let $x, y \in \mu_0$. Then $\mu(x+y) \geq min\{\mu(x), \mu(y)\} = \mu(0)$. This shows that $x+y \in \mu_0$. If $x \in \mu_0$ and $y, z \in S$, then $\mu(xyz) \geq \mu(x) = \mu(0)$ and so $\mu(xyz) = \mu(0)$. Hence $xyz \in \mu_0$. Similarly $yxz, yzx \in \mu_0$. Consequently μ_0 is an ideal in S. Let I, J and K be ideals of S such that $IJK \subseteq \mu_0$. Define the fuzzy ideals f, g and h by $f = \chi_I$, $g = \chi_J$ and $h = \chi_K$. Let us now show that $f \circ g \circ h(x) \leq \mu(x)$ for all $x \in S$. If $f \circ g \circ h(x) = 0$, there is nothing to show. Now $f \circ g \circ h(x) = \sup\limits_{x=abc} \{min\{f(a), g(b), h(c)\}\}$ and we have to consider only the cases where $min\{f(a), g(b), h(c)\} \neq 0$. For all these cases $f(a) = g(b) = h(c) = 1$. Now $f(a) = g(b) = h(c) = 1$ implies $a \in I$, $b \in J$ and $c \in K$, so that $x \in IJK \subseteq \mu_0$. Hence $\mu(x) = \mu(0)$, so $f \circ g \circ h(x) \leq \mu(x)$ for all $x \in S$. As μ is a fuzzy prime ideal and f, g, h are fuzzy ideals, it follows that $f \subseteq \mu$ or $g \subseteq \mu$ or $h \subseteq \mu$. Suppose $f \subseteq \mu$. Then $\chi_I \subseteq \mu$. We shall show that $I \subseteq \mu_0$. Suppose $I \nsubseteq \mu_0$. There exists $a \in I$ such that $a \notin \mu_0$. This means that $\mu(a) \neq \mu(0)$. Hence $\mu(a) < \mu(0)$. Then $f(a) = \chi_I(a) = 1 = \mu(0) > \mu(a)$ contradicts the assumption that $f \subseteq \mu$. Hence $I \subseteq \mu_0$. Similarly if $g \subseteq \mu$ and $h \subseteq \mu$, we can show that $J \subseteq \mu_0$ and $K \subseteq \mu_0$. Hence μ_0 is a prime ideal. □

Lemma 3.7. *If μ is a fuzzy ideal of S and $a \in S$ then $\mu(x) \geq \mu(a)$ for all $x \in \langle a \rangle$.*

Proof. Let μ be a fuzzy ideal of S and $x \in \langle a \rangle$ for any $a \in S$.
Clearly $x = \sum p_i q_i a + \sum a r_j s_j + \sum u_k a v_k + \sum p'_l q'_l a r'_l s'_l + na$
for some $p_i, q_i, r_j, s_j, u_k, v_k, p'_l, q'_l, r'_l, s'_l \in S$ and for some $n \in Z_0^+$.
Then $\mu(x) = \mu(\sum p_i q_i a + \sum a r_j s_j + \sum u_k a v_k + \sum p'_l q'_l a r'_l s'_l + na)$
$\geq min\{\mu(\sum p_i q_i a), \mu(\sum a r_j s_j), \mu(\sum u_k a v_k), \mu(\sum p'_l q'_l a r'_l s'_l), \mu(na)\}$
consider $\mu(\sum\limits_{i=1}^{k} p_i q_i a) \geq min\{\mu(p_1 q_1 a), \mu(p_2 q_2 a), ..., \mu(p_k q_k a)\}$
$\geq min\{\mu(a), \mu(a), ..., \mu(a)\} = \mu(a)$.
Similarly $\mu(\sum a r_j s_j) \geq \mu(a)$, $\mu(\sum u_k a v_k) \geq \mu(a)$, $\mu(\sum p'_l q'_l a r'_l s'_l) \geq \mu(a)$, $\mu(na) \geq \mu(a)$. Therefore $\mu(x) \geq \mu(a)$. □

Theorem 3.8. *Let S be a ternary semiring with zero and let μ be a fuzzy subset of S such that $|Im(\mu)| = 2$, $\mu(0) = 1$ and the set $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ is a prime ideal of S. Then μ is a fuzzy prime ideal of S.*

Proof. Let $Im(\mu) = \{t, 1\}, t < 1$ and $\mu(0) = 1$. Let $x, y, z \in S$. If $x, y \in \mu_0$, then $x + y \in \mu_0$ and $\mu(x+y) = 1 = min(\mu(x), \mu(y))$. If $x \in \mu_0$ and $y \notin \mu_0$, then $\mu(y) = t$ and $\mu(x+y) \geq t = min(\mu(x), \mu(y))$. If $x \notin \mu_0$ and $y \notin \mu_0$, then $\mu(x) = \mu(y) = t$ and thus $\mu(x+y) \geq t = min(\mu(x), \mu(y))$. Hence $\mu(x+y) \geq min(\mu(x), \mu(y))$ for all $x, y \in S$. Now if $x \in \mu_0$, then $xyz, yxz, yzx \in \mu_0$ and $\mu(xyz) = \mu(yxz) = \mu(yzx) = 1 = \mu(x)$. If $x \notin \mu_0$, then $\mu(xyz) \geq t = \mu(x)$, $\mu(yxz) \geq t = \mu(x)$ and $\mu(yzx) \geq t = \mu(x)$. Hence μ is a fuzzy ideal of S. Let f, g and h be fuzzy ideals of S such that $f \circ g \circ h \subseteq \mu$. Suppose that $f \nsubseteq \mu$, $g \nsubseteq \mu$ and $h \nsubseteq \mu$. Then there exists $x, y, z \in S$ such that $f(x) > \mu(x)$, $g(y) > \mu(y)$ and $h(z) > \mu(z)$. Clearly $\mu(x) = \mu(y) = \mu(z) = t$ implies $x \notin \mu_0$, $y \notin \mu_0$ and $z \notin \mu_0$ hence $\langle x \rangle \langle y \rangle \langle z \rangle \nsubseteq \mu_0$. Now, since μ_0 is a prime ideal of S there exist $x_1 \in \langle x \rangle$, $y_1 \in \langle y \rangle$ and $z_1 \in \langle z \rangle$ such that $x_1 y_1 z_1 \notin \mu_0$. Let $a = x_1 y_1 z_1$. Then $\mu(a) = \mu(x) = \mu(y) = \mu(z) = t$.

Now, $f \circ g \circ h(a) = \sup_{a=uvw} \{min(f(u), g(v), h(w))\} \geq min(f(x_1), g(y_1), h(z_1)) \geq min(f(x), g(y), h(z)) > t = \mu(a)$. That is, $f \circ g \circ h \nsubseteq \mu$. This contradicts the assumption that $f \circ g \circ h \subseteq \mu$. Thus $f \subseteq \mu$ or $g \subseteq \mu$ or $h \subseteq \mu$. Thus μ is a fuzzy prime ideal of S. □

Corollary 3.9. *Let I be a prime ideal of S. Then χ_I, the characteristic function of I, is a fuzzy prime ideal of S.*

Theorem 3.10 is an immediate consequence of Theorem 3.3, Theorem 3.5, Theorem 3.6 and Theorem 3.8.

Theorem 3.10. *Let μ be a fuzzy subset of S. μ is a fuzzy prime ideal of S if and only if $|Im(\mu)| = 2$, $\mu(0) = 1$ and the set $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ is a prime ideal of S.*

Theorem 3.11. *If μ is a fuzzy prime ideal of S then the level set μ_t is a prime ideal in S for all $t \in [0, 1]$.*

Proof. If μ is a fuzzy prime ideal then the values of μ has only two elements 1 and s where $s \in [0, 1)$. Since μ_1 is a prime ideal, then $\mu_t = \mu_1$ for all $t > s$ is a prime ideal and $\mu_t = S$ for all $t \leq s$. Thus μ_t is a prime ideal in S for all $t \in [0, 1]$. □

Remark 3.12. The following example shows that the converse of the above theorem is not true.

Example 3.13. Consider the ternary semiring $S = Z_0^-$ with usual addition and ternary multiplication. Define $\mu : Z_0^- \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 \quad if \quad x = 0 \\ 0.6 \quad if \quad x \in \langle -2 \rangle \quad and \quad x \neq 0 \\ 0 \quad otherwise. \end{cases}$$

For any $t \in [0, 1]$, $\mu_t = \{0\}$ or $\langle -2 \rangle$ or S. Hence μ_t is a prime ideal in S for all $t \in [0, 1]$. But μ is not a fuzzy prime ideal of S as $|Im\mu| > 2$.

4. Fuzzy m-systems

Definition 4.1. A fuzzy subset μ of S is said to be a fuzzy m-system if for any $r, s, t \in [0, 1)$ and $x, y, z \in S$, $\mu(x) > r, \mu(y) > s, \mu(z) > t$ implies that there exist $x_1 \in \langle x \rangle$, $y_1 \in \langle y \rangle$ and $z_1 \in \langle z \rangle$ such that $\mu(x_1 y_1 z_1) > max\{r, s, t\}$.

Example 4.2. Consider the ternary semiring $(Z_6^-, +, \cdot)$ as defined in Example 3.2. Let a fuzzy subset $\mu : Z_6^- \to [0, 1]$ be defined by $\mu(0) = 0$, $\mu(-1) = 1$, $\mu(-2) = 0$, $\mu(-3) = 1$, $\mu(-4) = 0$ and $\mu(-5) = 1$. Then μ is a fuzzy m-system of Z_6^-.

Example 4.3. Consider the ternary semiring $(Z_6^-, +, \cdot)$ as defined in Example 3.2. Let a fuzzy subset $\mu : Z_6^- \to [0, 1]$ be defined by $\mu(0) = 0$, $\mu(-1) = 0.4$, $\mu(-2) = 0.4$, $\mu(-3) = 0$, $\mu(-4) = 0.4$ and $\mu(-5) = 0.4$. Then μ is a fuzzy m-system of Z_6^-.

Theorem 4.4. *Let M be a subset of S. M is an m-system in S if and only if the characteristic function of M, χ_M is a fuzzy m-system of S.*

Proof. Let M be an m-system in S. For any $r, s, t \in [0, 1)$, suppose there exist $a, b, c \in S$ such that $\chi_M(a) > r,\ \chi_M(b) > s,\ \chi_M(c) > t$. Hence $a, b, c \in M$. As M is an m-system in S, there exist $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$ and $c_1 \in \langle c \rangle$ such that $a_1b_1c_1 \in M$, and hence $\chi_M(a_1b_1c_1) = 1$. Thus $\chi_M(a_1b_1c_1) > max\{r, s, t\}$.
Conversely, let us assume that χ_M is a fuzzy m-system of S. Let $a, b, c \in M$. Then $\chi_M(a) = \chi_M(b) = \chi_M(c) = 1$. Thus for any $r, s, t \in [0, 1),\ \chi_M(a) > r,\ \chi_M(b) > s,\ \chi_M(c) > t$. Hence there exist $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$ and $c_1 \in \langle c \rangle$ such that $\chi_M(a_1b_1c_1) > max\{r, s, t\}$. Therefore $\chi_M(a_1b_1c_1) = 1$ and hence $a_1b_1c_1 \in M$. □

Remark 4.5. Let μ be a fuzzy subset in S. μ holds a property like subgroup, ideal etc., if and only if its level set μ_t in S also satisfies the same property in S. However, μ is a fuzzy subset in S such that the level set μ_t in S is an m-system in S, for all $t \in [0, 1]$, does not imply μ is a fuzzy m-system of S as the following example shows.

Example 4.6. Consider the ternary semiring $(Z_6^-, +, \cdot)$ as defined in Example 3.2. Define $\mu : Z_6^- \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 \quad if \quad x = -1 \\ 0.5 \quad if \quad x = -3 \\ 0.4 \quad if \quad x = -5 \\ 0 \quad otherwise. \end{cases}$$

For any $t \in [0, 1],\ \mu_t = \{-1\}$ or $\{-1, -3\}$ or $\{-1, -3, -5\}$ or $\{0, -1, -2, -3, -4, -5\}$. Hence μ_t is an m-system in S for all t. But μ is not a fuzzy m-system in S, since $\mu(-1) > 0.9,\ \mu(-3) > 0.4,\ \mu(-5) > 0.3$ but there is no $a_1 \in \langle -1 \rangle$, $b_1 \in \langle -3 \rangle$ and $c_1 \in \langle -5 \rangle$ such that $\mu(a_1b_1c_1) > max\{0.9, 0.4, 0.3\}$.

Theorem 4.7. *If μ is a fuzzy m-system of S then $\mu^r = \{x \in S/\mu(x) > r\}$ is an m-system in S for all $r \in [0, 1)$.*

Proof. Let μ be a fuzzy m-system of S. Let $x, y, z \in \mu^r$, for some $r \in [0, 1)$. This implies that $\mu(x) > r$, $\mu(y) > r$ and $\mu(z) > r$. As μ is a fuzzy m-system of S, there exist $x_1 \in \langle x \rangle$, $y_1 \in \langle y \rangle$ and $z_1 \in \langle z \rangle$ such that $\mu(x_1y_1z_1) > r$ implies $x_1y_1z_1 \in \mu^r$. Thus μ^r is an m-system in S. □

Theorem 4.8. *Let μ be a fuzzy subset of S with $x_1 \in \langle x \rangle$ implies $\mu(x_1) \geq \mu(x)$ then μ is a fuzzy m-system of S.*

Proof. Let $\mu(x) > r$, $\mu(y) > s$ and $\mu(z) > t$, for some $r, s, t \in [0, 1)$ and $x, y, z \in S$. Let us assume that $t > s > r$. Since $xyz \in \langle z \rangle$ then $\mu(xyz) \geq \mu(z) > t = max\{r, s, t\}$. Thus μ is a fuzzy m-system of S. □

Theorem 4.9. *Let μ be a fuzzy ideal of S. μ is a fuzzy prime ideal of S if and only if $1 - \mu$ is a fuzzy m-system of S.*

Proof. Let us assume that μ is a fuzzy prime ideal of S. For any $r, s, t \in [0, 1)$, suppose there exist $a, b, c \in S$ such that $1 - \mu(a) > r, 1 - \mu(b) > s$ and $1 - \mu(c) > t$. Hence $\mu(a) < 1 - r, \mu(b) < 1 - s$ and $\mu(c) < 1 - t$. As μ is a fuzzy prime ideal of S, $Im(\mu) = \{1, \alpha\}$, $\alpha \in [0, 1)$. Thus $\mu(a) = \mu(b) = \mu(c) = \alpha$ and $\alpha < 1 - r$, $\alpha < 1 - s$, $\alpha < 1 - t$. Let $P = \{x \in S/\mu(x) = 1\}$. Then by Theorem 3.10, P is a prime ideal in S and $a, b, c \notin P$. This implies $a, b, c \in S \backslash P$ which is an m-system in S. Thus there exist $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$ and $c_1 \in \langle c \rangle$ such that $a_1b_1c_1 \in S \setminus P$ which implies $\mu(a_1b_1c_1) = \alpha$. Now $\mu(a_1b_1c_1) = \alpha < min\{(1-r), (1-s), (1-t)\} = 1 - max\{r, s, t\}$. Now $max\{r, s, t\} < 1 - \mu(a_1b_1c_1)$.

Conversely, let us assume that $1 - \mu$ is a fuzzy m-system of S. Let μ_1, μ_2 and μ_3 be any three fuzzy ideals such that $\mu_1 \circ \mu_2 \circ \mu_3 \subseteq \mu$. Suppose that $\mu_1 \nsubseteq \mu$, $\mu_2 \nsubseteq \mu$ and $\mu_3 \nsubseteq \mu$. Now $\mu_1 = \bigcup\limits_{a_u \in \mu_1} a_u$, $\mu_2 = \bigcup\limits_{b_v \in \mu_2} b_v$ and $\mu_3 = \bigcup\limits_{c_w \in \mu_3} c_w$. Then there exist $a_r \in \mu_1$, $b_s \in \mu_2$, and $c_t \in \mu_3$, $r, s, t \in [0, 1)$, such that $\mu(a) < r, \mu(b) < s$ and $\mu(c) < t$. This implies that $1 - \mu(a) > 1 - r$, $1 - \mu(b) > 1 - s$ and $1 - \mu(c) > 1 - t$. As $1 - \mu$ is a fuzzy m-system of S, there exist $a_1 \in \langle a \rangle$, $b_1 \in \langle b \rangle$ and $c_1 \in \langle c \rangle$ such that $(1 - \mu)(a_1b_1c_1) > max\{(1 - r), (1 - s), (1 - t)\} = 1 - min\{r, s, t\}$. Thus $\mu(a_1b_1c_1) < min\{r, s, t\}$ and $(a_1b_1c_1)_{min\{r,s,t\}} \notin \mu$. Now by Lemma 2.16 and Lemma 2.17, $(a_1b_1c_1)_{min\{r,s,t\}} = (a_1)_r \circ (b_1)_s \circ (c_1)_t \in \mu_1 \circ \mu_2 \circ \mu_3 \subseteq \mu$, a contradiction. Therefore μ is a fuzzy prime ideal of S. □

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