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Possibilistic programming with L-R fuzzy number coefficients

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Abstract. In this paper, we consider the fuzzy possibilistic methods for solving fuzzy linear programming problems. Possibilistic distribution on a possibilistic linear function with triangular and L-R fuzzy numbers has been considered. We apply possibilistic programming on fuzzy linear programming with coefficients of L-R fuzzy numbers and we compare it with the proposed method by Inuiguchi on triangular fuzzy numbers.

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1. Introduction

The concept of fuzzy linear programming (FLP) on a general level was first proposed by Bellman and Zadeh [1], Tanaka et al. [8] and Zimmermann ([10, 11]). In the framework of the fuzzy decision of bellman and Zadeh, fuzzy mathematical programming has been developed not only on a general level but also on a more practical level. Using the fuzzy decision proposed by bellman and Zadeh with linear, hyperbolic or piecewise linear membership functions has been proved that there exist equivalent linear programming problems. In such a situation, the resulting problem becomes a nonlinear programming problem and cannot be solved by a linear programming technique.

Relations between the FLP and possibility and necessity theory have been studied by Some authors, see, e.g., [5, 6].

In the first part of the paper, we introduce possibility and necessity measures and so possibilistic programming on triangular fuzzy numbers proposed by Inuiguchi [8]. After this description, we apply possibilistic programming on FLP problem with L-R fuzzy number coefficients.

2. Preliminaries

In this section, we present some definitions [7].

2.1. **Fuzzy sets theory.**

Let X be an universal set. A fuzzy set $\tilde{A}$ in X is characterized by its membership function $\mu_{\tilde{A}} : X \to [0,1]$. The value $\mu_{\tilde{A}}(x)$ at X represents the grade of membership of x in $\tilde{A}$ and is interpreted as the degree in which x belongs to $\tilde{A}$.

Definition 2.1. The α-cut of the fuzzy set $\tilde{A}$ is the crisp set $\tilde{A}_\alpha$ defined by

$$\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Definition 2.2. A fuzzy set $\tilde{A}$ in X is said to be a convex fuzzy set if and only if its α-cut sets are convex.

Definition 2.3. Let $f : X \to Y$ be a mapping from a set X to a set Y. The extension principle of Zadeh states that the fuzzy set $\tilde{B}$ in Y induced by the fuzzy set $\tilde{A}$ in X through f as follows:

$$B = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\},$$

with

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{x \in X, y=f(x)} \mu_{\tilde{A}}(x) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi, \end{cases}$$

where f^{-1} is the inverse image of y.

Definition 2.4. A fuzzy number is a normalized convex fuzzy set of the real line $\mathbb{R}^1$ whose membership function is piecewise continues. A fuzzy number $\tilde{M}$ is called positive (negative), denoted by $\tilde{M} > 0$ ($\tilde{M} < 0$), if its membership function satisfies, $\mu_{\tilde{M}}(x) = 0, \ \forall x < 0 (\forall x > 0)$.

Definition 2.5. A fuzzy number $\tilde{A}$ is called a L-R fuzzy number if its membership function has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \leq a, \alpha > 0 \\ R(\frac{x-a}{\beta}) & x \geq a, \beta > 0 \end{cases},$$

where L(.) and R(.) are piecewise continuous functions, L(.) is increasing, R(.) is decreasing and $L(0) = R(0) = 1$. The fuzzy number $\tilde{A}$ described above will be represented as $\tilde{A} = (a, \alpha, \beta)_{LR}$. Here L and R are called as the left and right reference functions, a is the mean value of A, α and β are called the left and right spread, respectively.

Definition 2.6. A fuzzy number $\tilde{A}$ is called a triangular fuzzy number (TFN) if its membership function $\mu_{\tilde{A}}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a^l, x > a^u \\ \frac{x-a^l}{a-a^l} & a^l \leq x \leq a \\ \frac{a^u-x}{a^u-a} & a < x \leq a^u. \end{cases}$$

The TFN $\tilde{A}$ is denoted by the triplet $\tilde{A} = (a^l, a, a^u)$.

2.2. **Possibility theory.**
Suppose we have two fuzzy numbers $\tilde{A}$ and $\tilde{B}$. Using the extension principle of Zadeh, the crisp inequality $x \leq y$ can be extended to obtain the truth value of the assertion that $\tilde{A}$ is less than or equal to $\tilde{B}$, as follows:

$$T(\tilde{A} \leq \tilde{B}) = \sup_{x \leq y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)).$$

This truth value $T(\tilde{A} \leq \tilde{B})$ is also called the grade of possibility of dominance of $\tilde{B}$ on $\tilde{A}$ and is denoted by $\text{pos}(\tilde{A} \leq \tilde{B})$.
In a similar way, the grade (or degree) of possibility that the assertion "$\tilde{A}$ is greater than or equal to $\tilde{B}$" is true, is given by

$$Pos(\tilde{A} \geq \tilde{B}) = \sup_{x \geq y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)).$$

Also the degree of possibility that the assertion "$\tilde{A}$ is equal to $\tilde{B}$" is denoted by $\text{pos}(\tilde{A} = \tilde{B})$, and is defined as

$$Pos(\tilde{A} = \tilde{B}) = \sup_{x} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)).$$

According to above discussion, we can define $\tilde{A} \leq \tilde{B}$ if and only if $Pos(\tilde{A} \leq \tilde{B}) \geq Pos(\tilde{B} \leq \tilde{A})$.
Related to the number "$Pos(\tilde{A} \leq \tilde{B})$" there is another number "$Nes(\tilde{A} \leq \tilde{B})$" which measures the grade (or degree) of necessity of dominance of $\tilde{B}$ on $\tilde{A}$, given by

$$Nes(\tilde{A} \leq \tilde{B}) = 1 - Pos(\tilde{A} \geq \tilde{B}).$$

The number"$Nes(\tilde{A} \leq \tilde{B})$" can also be used for ranking of fuzzy numbers. For this, we define $\tilde{A} \leq \tilde{B}$ if and only if $Nes(\tilde{A} \leq \tilde{B}) \geq Nes(\tilde{B} \leq \tilde{A})$.
In case $\tilde{A} = (a^l, a, a^u)$ and $\tilde{B} = (b^l, b, b^u)$ are TFN, then by actual computation of $Nes(\tilde{A} \geq \tilde{B})$, it can be defined that $\tilde{A}(\leq)\tilde{B}$ with respect to $Nes(\tilde{A} \leq \tilde{B})$ approach if $a + a^l \leq b + b^l$.

3. Possibilistic programming

In this section, we consider possibilistic programming with LR fuzzy number coefficients. Firstly, we define possibility and necessity measures.

3.1. **Possibility and Necessity measures.**
A possibilistic linear function value cannot be determined uniquely since its coefficients are ambiguous, i.e. non-deterministic. Thus maximizing a possibilistic objective function with possibilistic constraint function value that is not greater than a certain value do not make sense. For two fuzzy sets $\tilde{A}$ and $\tilde{B}$ Possibility and necessity measures of the event that a is in a fuzzy set $\tilde{B}$ are defined as follows[2, 9]:

$$\Pi_{\tilde{A}}(\tilde{B}) = \sup_{r} \min(\mu_{\tilde{A}}(r), \mu_{\tilde{B}}(r)), \tag{1}$$

$$N_{\tilde{A}}(\tilde{B}) = \inf_{r} \max(1 - \mu_{\tilde{A}}(r), \mu_{\tilde{B}}(r)). \tag{2}$$

Where $\mu_{\tilde{B}}$ is the membership function of fuzzy set $\tilde{B}$. $\Pi_{\tilde{A}}(\tilde{B})$ evaluates to what extent it is possible that the possibilistic variable a restricted by the possibility distribution $\mu_{\tilde{A}}$ is in the fuzzy set $\tilde{B}$. On the other hand, $N_{\tilde{A}}(\tilde{B})$ evaluates to what extent it is certain that the possibilistic variable a restricted by the possibility distribution μ_A is in the fuzzy set $\tilde{B}$. Let a be a possibilistic variable. In context to the above example, let $B = (-\infty, g]$, i.e. B be a crisp (non fuzzy) set of real numbers which is not greater than g. We obtain the following indices by possibility and necessity measures defined by (1) and (2):

$$Pos(a \leq g) = \Pi_{\tilde{A}}((-\infty, g]) = \sup\{\mu_{\tilde{A}}(r) : r \leq g\}, \tag{3}$$
$$Nes(a \leq g) = N_{\tilde{A}}((-\infty, g]) = 1 - \sup\{\mu_{\tilde{A}}(r) : r > g\}. \tag{4}$$

$Pos(a \leq g)$ and $Nes(a \leq g)$ show the possibility and necessity degrees to what extent a is not greater than g. Those indices are depicted in Figure 1.

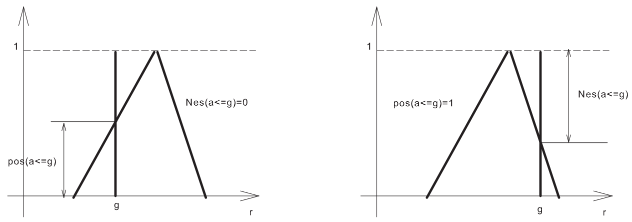

FIGURE 1. Possibility and necessity degree of $a \leq g$

Similarly, letting $B = [g, +\infty)$, we obtain the following two indices;

$$Pos(a \geq g) = \Pi_{\tilde{A}}([g, +\infty)) = \sup\{\mu_{\tilde{A}}(r) : r \geq g\}, \tag{5}$$
$$Nes(a \geq g) = N_{\tilde{A}}([g, +\infty)) = 1 - \sup\{\mu_{\tilde{A}}(r) : r < g\}. \tag{6}$$

$Pos(a \geq g)$ and $Nes(a \geq g)$ show the possibility and necessity degrees to what extent a is not smaller than g. Those indices are depicted in Figure 2[3, 4].

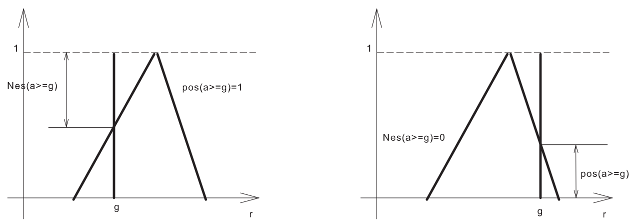

FIGURE 2. Possibility and necessity degrees of $a \geq g$

3.2. **Possibilistic programming formulation.**
Consider the following linear programming problem

$$\begin{aligned} \min\ & z = \mathbf{cx} \\ s.t.\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0, \end{aligned} \tag{7}$$

where $\mathbf{c} = (c_1, \dots, c_n)$ is an n-dimensional row vector, $\mathbf{x} = (x_1, \dots, x_n)^t$ is an n-dimensional column vector, $\mathbf{b} = (b_1, \dots, b_m)^t$ is an m-dimensional column vector and $A = [a_{i,j}]$ is an $m \times n$ matrix. The L-R fuzzy number $A_{i,j}$ can be determined by a center a^c_{ij} with a left spread $w^l_{a_{ij}}$ and a right spread $w^r_{a_{ij}}$, it is represented as $A_{i,j} =< a^c_{ij}, w^l_{a_{ij}}, w^r_{a_{ij}} >$. c_j is estimated as a L-R fuzzy number $c_j =< c^c_j, w^l_{c_j}, w^r_{c_j} >$.
The fuzzy number which restricts the possibilistic linear function value is defined by the extension principle. Applying the extension principle, for example, to the objective function of problem (7), $f_0(x_1, \dots, x_n) = \sum_{j=1}^n c_j x_j$, the fuzzy number $F_0(x_1, \dots, x_n)$ which restricts $f_0(x_1, \dots, x_n)$ is defined by the following membership function:

$$\mu_{F_0(x_1,\dots,x_n)}(r) = \sup_{p_1,\dots,p_n} \min(\mu_{c_1}(p_1), \dots, \mu_{c_n}(p_n)),$$

where $r = p_1x_1 + \dots + p_nx_n$. Taking into consideration the fact that c_j is L-R fuzzy numbers $< c^c_j, w^l_{c_j}, w^r_{c_j} >$, the fuzzy number $F_0(x_1, \dots, x_n)$ also becomes a L-R fuzzy number, i.e.,

$$\begin{aligned} F_0(x_1, \dots, x_n) &= < \sum_{j=1}^n c^c_j x_j, \sum_{j=1}^n w^l_{c_j} |x_j|, \sum_{j=1}^n w^r_{c_j} |x_j| > \\ &= < \sum_{j=1}^n c^c_j x_j, \sum_{j=1}^n w^l_{c_j} x_j, \sum_{j=1}^n w^r_{c_j} x_j > . \end{aligned} \tag{8}$$

The second equality is from the non-negativity of x_j's of problem (7). Let $F_i(x_1, \dots, x_n)$ be a fuzzy number which restricts the left-hand side value of the i-th constraint of (7), therefore for $i = 1, \dots, m$,

$$F_i(x_1, \dots, x_n) =< \sum_{j=1}^n a^c_{ij} x_j, \sum_{j=1}^n w^l_{a_{ij}} x_j, \sum_{j=1}^n w^r_{a_{ij}} x_j > . \tag{9}$$

Now, we give specific meanings of maximizing a possibilistic linear function value and the condition that a possibilistic linear function value is not greater than a given fuzzy number, or particularly, a crisp number, so that the ill-posed problem can be transformed to a usual linear programming problem. We interpret the constraints and the objective function.

3.2.1. **Constraints:**
Assume that the constraints of problem (7) should be satisfied with high certainly. If the decision maker feels that a certainty degree not less than a is high enough, the constraints of problem (7) can be treated as follows:

$$Nes(\sum_{j=1}^n \tilde{a}_{ij} x_j \leq b_i) \geq a,\ i = 1, 2, \dots, m, \quad x_1, \dots, x_n \geq 0. \tag{10}$$

Since possibilistic linear function value $f_i(x_1, \ldots, x_n)$ is a possibilistic variable restricted by $F_i(x_1, \ldots, x_n)$, we can substitute $f_i(x_1, \ldots, x_n)$ for a and $F_i(x_1, \ldots, x_n)$ for A in (3)-(6), thus, we get the possibility and certainly degrees to what extent a possibilistic linear function value is not greater (smaller) than a given real number. From (9), the fuzzy number $F_i(x_1, \ldots, x_n)$ restricting $f_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \tilde{a}_{ij} x_j$ is a L-R fuzzy number $< \sum_{j=1}^{n} a_{ij}^c x_j, \sum_{j=1}^{n} w_{a_{ij}}^l x_j, \sum_{j=1}^{n} w_{a_{ij}}^r x_j >$. This fuzzy number and the index $Nes(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i)$ are depicted in Figure 3. In order to satisfy $Nes(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i) \geq a$, point p Should be under line l. This is equivalent to the fact that t_i is not greater than b_i. By attention to Figure 3, we obtain

$$t_i = \sum_{j=1}^{n} a_{ij}^c x_j + R^{-1}(1-a) \sum_{j=1}^{n} w_{a_{ij}}^r x_j \leq b_i. \quad i = 1, \ldots, m \tag{11}$$

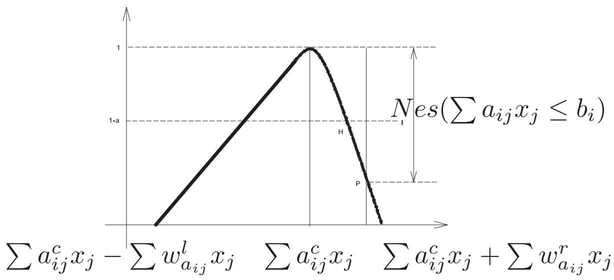

FIGURE 3. $Nes(\sum_{j=1}^{n} a_{ij} x_j \leq b_i)$

3.2.2. **Objective function:**

Assume that the decision maker feels the a certainty is high enough, then maximization of the objective function can be treated as[3]:

$$\begin{aligned} &\max \quad u \\ &s.t.\ Nes(\sum_{j=1}^{n} c_j x_j \geq u) \geq a, \end{aligned} \tag{12}$$

Problem (12) is illustrated in Figure 4. As shown in the Figure, u is maximized under the condition that point p is under line l. By the same discussion with the previous section, problem (12) is equivalent to

$$\begin{aligned} &\max \quad u \\ &s.t.\ \sum_{j=1}^{n} c_j^c x_j - L^{-1}(1-a) \sum_{j=1}^{n} w_{c_j}^l x_j \geq u, \end{aligned}$$

Finally, adding the constraints (11), problem (7) is formulated as the following

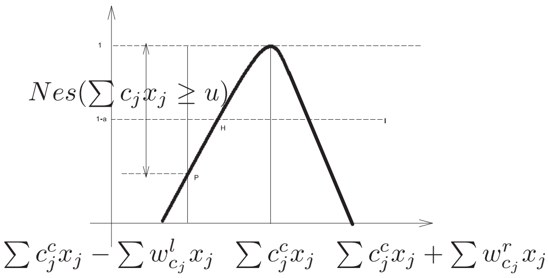

FIGURE 4. $Nes(\sum_{j=1}^{n} c_j x_j \geq u)$

linear programming problem:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j^c x_j - L^{-1}(1-a) \sum_{j=1}^{n} w_{c_j}^l x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij}^c x_j + R^{-1}(1-a) \sum_{j=1}^{n} w_{a_{ij}}^r x_j \leq b_i. \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
\tag{13}
$$

Note 3.1. *Assume A_{ij} and c_j are symmetric triangular fuzzy numbers as follows:*

$$A_{ij} = < a_{ij}^c, w_{a_{ij}}^c >, \quad c_j = < c_j^c, w_{c_j}^c > .$$

Similarly to the last section, if we set $L(x) = R(x) = 1 - x$, then problem (7) is formulated as the following linear programming problem:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j^c x_j - a \sum_{j=1}^{n} w_{c_j} x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij}^c x_j + a \sum_{j=1}^{n} w_{a_{ij}}^c x_j \leq b_i. \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
\tag{14}
$$

4. Example

Example 4.1. Consider the following problem:

$$
\begin{aligned}
\max \quad & z = x_1 + 2x_2 \\
s.t. \; & 2x_1 + 6x_2 \leq 27 \\
& 8x_1 + 6x_2 \leq 45 \\
& 3x_1 + x_2 \leq 15 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

$A_1 =< 2, 0.3, 0.7 >$	$B_1 =< 6, 0.2, 0.5 >$
$A_2 =< 8, 1, 1.5 >$	$B_2 =< 6, 0.1, 0.3 >$
$A_3 =< 3, 0.4, 0.5 >$	$B_3 =< 1, 0.2, 0.3 >$
$C_1 =< 1, 0.8, 1 >$	$C_2 =< 2, 0.5, 0.7 >$

Table 4.1. L-R fuzzy numbers of Example 4.1

a	L-R numbers	triangular numbers
0	z=12.5680	z=10
0.1	z=12.5085	z=9.4538
0.2	z=12.4790	z=8.9205
0.3	z=12.3792	z=8.3996
0.4	z=12.2088	z=7.8909
0.5	z=11.9676	z=7.5600
0.6	z=11.0562	z=7.2857
0.7	z=11.2762	z=7.0157
0.8	z=10.8322	z=6.7500
0.9	z=10.3396	z=6.4884
1	z=10.0000	z=6.2308

Table 4.2. The optimal values of the objective function of Example 4.1

By solving this problem, we obtain $(x_1, x_2) = (3, 3.5)$. we will solve this problem with possibilistic programming method.
Let the above problem be a possibilistic programming problem as following:

$$\begin{aligned}
\max \quad & z = c_1x_1 + c_2x_2 \\
s.t. \ & a_1x_1 + b_1x_2 \leq 27 \\
& a_2x_1 + b_2x_2 \leq 45 \\
& a_3x_1 + b_3x_2 \leq 15 \\
& x_1, x_2 \geq 0,
\end{aligned}$$

where a_i, b_i for $i = 1, 2, 3$, and c_j for $j = 1, 2$, are possibilistic variables restricted by fuzzy numbers A_i, B_i for $i = 1, 2, 3$, and C_j for $j = 1, 2$, respectively, such that A_i, B_i and C_j are as shown in Table 4.1. From (13), we obtain the following problem

$$\begin{aligned}
\max \quad & z = x_1 + 2x_2 - L^{-1}(1-a)(0.8x_1 + 0.5x_2) \\
s.t. \ & 2x_1 + 6x_2 + R^{-1}(1-a)(0.7x_1 + 0.5x_2) \leq 27 \\
& 8x_1 + 6x_2 + R^{-1}(1-a)(1.5x_1 + 0.3x_2) \leq 45 \\
& 3x_1 + x_2 + R^{-1}(1-a)(0.5x_1 + 0.3x_2) \leq 15 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

We set $L(x) = \begin{cases} 0 & x < -1 \\ \sqrt{1+x} & -1 \leq x \leq 0 \end{cases}$ and $R(x) = \begin{cases} 1 - x^2 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$.
As shown in Table 4.2, solving the above problem by simplex method, the optimal values of the objective function obtained of possibilistic programming on L-R

fuzzy numbers method are better than the obtained optimal values of possibilistic programming on triangular fuzzy numbers method.

5. Conclusion

In this paper, we used possibilistic programming with coefficients of L-R fuzzy numbers, and we compared this method with the proposed method by Inuiguchi on triangular fuzzy numbers. As observed in Example 4.1, the obtained values of the objective function by the possibilistic programming on L-R fuzzy numbers are better than the obtained values of the possibilistic programming on triangular fuzzy numbers method.

References

[1] R. E. Bellman and L. A. Zadeh, Decision-making in a fuzzy environment, Management Sci. 17 (1970) B141–B164.

[2] D. Dubios and H. Prade, Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum Press, New York. 1988.

[3] M. Inuiguchi, Fuzzy linear programming: what, why and how, Tatra Mt. Math. Publ. 13 (1997) 123–167.

[4] M. Inuiguchi and J. Ramik, Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, Fuzzy Sets and Systems 111 (2000) 3–28.

[5] H. Li and Z. Gong, Fuzzy linear programming with possibility and necessity relation, Advances in Intelligent and soft computing 78 (2010) 305–311.

[6] A. H. Marbini, P. J. Agrell, M, Tavana and A. Emroznejad, A stepwise fuzzy linear programming model with possibility and necessity relations, J. Intell. Fuzzy Systems 25(1) (2013) 81–93.

[7] M. Sakawa, Fuzzy sets and interactive multiobjective optimization, Plenum Press, New york. 1993.

[8] H. Tanaka, T. Okuda and K. Asai, On fuzzy mathematical programming, J. Cybernet. 3(4) (1973) 37–46.

[9] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–28.

[10] H. J. Zimmermann, Description and optimization of fuzzy systems, Int. J. General Systems 2 (1976) 209–215.

[11] H. J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy sets and Systems 1 (1978) 45–55.

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