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Soft topological groups and rings

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Abstract. In this paper we introduce the concepts of a soft topological group and a soft topological ring over a group and a ring respectively. Moreover, some notions like soft topological homomorphisms and closures of soft topological rings are given.We then finish with an introduction to notion of soft topological divisors of zero.

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1. Introduction

Most of our traditional tools for modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, there are many convoluted problems in economics, engineering, medical sciences, environment, social sciences, etc., that involve data which are not always crisp. We cannot use classical methods because of various types of uncertainties present in these problems. There are several theories, for example, theory of fuzzy sets [16], vague set theory, rough set theory, interval mathematics theory and other mathematical tools. But each of these theories has inherent difficulties as pointed out by Molodtsov [11]. Consequently, to triumph over these difficulties, Molodstov proposed a completely new approach for modeling, vagueness and uncertainty. This so-called soft set theory is free from difficulties affecting existing methods. The cause for these difficulties is, perhaps, the lack of the parameterization tool of the theory as it was indicated by Molodtsov in [10]. Molodtsov initiated the idea of soft sets as an innovative mathematical tool which overcome the troubles mentioned earlier. In [10], he presented the primary results of new theory and effectively applied it into a number of directions such as smoothness of functions, operations research, game theory, Riemann-integration, Perron integration, theory of probability etc. A soft set is a collection of fairly accurate description of an object. He also showed how soft set theory is free from parametrization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. Soft systems supply a very universal outline with the participation of parameters.A soft set can be represented as an information system and parameters behave as primitive attributes having values 0 or 1. It also is interesting to see that soft sets are closely related to many other soft computing models such as rough sets and fuzzy sets. Feng et al. [7] first considered the combination of soft sets, fuzzy sets and rough sets. Using soft sets as the granulation structures, Feng et al. [8] initiated soft approximation spaces and soft rough sets, which extended Pawlak's rough set model using soft sets. In some cases Feng's soft rough set model could provide better approximations than classical rough sets.

Appliance of soft set theory in algebraic structures was introduced by Aktaş and Çağman [2]. They discussed the notion of soft groups and consequently obtained some fundamental properties. Shabir and Ali [14], studied soft semigroups and soft ideals which characterize (generalized) fuzzy ideals with entrance of a semigroup. On the other hand, in [1] Acar, et al. introduced the basic notion of a soft ring, which is in fact a parameterized family of subrings (ideals) of a ring, over a ring.

In [4] soft subrings and soft ideals over a ring are introduced, moreover in [4] soft subfield over a field and soft sub-module over a left R-module has been introduced. Celik, et. al [5] defined a new binary relation and some new operations on soft sets, also they introduced the notion of a subrings (ideals) of a given ring. However, in [13] Sk. Nazmul and Sk Samnta introduce the basic idea of a soft topological group, its subsystem and morphism over a topological group.

The main purpose of this paper is to introduce basic notions of soft topological groups, which are actually a parametrized family of subgroups of a group that also possess the topological properties, over an algebraic group. Further we extend this study to soft topological rings and deal with some of their algebraic properties by giving examples of topologies on groups and rings that illustrate the main concepts. It is also provided in this paper that under what condition every soft group (respectively soft ring) becomes a soft topological group (respectively soft topological ring).

Moreover, the concept of the soft topological homomorphism is introduced and illustrated with an example. We also given the idea of closure of a soft topological ring, which helps in defining a soft topological divisor of zero.

2. Background

Molodtsov [10] defined the notion of a soft set in the following way:

Definition 2.1 ([10])**.** Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $a \in A$, $F(a)$ may be considered as the set of $a-$approximate elements of the soft set (F, A). Clearly a soft set is not a set.

Definition 2.2 ([12])**.** For two soft sets (F, A) and (K, B) over a common universe U, we say that (F, A) is a soft subset of (K, B) (i.e., $(F, A)\widetilde{\subset}(K, B)$) if

(a) $A \subset B$ and

(b) $F(a)$ and $K(a)$ are identical approximations for all $a \in A$.

(F, A) is said to be a soft super set of (K, B), if (K, B) is a soft subset of (F, A) and it is denoted by $(F, A)\widetilde{\supset}(K, B)$.

Definition 2.3 ([12])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([12])**.** A soft set (F, A) over U is said to be a $NULL$ soft set, denoted by Φ, if $F(a) = \varnothing$ (null set) for all $a \in A$, .

Definition 2.5 ([12])**.** A soft set (F, A) over U is said to be $ABSOLUTE$ soft set, denoted by $\widetilde{A}$, if $F(a) = U$ for all $a \in A$.

Definition 2.6 ([12])**.** If (F, A) and (G, B) are two soft sets, then "(F, A) and (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H((a, b)) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.

Definition 2.7 ([12])**.** If (F, A) and (G, B) are two soft sets then " (F, A) or (G, B)" denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (O, A \times B)$, where, $O((a, b)) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$.

Definition 2.8 ([12])**.** Union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C) , where $C = A \cup B$ and for all $a \in C$,

$$H(a) = \begin{cases} F(a) & if\ a \in A - B \\ G(a) & if\ a \in B - A \\ F(a) \cup G(a) & if\ a \in A \cap B \end{cases}$$

We write $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 2.9 ([6])**.** Bi-intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$, denoted by $(F, A)\widetilde{\sqcap}(G, B)$, is defined as $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$, where $C = A \cap B$ and $H(a) = F(a) \cap G(a)$ for all $a \in C$.

Definition 2.10 ([9])**.** Extended intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $a \in C$,

$$H(a) = \begin{cases} F(a) & if\ a \in A - B \\ G(a) & if\ a \in B - A \\ F(a) \cap G(a) & if\ a \in A \cap B \end{cases}$$

We write $(F, A) \cap_E (G, B) = (H, C)$.

Definition 2.11 ([5])**.** The extended sum of two soft sets (F, A) and (G, B) over a ring R is denoted by $(F, A) \oplus_{\cup} (G, B)$, is defined as $(F, A) \oplus_{\cup} (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(a) = \begin{cases} F(a) & if\ a \in A - B \\ G(a) & if\ a \in B - A \\ F(a) + G(a) & if\ a \in A \cap B \end{cases}$$

for all $a \in C$.

Definition 2.12 ([5]). The restricted sum of two soft sets (F, A) and (G, B) over a ring R is denoted by $(F, A) \oplus_\cap (G, B)$, is defined as $(F, A) \oplus_\cap (G, B) = (H, C)$, where $C = A \cap B$ and $H(a) = F(a) + G(a)$ for all $a \in C$.

Definition 2.13 ([5]). The extended product of two soft sets (F, A) and (G, B) over a ring R is denoted by $(F, A) \odot_\cup (G, B)$, is defined as $(F, A) \odot_\cup (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(a) = \begin{cases} F(a) & if\ a \in A - B \\ G(a) & if\ a \in B - A \\ F(a) \cdot G(a) & if\ a \in A \cap B \end{cases}$$

for all $a \in C$.

Definition 2.14 ([5]). The restricted product of two soft sets (F, A) and (G, B) over a ring R is denoted by $(F, A) \odot_\cap (G, B)$, is defined as $(F, A) \odot_\cap (G, B) = (H, C)$, where $C = A \cap B$ and $H(a) = F(a) \cdot G(a)$ for all $a \in C$.

Definition 2.15 ([2]). Let (F, A) be a soft set over G. Then (G, A) is said to be a soft group over G if and only if $F(a)$ is a subgroup of G for all $a \in A$.

Definition 2.16 ([2]). Let (F, A) and (H, B) be two soft groups over G. Then (H, B) is called a soft subgroup of (F, A) , written $(H, B)\tilde{<}(F, A)$, if

(a) $B \subset A$ and

(b) $H(a)$ is an subgroup of $F(k)$ for all $b \in B$.

From now on, R denotes a unitary commutative ring and all soft sets are considered over R.

Definition 2.17 ([1]). Let (F, A) be a soft set. The set Supp $(F, A) = \{a \in A : F(a) \neq \varnothing\}$ is called the support of the soft set (F, A). A soft set is said to be non-null if its support is not equal to the empty set.

Definition 2.18 ([1]). Let (F, A) be a non-null soft set over a ring R. Then (F, A) is called a soft ring over R if $F(a)$ is a subring of R for all $a \in A$.

In classical algebra, the notion of ideals are very important. For this reason, in [1, Definition 4.1], there is an introduction of soft ideals of a soft ring. Note that, if I is an ideal of a ring R, we write $I \lhd R$.

Definition 2.19 ([1]). Let (F, A) be a soft ring over R. A non-null soft set (γ, I) over R is called soft ideal of (F, A), which will be denoted by $(\gamma, I)\tilde{\lhd}(F, A)$, if it satisfies the following conditions:

(a) $I \subset A$.

(b) $\gamma(a)$ is an ideal of $F(a)$ for all $a \in$ Supp (γ, I).

Definition 2.20 ([15]). (a) Let (F, A) be a soft ring over the ring R. Let $\{0\} \neq F(a) \in (F, A)$, then $F(a)$ is said to be a soft left (respectively soft right) zero divisor in (F, A) if there exist some elements $\{0\} \neq F(b) \in (F, A)$ such that $F(a) \cdot F(b) = \{0\}$ (respectively $F(b) \cdot F(a) = \{0\}$).

(b) A soft zero divisor in (F, A) is either a soft left zero divisor or a soft right zero divisor.

3. Soft topological groups

Throughout this section, G denotes a group and (G, τ) denotes a topological group. We start by the following definition.

Definition 3.1. Let τ be a topology defined on a group G. Let (F, A) be a non-null soft set defined over G. Then the triplet (F, A, τ) is called soft topological group over G if

(a) $F(a)$ is a subgroup of G for all $a \in A$.

(b) the mapping $(x, y) \to x - y$ of the topological space $F(a) \times F(a)$ onto $F(a)$ is continuous for all $a \in A$, .

If G is a topological group, then Definition 3.1 coincide with [13, Definition 3.1].

In terms of neighborhoods, conditions (b) of Definition 3.1 implies that for any $x, y \in F(a)$ and arbitrary neighborhood N of $x - y$ there exist neighborhoods N_1 and N_2 of elements x and y, respectively, such that $N_1 - N_2 \subset N$.

Example 3.2. Take $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$, $A = \{e_1, e_2, e_3\}$ and base for the topology τ is $\mathcal{B} = \{\{e\}, \{(12)\}, \{(123)\}, \{(132)\}\}$. The set valued function F is defined by $F(e_1) = \{e\}$, $F(e_2) = \{e, (12)\}$ and $F(e_3) = \{e, (123), (132)\}$. Clearly $F(a)$ is a subgroup of G, for all $a \in A$. Also condition (b) of Definition 3.1 is satisfied. Hence (F, A, τ) is a soft topological group.

Remark 3.3. Let G be a group. Then every soft group can be transformed into a soft topological group over G by endowing G with discrete or anti-discrete topology. It is easy to verify that any soft group satisfies the condition (b) of Definition 3.1 in both topologies. In this manner any soft group can be considered as a soft topological group in the discrete or anti-discrete topology.

Theorem 3.4. *Every soft group over a topological group (non-discrete) is a soft topological group.*

Proof. Let (G, τ) be topological group and (F, A) be a soft group over G. So for all $a \in A$, $F(a)$ is a subgroup of G. Since G is a topological group and the mapping $(a, b) \to a - b$ of the topological space $G \times G$ onto G is continuous, so its restriction from $F(a) \times F(a)$ onto $F(a)$ is also continuous. Hence (F, A, τ) is a soft topological group over (G, τ). □

Remark 3.5. Every Soft group over a group need not to be a soft topological group. Since in Example 3.2, (F, A, τ) is a soft topological group with defined topology. Take$\tau = \{\varnothing, \{e\}, \{e, (12)\}, \{e, (123), (132)\}, S_3\}$. Then $F(e_3) = \{e, (123), (132)\}$ does not satisfies the condition (b) of Definition 3.1. Hence (F, A, τ) is not a soft topological group. Thus, this example demonstrates that the condition of being a topological group for the group A in Theorem 3.4 is important.

Theorem 3.6. *Let (F, A, τ) and (K, B, τ) be soft topological groups over G.*

(1) *The Bi-intersection $(F, A, \tau)\widetilde{\sqcap}(K, B, \tau)$ is a topological group over G if it is non-null.*

(2) *The extended intersection $(F, A, \tau)\cap_E(K, B, \tau)$ is a soft topological group over G.*

Proof. (1) Since (F, A, τ) and (K, B, τ) are soft topological groups over G. Therefore by Definition 2.9 their Bi-intersection over G is the soft topological set (H, C, τ), where $C = A \cap B$ and for all $c \in C$, it is defined as $H(c) = F(c) \cap K(c)$. Since both $F(c)$ and $K(c)$ are subgroups, therefore $H(c)$ is an subgroup of G, for all $c \in A \cap B$. Also $H(c) \subset F(c)$ or $K(c)$ and condition (b) of Definition 3.1 holds for $F(c)$. So it also holds for $H(c)$, for all c$\in C$. Hence $(F, A, \tau)\widetilde{\sqcap}(K, B, \tau)$ is a topological group over G.

(2) Obvious. □

Theorem 3.7. *Let (F, A, τ) and (K, B, τ) be soft topological groups over R, where τ is a topology defined over G. we have the following:*

(1) *Then $(F, A, \tau) \wedge (K, B, \tau)$ is a soft topological group over G if it is non-null.*

(2) *If A and B are disjoints, then $(F, A, \tau)\tilde{\cup}(K, B, \tau)$ is a topological group over G.*

Proof. The proof is similar to the proof of Theorem 3.6. □

If G is a topological group, then Theorem 3.7 coincide with [13, Theorem 3.4].

Definition 3.8. Let (F, A, τ) be a soft topological group over G. Then (F, A, τ) is said to be soft trivial if $F(a) = \{0\}$ for all $a \in A$ and whole if $F(a) = G$ for all $a \in A$.

If G is a topological group, then Definition 3.8 coincides with [13, Definatin 3.5].

Definition 3.9. Let (F, A, τ) be a soft topological group over G. Then (K, B, τ) is said to be a soft topological subgroup (resp. normal subgroup) of (F, A, τ) if

(a) $B \subset A$ and $K(b)$ is a subgroup (resp. normal subgroup) of $F(b)$ for all $b \in \sup(K, B)$,

(b) the mapping $(x, y) \to x - y$ of the topological space $K(b) \times K(b)$ onto $K(b)$ is continuous for all $b \in \sup(K, B)$.

If G is a topological group and $A = B$, then Definition 3.9 coincides with [13, Definatin 3.7].

Definition 3.10. Let (F, A, τ) and $(K, B, \tau^{'})$ be the soft topological groups over G and $G^{'}$, where τ and $\tau^{'}$ are topologies defined over G and $G^{'}$ respectively. Let $f : G \to G^{'}$ and $g : A \to B$ be two mappings. Then the pair (f, g) is called a soft topological group homomorphism if the following conditions are satisfied:

(a) f is group epimorphism and g is surjection.

(b) $f(F(a)) = K(g(a))$.

(c) $f_a : (F(a), \tau_{F(a)}) \to (K(g(a)), \tau^{'}_{K(g(a))})$ is continuous.

Then (F, A, τ) is said to be soft topologically homomorphic to $(K, B, \tau^{'})$ and denoted by $(F, A, \tau) \sim (K, B, \tau^{'})$.

If f is a group isomorphism, g is bijective and f_a is continuous as well as open, then the pair (f, g) is called a soft topological group isomorphism. In this case (F, A, τ) is soft topologically isomorphic to $(K, B, \tau^{'})$, which is denoted by $(F, A, \tau) \simeq (K, B, \tau^{'})$.

Example 3.11. Let (F, A) and (K, B) be the two soft homomorphic groups defined over G and $G^{'}$ respectively. Then (F, A) is soft topologically homomorphic to (K, B)

with discrete or anti-discrete topology. So any soft homomorphic groups can be considered as soft topological homomorphic groups in the discrete or anti-discrete topology.

Two soft topological groups may be isomorphic as soft groups,but not isomorphic as soft topological groups.

Example 3.12. Let G be the additive Abelian group of real numbers with the discrete topology τ, and $G\prime$ the additive group of real numbers with its natural topology (τ', interval topology). We shall define a mapping from G to $G\prime$ which associate with every real number $x \in G$ the same real number $x\prime \in G'$, write $f(x) = x\prime$, and define a natural bijection g from $\mathbb{R}$ to $\mathbb{R}$. Let us consider the set-valued functions $F : \mathbb{R} \to P(G)$ given by $F(x) = x\mathbb{Z}$, for $x \in A$ and $K : \mathbb{R} \to P(G')$ defined by $K(y) = \mathbb{R}$, for $y \in B$. It is obvious that (F, A, τ) and $(K, B, \tau^{'})$ are soft topological groups. Also for every $a \in A$, $f(F(a)) = K(g(a))$. Algebrically (F, A) and (K, B) are soft isomorphic but not as soft topological groups. Because algebrically f is even isomorphism but f is not an open mapping. Moreover, for any fix integer a, f_a is not an open mapping form $(F(a), \tau_{F(a)})$ to $(K(g(a)), \tau^{'}_{K(g(a))})$.

Definition 3.13. Let (F, A, τ) be a soft topological group over G. Then we can associate with (F, A, τ) a soft set over G denoted by $([F]_G, A, \tau)$, named closure of (F, A, τ), and defined as:

$$[F]_G(a) = [F(a)]_G$$

where $[F(a)]_G$ is the closure of $F(a)$ in topology defined on G.

Theorem 3.14. *Let (F, A, τ) be a soft topological group over a topological group (G, τ).*

(1) *Then $([F]_G, A, \tau)$ is also a soft topological group over (G, τ).*

(2) *Then $(F, A, \tau)\tilde{\subset}([F]_G, A, \tau)$*

(3) *If (F, A, τ) and (K, B, τ) are soft topological sets over (G, τ), then $([F]_G, A, \tau) \oplus_{\cup}([K]_{G}, B, \tau)\tilde{\subseteq}[(F, A, \tau) \oplus_{\cup} (K, B, \tau)]_G$.*

Proof. (1) Since (F, A, τ) is a soft topological group over (G, τ). Therefore $(F(a), \tau_{F(a)})$ is a topological subgroup of (G, τ) for all $a \in A$. So $F(a)$ is a subgroup of G and from [3, Proposition 1.4.5] closure of any subgroup of a topological group is also a subgroup of G. Therefore $[F(a)]_G$ is a subgroup of topological group G together with the topology defined on G. So $([F(a)]_G, \tau_{F(a)})$ is a topological subgroup of (G, τ). Hence $([F]_G, A, \tau)$ is also a soft topological group over (G, τ).

(2) Obvious.

(3) If $a \in A - B$

$$\begin{aligned} [([F]_G, A, \tau) \oplus_{\cup} ([K]_{G}, B, \tau)](a) &= ([F]_G, A, \tau)(a) = [F(a)]_G \\ &= [(F, A, \tau) \oplus_{\cup} (K, B, \tau)]_G(a) \end{aligned}$$

If $a \in B - A$

$$\begin{aligned} [([F]_G, A, \tau) \oplus_{\cup} ([K]_{G}, B, \tau)](a) &= ([K]_G, B, \tau)(a) = [K(a)]_G \\ &= [(F, A, \tau) \oplus_{\cup} (K, B, \tau)]_G(a) \end{aligned}$$

If $a \in A \cap B$

$$\begin{aligned}[([F]_G, A, \tau) \oplus_\cup ([K]_{G,}B, \tau)](a) &= [F]_G(a) + [K]_G(a) \\ &= [F(a)]_G + [K(a)]_G \\ &\subseteq [F(a) + K(a)]_G \\ &= [(F, A, \tau) \oplus_\cup (K, B, \tau)]_G(a)\end{aligned}$$

Therefore $([F]_G, A, \tau) \oplus_\cup ([K]_{G,}B, \tau)\tilde{\subseteq}[(F, A, \tau) \oplus_\cup (K, B, \tau)]_G$. □

The following example depicts that containment in Theorem 3.14 is proper.

Example 3.15. Consider the additive group of real numbers with its natural topology (interval topology τ). Define the set-valued functions $F : \mathbb{Z} \to P(\mathbb{R})$ given by $F(a) = \sqrt{a}\mathbb{Z}$, for $a \in A$ and define $K : \mathbb{Z}^+ \to P(\mathbb{R})$ by $K(b) = \mathbb{Z}$, for $y \in B = \mathbb{Z}^+$. For any $a \in \mathbb{Z}^+$, $[([F]_\mathbb{R}, A, \tau) \oplus_\cup ([K]_{\mathbb{R},}B, \tau)](a) = \mathbb{Z} + \sqrt{a}\mathbb{Z}$ because the discrete topologies are induced onto $\mathbb{Z}$ and $\sqrt{a}\mathbb{Z}$ which is why these subgroups are closed in $\mathbb{R}$. Moreover $[(F, A, \tau) \oplus_\cup (K, B, \tau)]_\mathbb{R}(a) = \mathbb{R}$ because $\mathbb{Z}+\sqrt{a}\mathbb{Z}$ is dense in $\mathbb{R}$ being a non-discrete subgroup of $\mathbb{R}$. Thus $[(F, A, \tau) \oplus_\cup (K, B, \tau)]_\mathbb{R}\tilde{\nsupseteq}([F]_\mathbb{R}, A, \tau) \oplus_\cup ([K]_{\mathbb{R},}B, \tau)$.

Definition 3.16. Let (F, A, τ) be a soft topological group over a group G. Then (F, A, τ) is said to be closed if $[F]_G(a) = F(a)$, where $([F]_G, A, \tau)$ being the corresponding soft topological set.

Definition 3.17. Let (F, A, τ) be a soft topological group over a group G. Then (F, A, τ) is said to be dense if $[F]_G(a) = G$, where $([F]_G, A, \tau)$ being the corresponding soft topological set.

4. Soft topological rings

From now on, R denotes a (unitary) commutative ring and (R, τ) denotes a topological ring.

Definition 4.1. Let τ be a topology defined on a R. Let (F, A) be a non-null soft set defined over R. Then the triplet (F, A, τ) is called soft topological ring over R if

(a) $F(a)$ is a subring of R for all $a \in A$.

(b) the mapping $(x, y) \to x - y$ of the topological space $F(a) \times F(a)$ onto $F(a)$ is continuous for all $a \in A$.

(c) the mapping $(x, y) \to x.y$ of the topological space $F(a) \times F(a)$ to $F(a)$ is continuous for all $a \in A$.

In terms of neighborhoods, conditions (b) and (c) implies that for any $x, y \in F(a)$ and arbitrary neighborhoods N of $x+y$ (resp. $x.y$) there exist neighborhoods N_1 and N_2 of elements x and y, respectively, such that $N_1 + N_2 \subset N$ (resp. $N_1.N_2 \subset N$).

Example 4.2. Take $R = \mathbb{Z}_4$, $A = \{2, 3\}$ and $\tau = \{\Phi, \{0\}, \{0, 2\}, \{0, 1\}, \{0, 1, 2\}, \mathbb{Z}_4\}$. Let us consider the set-valued function $F : A \to P(R)$ defined by $F(a) = \{b \in R : a.b = 0\}$. Then $F(2) = \{0, 2\}$ and $F(3) = \{0\}$ which are subrings of R and conditions (b) and (c) of Definition 4.1 are also satisfied. Hence (F, A, τ) is soft topological ring over R.

Remark 4.3. Let R be a ring. Then every soft ring can be transformed into a soft topological ring over R by endowing R with discrete or anti-discrete topology. It is easy to verify that any soft ring satisfies the conditions (b) and (c) of Definition 4.1 in both topologies. In this manner any soft ring can be considered as a soft topological ring in the discrete or anti-discrete topology.

Theorem 4.4. *Every soft ring over a topological ring (non-discrete) is a soft topological ring.*

Proof. Let (R, τ) be topological ring. Let (F, A) be a soft ring over R. So for all $a \in A$, $F(a)$ is a subring of R. Since R is a topological ring and from [3, Remark 1.4.4] subring of topological ring is itself a topological ring. So for every $F(a)$ condition (b) and (c) of Definition 4.1 holds, being subring of R. Hence (F, A, τ) is a soft topological ring over (R, τ). □

Example 4.5. Take $A = \mathbb{Z}^+$, $R = \mathbb{R}$ and $\tau =$ The interval topology defined on $\mathbb{R}$ and (F, A) be soft set defined by $F(a) = \mathbb{Q}[\sqrt{a}]$ if a not a perfect square integer and $F(a) = \mathbb{Q}$ if otherwise. Since $F(a)$ is a subring of $\mathbb{R}$ for each $a \in \mathbb{Z}^+$. Therefore $(F(a), \tau_{F(a)})$ is topological subring of $(\mathbb{R}, \tau)$ for each $a \in \mathbb{Z}^+$. Hence (F, A, τ) is a soft topological ring over $(\mathbb{R}, \tau)$.

Remark 4.6. Every Soft ring over a ring need not to be a topological ring. For instance, consider $R = \mathbb{Z}_8$, $A = \{0, 1, 2\}$ and $\tau = \{\Phi, \{0\}, \{0, 2\}, \{0, 4\}, \{0, 2, 4, 6\}, \mathbb{Z}_8\}$. Let us consider the set-valued function $F : A \to P(R)$defined by $F(a) = \{b \in R : a.b = \{0, 4\}\}$. Then $F(0) = R$ and $F(1) = \{0, 4\}$ and $F(2) = \{0, 2, 4, 6\}$ which are all subrings of R. Condition (2) and (3) of Definition 4.1 are satisfied for $F(1)$ and $F(2)$ but not for $F(0)$. Hence (F, A, τ) is not soft topological ring.

Theorem 4.7. *Let (F, A, τ) and (G, B, τ) be soft topological rings over R, where τ is a topology defined over R.*

(1) *Then the Bi-intersection $(F, A, \tau)\widetilde{\sqcap}(G, B, \tau)$ is a topological ring over R if it is non-null.*

(2) *Then extended intersection $(F, A, \tau) \cap_E (G, B, \tau)$ is a soft topological ring over R.*

Proof. (1) Since (F, A, τ) and (G, B, τ) are soft topological rings over R. Therefore by Definition 2.9 their Bi-intersection over R is the soft topological set (H, C, τ), where $C = A \cap B$ and for all $c \in C$, and it is defined as $H(c) = F(c) \cap G(c)$. Since both $F(c)$ and $G(c)$ are subrings of R, therefore $H(c)$ is a subring of R for all $c \in A \cap B$. Also $H(c) \subset F(c)$,$H(c) \subset G(c)$, and condition (b) of Definition 4.1 holds for $F(c)$. So it also holds for $H(c)$ for all $c \in C$. Hence $(F, A, \tau)\widetilde{\sqcap}(G, B, \tau)$ is a topological ring over R.

(2) Obvious result. □

Theorem 4.8. *If (F, A, τ) and (G, B, τ) are two soft topological rings over R. Then $(F, A, \tau) \wedge (G, B, \tau)$ is a soft topological ring over R if it is non-null.*

Proof. The proof is easily seen by Definitions 2.6 and 4.1. □

Definition 4.9. Let (F, A, τ) be a soft topological ring over R, then (F, A, τ) is said to be soft trivial if $F(a) = \{0\}$ for all $a \in A$ and soft whole if $F(a) = R$ for all $a \in A$

Definition 4.10. Let (F, A, τ) be a soft topological rings over R. Then (G, B, τ) is said to be a soft topological subring (resp. ideal) of (F, A, τ) if

(a) $B \subset A$ and $G(a)$ is a subring (resp. ideal) of $F(a)$ for all $a \in \sup(G, B)$.

(b) the mapping $(x, y) \to x - y$ of the topological space $G(a) \times G(a)$ onto $G(a)$ is continuous for all $b \in \sup(G, B)$.

(c) For all $b \in B$, the mapping $(x, y) \to x.y$ of the topological space $G(a) \times G(a)$ to $G(a)$ is continuous.

Example 4.11. In Example 4.5, Take $B = 2\mathbb{Z}^+$ and let us consider the set-valued function $G : B \to P(R)$ defined by $G(b) = b\mathbb{Z}$. $G(b)$ is a subring of $F(b)$ for all $b \in \sup(G, B) = 2\mathbb{Z}^+$. Also conditions (b) and (c) of Definition 4.10 are satisfied. Hence (F, B, τ) is a soft topological subring of (F, A, τ).

Example 4.12. Take $A = \mathbb{Z}^+$, $R = \mathbb{R}$ and $(\mathbb{R}, \tau)$ is a topological ring, where τ is the interval topology defined on $\mathbb{R}$. Let (F, A) be a soft set defined by $F(a) = \mathbb{Q}[\pi]$ if $7/a$ and $F(a) = \mathbb{Q}$ otherwise. Here $\mathbb{Q}[\pi]$ is the subring of $\mathbb{R}$ generated by the subset $\mathbb{Q} \cup \{\pi\}$. Clearly (F, A, τ) is a soft topological ring over $(\mathbb{R}, \tau)$. Now take $B = 7\mathbb{Z}^+$ and define a soft set (G, B) by $G(b) = \pi^b.\mathbb{Q}[\pi]$ for $b \in B$. Each $G(b)$ is a proper ideal of $F(b)$, for all $b \in B$. Since $(\mathbb{R}, \tau)$ is a topological ring. $\Longrightarrow (G(y), \tau_{G(y)})$is topological ideal of $(F(y), \tau_{F(y)})$, for all y$\in B$. Hence (G, B, τ) is a soft topological ideal of (F, A, τ).

Theorem 4.13. *Let (F, A, τ) and (G, B, τ) be soft topological rings over R.*

(1) If $G(a) \subset F(a)$, for all $a \in B \subset A$, then (G, B) is a soft topological subring of (F, A, τ).

(2) Then $(F, A, \tau)\widetilde{\sqcap}(G, B, \tau)$ is a soft topological subring (ideal) of both (F, A, τ) and (G, B, τ).

Proof. The proof is straightforward. □

Theorem 4.14. *Every soft subring (resp. ideal) of a soft topological ring is a soft topological subring (resp. ideal).*

Proof. Let (F, A, τ) be a soft topological ring over R and (G, B) be a soft subring (resp. ideal) of (F, A). Then for each $b \in \sup(G, B)$, $G(b)$ is a subring (resp. ideal) of $F(b)$. And conditions (b) and (c) of Definition 4.1 are hold for every subset of $F(b)$. So, also hold for $G(b)$, being a subring (resp. ideal) of $F(a)$. Hence (G, B, τ) is a soft topological subring (resp. ideal) of (F, A, τ). □

5. Idealistic soft topological rings

Definition 5.1. Let τ be a topology defined on R and (F, A) be a non-null soft set defined over R. Then the triplet (F, A, τ) is called idealistic soft topological ring over R if

(a) $F(a)$ is an ideal of R for all $a \in A$.

(b) the mapping $(x, y) \to x - y$ of the topological space $F(a) \times F(a)$ onto $F(a)$ is continuous for all $a \in A$.

(c) the mapping $(r, y) \to r.y$ of the topological space $R \times F(a)$ to $F(a)$ is continuous for all $a \in A$.

Example 5.2. Every idealistic soft ring can be transformed into a soft topological ring over R by endowing R with discrete or anti-discrete topology. It is easy to verify that any idealistic soft ring satisfies the conditions (b) and (c) of Definition 5.1 in both topologies. In this manner any ring can be considered as a soft topological ring in the discrete or anti-discrete topology.

Theorem 5.3. *Every idealistic soft ring over a topological ring (non-discrete) is an idealistic soft topological ring.*

Proof. Let (R, τ) be a topological ring. Let (F, A) be an idealistic soft ring over R. So $F(a)$ is an ideal of R for all $a \in A$. R is a topological ring and ideal of a topological ring is itself a topological ideal and satisfies the conditions (b) and (c) of Definition 5.1. So every $F(a)$ satisfies conditions (b) and (c) of Definition 5.1, being an ideal of R. Hence (F, A, τ) is a soft topological ring over (R, τ). $\square$

Theorem 5.4. *Let (F, A, τ) and (G, B, τ) be idealistic soft topological rings over R.*

(1) *Then $(F, A, \tau)\tilde{\wedge}(G, B, \tau)$ is an idealistic soft topological ring over R if it is non-empty.*

(2) *Then Bi-intersection $(F, A, \tau)\widetilde{\sqcap}(G, B, \tau)$ is an idealistic soft topological ring over R if it is non-empty.*

(3) *If A and B are disjoints, then $(F, A, \tau)\tilde{\cup}(G, B, \tau)$ is an idealistic soft topological ring over R.*

Theorem 5.5. *Let (F, A, τ) and (G, B, τ) be idealistic soft topological rings over (R, τ). Then $(F, A, \tau) \oplus_{\cup} (G, B, \tau)$ is an idealistic soft topological ring over (R, τ).*

Proof. Since (F, A, τ) and (G, B, τ) are idealistic soft topological ring over (R, τ). Therefore by Definition 2.11 their extended sum over R is the soft topological set (H, C, τ), where $C = A \cup B$ and for all $c \in C$, it is defined as

$$H(c) = \begin{cases} F(c) & if\ c \in A - B \\ G(c) & if\ c \in B - A \\ F(c) + G(c) & if\ c \in A \cap B \end{cases}$$

In first two cases either $H(c) = F(c)$ or $H(c) = G(c)$. If $c \in A \cap B$, then $H(c) = F(c) + G(c)$. Since both $F(c)$ and $G(c)$ are ideals, therefore $H(c)$ is an ideal of R for all $c \in A \cap B$. Hence $H(c)$ is an ideal of R for all $c \in C$. By Theorem 5.3 $(H, C) = (F, A) \oplus_{\cup} (G, B)$ is an idealistic soft topological ring over (R, τ). $\square$

Remark 5.6. If (F, A, τ) and (G, B, τ) are idealistic soft topological rings over R. Then $(F, A, \tau) \oplus_{\cup} (G, B, \tau)$ need not to be an idealistic soft topological ring over R. For instance, consider the ring $R = \mathbb{Z}_{10}$, base for topology τ is $\mathcal{B} = \{\{0\}, \{2\}, \{4\}, \{5\}, \{6\}, \{8\}\}$, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. Let us consider the set valued function $F : A \to P(R)$ given by $F(a) = \{c : a.c \in \{0, 2, 4, 6, 8\}\}$. Then $F(1) = F(3) = F(5) = \{0, 2, 4, 6, 8\}$. Now $G : B \to P(R)$ is defined as $G(b) = \{d : b.d \in \{0, 5\}\}$. Since $(F, A, \tau) \oplus_{\cup} (G, B, \tau) = (H, C, \tau)$, where $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & if\ c \in A - B \\ G(c) & if\ c \in B - A \\ F(c) + G(c) & if\ c \in A \cap B \end{cases}$$

Now $H(1) = F(1) + G(1) = \mathbb{Z}_{10}$, $H(2) = G(2)$, $H(3) = F(3) + G(3) = \mathbb{Z}_{10}$ and $H(5) = F(5)$. Here $H(1)$ and $H(3)$ does not satisfies the condition (b) and (c) of Definition 5.1. Hence (H, C, τ) is not an idealistic soft topological ring over R.

Theorem 5.7. *Let (F, A, τ) and (G, B, τ) be idealistic soft topological rings over (R, τ). Then $(F, A, \tau) \odot_{\cup} (G, B, \tau)$ is an idealistic soft topological ring over (R, τ).*

Proof. Since (F, A, τ) and (G, B, τ) are idealistic soft topological ring over (R, τ), therefore by Definition 2.13 their extended product over R is the soft topological set (H, C, τ), where $C = A \cup B$ and for all $c \in C$ and it is defined as

$$H(c) = \begin{cases} F(c) & if\ c \in A - B \\ G(c) & if\ c \in B - A \\ F(c).G(c) & if\ c \in A \cap B \end{cases}$$

In first two cases either $H(c) = F(c)$ or $H(c) = G(c)$. If $c \in A \cap B$, then $H(c) = F(c).G(c)$. Since both $F(c)$ and $G(c)$ are ideals, therefore $H(c)$ is an ideal of R for all $c \in A \cap B$. Hence $H(c)$ is an ideal of R for all $c \in C$. By Theorem 5.3 $(H, C) = (F, A) \oplus_{\cup} (G, B)$ is an idealistic soft topological ring over (R, τ). □

Remark 5.8. If (F, A, τ) and (G, B, τ) are idealistic soft topological rings over the ring R. Then $(F, A, \tau) \odot_{\cup} (G, B, \tau)$ need not to be an idealistic soft topological ring over R.

Definition 5.9. Let (F, A, τ) be a soft topological ring over R and $f : R \to R^{'}$ be a ring homomorphism. Define the set $K_f(x)$ by

$K_f(a) = [K(f)]_{F(a)} = Ker(f) \cap F(a) = \{r \in F(a) : f(r) = 1_{R'}\}$ for all $a \in A$.

It is clear that (K_f, A, τ) is a soft topological ring over R.

Proposition 5.10. *(K_f, A, τ) is an ideal of soft topological ring (F, A, τ).*

Proof. Since $Ker(f)$ is an ideal of R and $F(a)$ is subring of R. Therefore $K_f(a) = Ker(f) \cap F(a)$ is an ideal of $F(a)$ for all $a \in A$. Hence by Theorem 4.14 (K_f, A, τ) is an ideal of of soft topological ring (F, A, τ). □

Definition 5.11. Let (F, A, τ) and $(G, B, \tau^{'})$ be the soft topological rings over R and R, where τ and $\tau^{'}$ are topologies defined over R and $R^{'}$, respectively. Let $f : R \to R^{'}$ and $g : A \to B$ be two mappings. Then the pair (f, g) is called a soft topological ring homomorphism if the following conditions are satisfied:

(a) f is ring epimorphism and g is surjection.
(b) $f(F(a)) = G(g(a))$.
(c) $f_a : (F(a), \tau_{F(a)}) \to (G(g(a)), \tau^{'}_{G(g(a))})$ is continuous.

Then (F, A, τ) is said to be soft topologically homomorphic to $(G, B, \tau^{'})$ and denoted by $(F, A, \tau) \sim (G, B, \tau^{'})$.

If f is a ring isomorphism, g is bijective and f_a is continuous as well as open, then (f, g) is called a soft ring isomorphism. In this case (F, A, τ) is softly topological isomorphic to $(G, B, \tau^{'})$, which is denoted by $(F, A, \tau) \simeq (G, B, \tau^{'})$.

Definition 5.12. Let (F, A, τ) and $(G, B, \tau^{'})$ be the soft topological rings over R and $R^{'}$ respectively. Let (F, A, τ) is soft topological homomorphic to $(G, B, \tau^{'})$ and (f, g) be the corresponding soft topological homomorphism. Then define

(a) $fF : B \to P(R^{'})$ by $(fF)b = f_a(F(a))$, where $b = g(a)$ for $b \in A$.
(b) $f^{-1}G : A \to P(R)$ by $(f^{-1}G)a = f_a^{-1}(G(g(a)))$, for all $a \in A$.

Theorem 5.13. *Let (F, A, τ) and $(G, B, \tau^{'})$ be soft topological rings over (R, τ) and $(R, \tau^{'})$ respectively. If (F, A, τ) is topological soft homomorphic to $(G, B, \tau^{'})$ and (f, g) be the corresponding soft topological homomorphisms. Then*
(1) *$(fF, B, \tau^{'})$ is soft topological ring over $(R^{'}, \tau^{'})$.*
(2) *$(f^{-1}G, A, \tau)$ is a soft topological ring over (R, τ).*

Proof. (1) Since g is surjective, so there exists $a \in A$ such that $b = g(a)$ for all $b \in B$. Also $f_a : (F(a), \tau_{F(a)}) \to (G(g(a)), \tau^{'}_{G(g(a))})$ is continuous. So $(fF)b = f_a(F(a))$ is a subring of $G(b)$ and hence $R^{'}$. Thus for all $a \in A$, $(f_a(F(a)), \tau^{'}_{f_a(F(a))})$ is topological subring of $(G(b), \tau^{'}_{G(b)})$ and $(G(b), \tau^{'}_{G(b)})$ is topological subring of $(R^{'}, \tau^{'})$. Consequently $(f_a(F(a)), \tau^{'}_{f_a(F(a))})$ is topological subring of $(R^{'}, \tau^{'})$. Hence, by Theorem 4.14 $(fF, B, \tau^{'})$ is soft topological ring over $(R^{'}, \tau^{'})$.

(2) f is the corresponding algebraic homomorphism from R to $R^{'}$. Also for each $a \in A$, $f_a : (F(a), \tau_{F(a)}) \to (G(g(a)), \tau^{'}_{G(g(a))})$ is continuous. Therefore $(f^{-1}G)a = f_a^{-1}(G(g(a)))$ is a subring of $F(a)$ and hence a subring of R. Thus by Theorem 4.14 $(f^{-1}G, A, \tau)$ is a soft topological ring over (R, τ). □

Theorem 5.14. *Let (F, A, τ) and $(G, B, \tau^{'})$ be idealistic soft topological rings over (R, τ) and $(R^{'}, \tau^{'})$ respectively. If (F, A, τ) is soft topological homomorphic to $(G, B, \tau^{'})$ and (f, g) be the corresponding soft topological homomorphism. Then*
(1) *$(fF, B, \tau^{'})$ is an idealistic soft topological ring over $(R^{'}, \tau^{'})$.*
(2) *$(f^{-1}G, A, \tau)$ is an idealistic soft topological ring over (R, τ).*

Proof. Proof is obvious. □

Theorem 5.15. *Let (F, A, τ) and $(G, B, \tau^{'})$ be the soft topological rings over (R, τ) and $(R^{'}, \tau^{'})$ respectively. Further assume that (F, A, τ) is soft topological homomorphic to $(G, B, \tau^{'})$ and (f, g) be the corresponding soft topological homomorphisms. If (H, A, τ) be a soft topological Ideal (resp. subring) of (F, A, τ). Then*
(1) *$(fH, B, \tau^{'})$ is a soft topological ideal(resp. subring) of $(fF, B, \tau^{'})$.*
(2) *$(f^{-1}H, A, \tau)$ is a soft topological ideal(resp. subring) of $(f^{-1}G, A, \tau)$.*

Proof. (1) Since g is surjective, so there exists $a \in A$ such that $b = g(a)$ for all $b \in B$. Also f_a is the corresponding algebraic homomorphism from $F(a)$ to $G(g(a))$. As $f_a(F(a))$ and $f_a(H(a))$ are the subrings of $G(g(a))$ and $H(a)$ is an ideal (resp. subring) of $F(a)$. So $f_a(H(a))$ is an ideal (resp. subring) of $f_a(F(a))$. Consequently $(f_a(H(a)), \tau_{f_a(H(a))})$ is a topological ideal of $(f_aF(a), \tau_{f_aF(a)})$. Thus by Theorem 5.3 $(fH, B, \tau^{'})$ is a soft topological ideal (resp. subring) of $(fF, B, \tau^{'})$.

(2) $(f^{-1}H)a = f_a^{-1}(H(a))$ for each $a \in A$. And f_a is the corresponding algebraic homomorphism from $F(a)$ to $G(g(a))$. So $f_a^{-1}(H(a))$ is an ideal (resp. subring) of $f_a^{-1}(G(g(a))$. Hence by Theorem 4.14 $(f^{-1}H, A, \tau)$ is a soft topological ideal (resp. subring) of $(f^{-1}G, A, \tau)$. □

6. Soft topological divisors of zero

We define the closure of a soft topological ring as follows:

Definition 6.1. Let (F, A, τ) be a soft topological ring over R. Then we can associate with (F, A, τ) a soft set over R, denoted by $([F]_R, A, \tau)$, named closure of (F, A, τ) and defined as:

$$[F]_R(a) = [F(a)]_R$$

where $[F(a)]_R$ is the closure of $F(a)$ in topology defined on R.

Theorem 6.2. *Let (F, A, τ) be a soft topological ring over (R, τ). Then*
(1) *$([F]_R, A, \tau)$ is a soft topological ring over (R, τ).*
(2) *$(F, A, \tau)\tilde{\subset}([F]_R, A, \tau)$.*

Proof. (1) Since (F, A, τ) is a soft topological ring over (R, τ). Therefore $(F(a), \tau_{F(a)})$ is a topological subring of (R, τ) for all $a \in A$. Hence $F(a)$ is a subring of R and from [3, Proposition 1.4.7] closure of any subring of a topological ring is also a subring of R. Therefore $[F(a)]_R$ is a subring of topological ring R together with the topology defined on R. So $([F(a)]_R, \tau_{F(a)})$ is a topological subring of (R, τ). Hence $([F]_R, A, \tau)$ is a soft topological ring over (R, τ).
(2) Obvious. □

Remark 6.3. If (F, A, τ) be a soft topological ring over R. Then $([F]_R, A, \tau)$ need not to be a soft topological ring over R. For instance, consider $R = \mathbb{Z}_6$,$A = \{e_1, e_2, e_3\}$ and $B = \{\{0\}, \{1\}, \{2\}, \{3\}, \mathbb{Z}_6\}$ be a base for the topology. Let us consider the set valued function $F : A \to P(R)$defined by $F(e_1) = \{0\}$, $F(e_2) = \{0, 1\}$ and $F(e_3) = \{0, 3\}$. Then clearly (F, A, τ) is a soft topological ring over $\mathbb{Z}_6$. But $([F]_R, A, \tau)$ is not a soft topological ring over R because $[F]_R(e_1) = \{0, 1, 2\}$ which is not a subring of R.

Theorem 6.4. *Let (F, A, τ) and (G, B, τ) be soft topological sets over (R, τ). Then we have the following:*
(1) $([F]_R, A, \tau) \odot_{\cup} ([G]_{R,}B, \tau)\tilde{\subseteq}[(F, A, \tau) \odot_E (G, B, \tau)]_R$.
(2) $[(F, A, \tau) \cap_E (G, B, \tau)]_R\tilde{\subseteq}([F]_R, A, \tau) \cap_E ([G]_{R,}B, \tau)$.
(3) $[(F, A, \tau)\tilde{\cup}(G, B, \tau)]_R = ([F]_R, A, \tau)\tilde{\cup}([G]_{R,}B, \tau)$.
(4) *If* $(F, A, \tau)\tilde{\subset}(G, B, \tau)$. *Then* $([F]_R, A, \tau)\tilde{\subset}([G]_{R,}B, \tau)$.

Proof. (1) If $a \in A - B$, then

$$\begin{aligned}[([F]_R, A, \tau) \odot_E ([G]_{R,}B, \tau)](a) &= ([F]_R, A, \tau)(a) = [F(a)]_R \\ &= [(F, A, \tau) \odot_E (G, B, \tau)]_R(a)\end{aligned}$$

If $a \in B - A$, then

$$\begin{aligned}[([F]_R, A, \tau) \odot_E ([G]_{R,}B, \tau)](a) &= ([G]_R, B, \tau)(a) = [G(a)]_R \\ &= [(F, A, \tau) \odot_E (G, B, \tau)]_R(a)\end{aligned}$$

If $a \in A \cap B$, then

$$\begin{aligned}[([F]_R, A, \tau) \odot_E ([G]_{R,} B, \tau)](a) &= [F]_R(a).[G]_R(a) \\ &= [F(a)]_R.[G(a)]_R \\ &\subseteq [F(a).G(a)]_R \\ &= [(F, A, \tau) \odot_\cup (G, B, \tau)]_R(a)\end{aligned}$$

Hence $([F]_R, A, \tau) \odot_\cup ([G]_{R,} B, \tau) \tilde{\subseteq} [(F, A, \tau) \odot_\cup (G, B, \tau)]_R$.

(2) If $a \in A - B$, then

$$\begin{aligned}[([F]_R, A, \tau) \cap_E ([G]_{R,} B, \tau)](a) &= ([F]_R, A, \tau)(a) = [F(a)]_R \\ &= [(F, A, \tau) \cap_E (G, B, \tau)]_R(a)\end{aligned}$$

If $a \in B - A$, then

$$\begin{aligned}[([F]_R, A, \tau) \cap_E ([G]_{R,} B, \tau)](a) &= ([G]_R, B, \tau)(a) = [G(a)]_R \\ &= [(F, A, \tau) \cap_E (G, B, \tau)]_R(a)\end{aligned}$$

If $a \in A \cap B$, then

$$\begin{aligned}[([F]_R, A, \tau) \cap_E ([G]_{R,} B, \tau)](a) &= [F(a)]_R \cap [G(a)]_R \\ &= [F(a)]_R \cap [G(a)]_R \\ &\supseteq [F(a) \cap G(a)]_R \\ &= [(F, A, \tau) \cap_E (G, B, \tau)]_R(a)\end{aligned}$$

Therefore $[(F, A, \tau) \cap_E (G, B, \tau)]_R \tilde{\subseteq} ([F]_R, A, \tau) \cap_E ([G]_{R,} B, \tau)$.

(3) Similar.

(4) Since $(F, A, \tau) \tilde{\subset} (G, B, \tau)$. So for all $a \in A$, $F(a) \subset G(a)$ and $[F(a)]_R \subset [G(a)]_R$. Hence $([F]_R, A, \tau) \tilde{\subset} ([G]_{R,} B, \tau)$. □

The containments in (3) and (4) in Theorem 6.4 are proper as can be seen in the following example:

Example 6.5. Define the set-valued functions $F : \mathbb{Z}^+ \to P(\mathbb{R})$ by $F(a) = a\mathbb{Z}$, for $a \in \mathbb{Z}^+$ and define $G : \mathbb{Z}^+ \to P(\mathbb{R})$ by $G(b) = (\mathbb{Z} \setminus \{0\})^{-1} \cup \{0\}$ for each $b \in \mathbb{Z}^+$. For any $a \in \mathbb{Z}^+$, $([F]_\mathbb{R}, A, \tau)(a) = a\mathbb{Z}$ and $([G]_{\mathbb{R},} B, \tau)(a) = (\mathbb{Z} \setminus \{0\})^{-1} \cup \{0\}$. It is clear that $(\mathbb{Z} \setminus \{0\})^{-1} \cup \{0\}$ is compact closed in $\mathbb{R}$ being a closed subset of the segment $[0; 1] \subset \mathbb{R}$. It is evident that for any $a \in \mathbb{Z}^+$ $(F, A, \tau) \odot_E (G, B, \tau)(a) = \mathbb{Q}$ and $[\mathbb{Q}]_\mathbb{R} = \mathbb{R}$. Hence $[(F, A, \tau) \odot_E (G, B, \tau)]_\mathbb{R} \tilde{\not\supseteq} ([F]_\mathbb{R}, A, \tau) \odot_E ([G]_{\mathbb{R},} B, \tau)$.

Definition 6.6. Let (F, A, τ) be a soft topological ring over a ring R . Then (F, A, τ) is said to be closed if $[F]_R(a) = F(a)$, where $([F]_R, A, \tau)$ being the corresponding soft topological set.

Definition 6.7. Let (F, A, τ) be a soft topological ring over a ring R . Then (F, A, τ) is said to be dense if $[F]_R(a) = R$, where $([F]_R, A, \tau)$ being the corresponding soft topological set.

Example 6.8. Take $A = \mathbb{Z}^+$, $R = \mathbb{R}$, $\tau =$The interval topology defined on $\mathbb{R}$ and define $F : A \to P(R)$ by $F(a) = a\mathbb{Z}$ for all $a \in A$. Clearly for all $a \in A$, $(F(a), \tau_{F(a)})$ are topological subrings of topological ring $(\mathbb{R}, \tau)$. So (F, A, τ) is a soft topological ring over (R,τ). Also $[F(a)]_R = [a\mathbb{Z}]_R = a\mathbb{Z}$ for all $a \in A$. So (F,A,τ) is closed soft topological ring over (R,τ) because $([F]_R, A, \tau) = (F,A,\tau)$.

If we define $G : A \to P(R)$ by $G(a) = \mathbb{Z}[\sqrt{a}]$ for all $a \in A$. (G,A,τ) is soft topological ring over (R,τ). Moreover, (G,A,τ) is dense soft topological ring over (R,τ).

Remark 6.9. Every Soft topological ring with discrete topology is closed and dense with anti-discrete topology.

Theorem 6.10. *A trivial soft topological ring over a Hausdorff ring is closed.*

Proof. Let (R,τ) be a Hausdorff ring and (F,A,τ) be a soft topological ring over (R,τ) such that $F(a) = \{0\}$, for all $a \in A$. Since R is a Hausdorff ring so $\{0\}$ is closed in R. Therefore for all $a \in A$, $[F(a)]_R = \{0\}$. □

Example 6.11. Take $R = \mathbb{Z}_6$, $A = \{0,1,2\}$ and $\tau = \{\varnothing, \mathbb{Z}_6\}$. Define $F : A \to P(R)$ by $F(a) = \{0\}$, for all $a \in A$. For every $a \in A$, $[F(a)]_R = \mathbb{Z}_6$. Hence (F,A,τ) is not closed. This example demonstrates that the condition of being Hausdorff for the ring R in Theorem 6.10 is important.

Theorem 6.12. *Let (F,A,τ) and (G,B,τ) be soft topological rings over the ring R. Then*

(1) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau)\tilde{\wedge}(G,B,\tau)$ is a dense soft topological ring over R.*

(2) *If (F,A,τ) and (G,B,τ) are closed soft topological rings, then $(F,A,\tau)\tilde{\vee}(G,B,\tau)$ is a closed soft topological ring over R.*

(3) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau)\tilde{\cup}(G,B,\tau)$ is a dense soft topological ring over R.*

(4) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau)\tilde{\vee}(G,B,\tau)$ is a dense soft topological ring R.*

(5) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \cap_\varepsilon (G,B,\tau)$ is a dense soft topological ring over R.*

(6) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \Cap (G,B,\tau)$ is a dense soft topological ring over R.*

(7) *If (F,A,τ) and (G,B,τ) are closed soft topological rings, then $(F,A,\tau)\tilde{\cup}(G,B,\tau)$ is a closed soft topological ring over R.*

(8) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \oplus_\cup (G,B,\tau)$ is a dense soft topological ring over R.*

(9) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \oplus_\cap (G,B,\tau)$ is a dense soft topological ring over R.*

(10) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \odot_\cup (G,B,\tau)$ is a dense soft topological ring over R.*

(11) *If (F,A,τ) and (G,B,τ) are dense soft topological rings, then $(F,A,\tau) \odot_\cap (G,B,\tau)$ is a dense soft topological ring over R.*

(12) *If (F,A,τ) is a closed soft topological ring and (G,B,τ) is a dense soft topological ring over R, then $(F,A,\tau) \cap_\varepsilon (G,B,\tau)$ is a closed soft topological ring over R.*

(13) *If* (F, A, τ) *is a closed soft topological ring and* (G, B, τ) *is a dense soft topological ring over* R*, then* $(F, A, \tau)\tilde{\cup}(G, B, \tau)$ *is a dense soft topological ring over* R*.*

Proof. Obvious. □

In topological ring theory, the concept of topological divisors of zero is well known. Now we use the concept of idealistic soft topological ring because motivations are due to the product of two ideals in a unitary commutative ring is again an ideal.

Definition 6.13. (a) Let (F, A, τ) be a soft topological ring over R. Let $\{0\} \neq F(a) \in (F, A, \tau)$, then $F(a)$ is said to be soft topological left (respectively soft topological right) divisor of zero in (F, A, τ) if there exist a soft subset (G, B) of soft set (F, A) such that

(i) $\{0\} \notin ([G]_R, B, \tau)$

(ii) $\{0\} \in \{[F(a).G(b)]_R : G(b) \in (G, B)\}$ (respectively $\{0\} \in \{[G(b).F(a)]_R : G(b) \in (G, B)\}$).

(b) An element of (F, A) is called soft topological divisor of zero if it is soft left and right topological divisor of zero.

Remark 6.14. A soft topological ring (F, A, τ) with anti-discrete topology τ has neither soft left nor soft right topological divisor of zero. For instance, let us consider a soft topological ring (F, A, τ) and $F(a)$ be a soft topological divisor. So there exist a soft subset (G, B) of (F, A) such that $\{0\} \notin ([G]_R, B, \tau)$. Since the topology is anti-discrete. So $[G]_R(b) = \Phi$ for all $b \in B$ and also $\{[F(a).G(b)]_R : G(b) \in (G, B)\} = \Phi$. Hence $F(a)$ can not be a soft topological divisor of zero. Hence (F, A, τ) has no soft topological divisor of zero.

Example 6.15. Take $R = \mathbb{Z}_4$, $A = \{1, 2\}$ and $\tau = \{\{0\}, \{2\}, \{0, 2\}, \mathbb{Z}_4\}$. Define $F : A \to P(R)$ by $F(1) = \{0\}$ and $F(2) = \{0, 2\}$. Clearly, (F, A, τ) is a soft topological ring over R. Here $F(2)$ is a soft topological divisor of zero in (F, A, τ).

Proposition 6.16. *In a soft topological ring with discrete topology, soft left (resp. right) zero divisors are soft left (resp. right) topological divisors of zero.*

Proof. Let (F, A, τ) be a soft topological ring, where τ is the discrete topology. Let $\{0\} \neq F(a)$ be a soft zero divisor. So there exist $F(b) \in (F, A, \tau)$ such that $F(a).F(b) = \{0\}$. Consider a soft subset $(G, b) = \{(F(b), b)\}$ of (F, A). Then clearly $\{0\} \notin ([G]_R, B, \tau)$ but

$$\{0\} \in \{[F(a).G(b)]_R : G(b) \in (G, B)\} = \{[F(a).F(b)]_R\}.$$

Hence the proof. □

The concept of the soft zero divisor and soft topological divisor of zero, in general, are different.

Proposition 6.17. *Let* (F, A, τ) *be a soft topological ring over a* R*. Let* $F(a_1)$ *be a soft left (right) topological divisors of zero in* (F, A, τ)*. Then for every* $F(a_2) \in (F, A, \tau)$ *the element* $F(a_2).F(a_1)$ *is a soft left topological divisor of zero (correspondingly, the element* $F(a_1).F(a_2)$ *is a soft right topological divisors of zero).*

Proof. Let $F(a_1)$ be a soft left topological divisor of zero in (F, A, τ) and (G, B) be a soft subset of (F, A) such that $\{0\} \notin ([G]_R, B, \tau)$, and $\{0\} \in \{[F(a_1).G(b)]_R : G(b) \in (G, B)\}$, then by using

$$\begin{aligned} \{0\} &\in \{F(a_2).[F(a_1).G(b)]_R : G(b) \in (G, B)\} \\ &\subseteq \{[(F(a_2).F(a_1)).G(b)]_R : G(b) \in (G, B)\}. \end{aligned}$$

$F(a_2).F(a_1)$ is a soft left topological divisor of zero. □

Analogously the case when $F(a_1)$ is a soft right topological divisor of zero in (F, A, τ).

Proposition 6.18. *Let (F, A, τ) be a soft topological ring over R and $F(a_1).F(a_2) \in (F, A, \tau)$. If $F(a_1).F(a_2)$ is a soft left (resp. right) topological divisor of zero in (F, A, τ), then either $F(a_1)$ or $F(a_2)$ is a soft left (resp. right) topological divisor of zero.*

Proof. Let $F(a_1).F(a_2)$ is a soft left topological divisor of zero in (F, A, τ) and (G, B) be a soft subset of (F, A) such that

$$\{0\} \notin ([G]_R, B, \tau) \text{ and } \{0\} \in \{[(F(a_1).F(a_2)).G(b)]_R : G(b) \in (G, B)\}$$

If $\{0\} \in \{[F(a_2).G(b)]_R : G(b) \in (G, B)\}$, then $F(a_2)$ is a soft left topological divisor of zero. If $\{0\} \notin \{[F(a_2).G(b)]_R : G(b) \in (G, B)\}$, then by using the fact that

$$\begin{aligned} \{0\} &\in \{[(F(a_1).F(a_2))G(b)]_R : G(b) \in (G, B)\} \\ &= \{0\} \in \{[F(a_1)(F(a_2).G(b))]_R : G(b) \in (G, B)\}, \end{aligned}$$

we have $F(a_1)$ is a soft left topological divisor of zero.

Similarly the case when $F(a_1).F(a_2)$ is a soft right topological divisor of zero in (F, A, τ). □

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