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Fuzzy ideal systems and some monoids

Waheed Ahmad Khan, Abdelghani Taouti

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Abstract. In this note we introduce weak ideal systems on fuzzy ideals of monoids. Moreover, utilizing fuzzy r-ideals and fractionary fuzzy r-ideals we characterize valuation monoids, almost valuation monoids, pseudovaluation monoids, almost pseud-ovaluation monoids, and pseudo-almost valuation monoids. We Also discuss few implications between said monoids. Finally, we fuzzify [14, Theorem 1] which reflects the linking bridge between different results in this note.

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Corresponding Author: Waheed Ahmad Khan (sirwak2003@yahoo.com)

1. Introduction and preliminaries

The concepts of fuzzy sets and fuzzy relations were first introduced by Zadeh[16]. Fuzzy subgroup and its properties were discussed by Rosenfeld [13]. After this, the notion of a fuzzy ideal of a ring was introduced by Liu, Malik, Mordeson and Mukherjee. Bhattacharya and Mukherjee have studied fuzzy relation on groups. Malik and Mordeson [9] came up with fuzzy relations on rings.

A fuzzy ideal ξ of a ring R is said to be fuzzy prime, if it is non-constant and for any two fuzzy ideal μ and ν of R, the condition $\mu \circ \nu \subseteq \xi$ implies that $\mu \subseteq \xi$ or $\nu \subseteq \xi$. It is well known that ξ is fuzzy prime if and only if $\xi(0) = 1$, ξ_* is a prime ideal of R and $|Im(\xi)| = 2$ [11, Theorem 3. 5. 5]. Fuzzy ideal ξ is called fuzzy primary if ξ is non-constant and for any two fuzzy ideals μ, ν of R, $\mu \circ \nu \subseteq \xi$ implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$[11, Theorem 3. 5. 5]. Reader may consult [11] for details of fuzzy R-submodule and defining operations. A fuzzy ideal μ in a noetherian ring R is called irreducible if $\mu \neq R$ and whenever $\mu_1 \wedge \mu_2 = \mu$ where μ_1 and μ_2 are fuzzy ideals of R, the $\mu_1 = \mu$ or $\mu_2 = \mu$[12, Definition 4.1].

We recall few standard definitions from the literature which have been introduced in [9], [6] and [13].

Let $\mu_t = \{x \in R : \mu_t(x) \geqslant t\}$, a level set for every $t \in [0,1]$. For a subset W of R let $\chi_W^{(t)}$ be the fuzzy subset of K (quotient field of R) such that $\chi_W^{(t)}(x) = 1$ if $x \in W$ and $\chi_W^{(t)}(x) = t$ if $x \in K\backslash W$, where $t \in [0,1)$. For $d \in K$ and $t \in [0,1]$, we let d_t denote the fuzzy subset of K defined by: for every $x \in K$, $d_t(x) = t$ if $x = d$ and $d_t(x) = 0$ otherwise. Let R be an integral domain, a fuzzy R-submodule β of K (quotient field of R) is called a fractionary fuzzy ideal of R if there exists $d \in R$, $d \neq 0$, such that $d_1 \circ\ \beta \subseteq \chi_R^{(t)}$ for some $t \in [0,\,1)$. Let β be a fractionary fuzzy ideal of R. Then $\beta|_R$ is a fuzzy ideal of R. If $\beta|_R$ is a prime(maximal) fuzzy ideal of R, then β is called a prime (maximal) fractionary fuzzy ideal of R. If $\beta(x) = 0$ for all $x \in K\backslash R$, then β is called an integral fractionary fuzzy ideal of R. Thus, if β is a prime (maximal) integral fractionary fuzzy ideal of R, then $Im(\beta) = \{0, t, 1\}$ for some $t \in [0, 1)$. A prime integral fractionary fuzzy ideal β of R is said to be strongly prime if for any fractionary fuzzy ideals μ, υ of R, $\mu\circ\ \upsilon \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\upsilon \subseteq \beta$[6, Definition 2.1]. Let β be an integral fractionary fuzzy ideal of R then is strongly primary fuzzy ideal of R if for any fractionary ideals μ and ν of R, $\mu \circ \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \sqrt{\beta}$ [6, Definition 4.1]. A proper fuzzy ideal μ of a ring R is said to be strongly irreducible if for each pair of fuzzy ideals θ and σ of R, if $\theta \wedge \sigma \subseteq \mu$ then either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$ [15, Definition 2].

Throughout this note a monoid means a commutative cancellative semigroup with identity with zero adjoined, we will represent the semigroup operation by ordinary multiplication notation and use 1 to denote the identity of the semigroup. We will implement [7] convention of allowing a zero element 0 with the property that $0x = 0$; yet $xy = 0$ implies $x = 0$ or $y = 0$. Here, a good representative example of a monoid is the multiplicative monoid of an integral domain. This close relationship has made it very natural to study results of a multiplicative nature known for integral domains in the monoid setting. If H is a monoid we represent $H^* = H\backslash\{0\}$, and $a, b \in H$ we write $a|_H b$ to denote a divides b in H, that is, $b = ac$ for some $c \in H$. We call $a, b \in H$ associates if $a|_H b$ and $b|_H a$. Associates of 1 in H are called units and the set of units of H is denoted by $H^\times$. Now H is said to be reduced if $H^\times = \{1\}$. The units of H can be shown to be invertible elements of H. Thus $H^\times$ is a subgroup of H and we can consider the quotient monoid $H/H^\times$ which is obviously reduced and is denoted by H_{red} . There is a quotient field that we have for a domain, in parallel there is a groupoid for a monoid H. We shall reserve $G(H)$ for the quotient groupoid of H. Quotient groupoid of a cancellative monoid H is a groupoid $G(H)$ such that $H \subset G(H)$ is a submonoid and $G(H) = \{c^{-1}a : a \in H,\, c \in H^*\}$ [7, Page 38]. As in the case of integral domains we can also define various ideal systems on a monoid H. This fact has been amply demonstrated in [7]. For the sake of completeness we included some basic definitions, we refer [7, Chapter 2] to readers for the properties of weak ideal systems of monoids.

Let R be an integral domain with quotient field K. A prime ideal P of R is called strongly prime if $xy \in P$, where $x,\, y \in K$, then $x \in P$ or $y \in P$ (alternatively P is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K\backslash R$ [4, Definition, page2]. Following [4] an integral domain said to be pseudo-valuation domain whose every prime ideal is strongly prime. Recall [1] an ideal I of D is strongly primary if, whenever $xy \in I$ with $x, y \in K$ implies $x \in I$ or $y^n \in I$ for some integer $n \geqslant$

1, and D is an almost pseudo-valuation domain (APVD) if each prime ideal of D is strongly primary. Following [2] a prime ideal P of R is called a pseudo-strongly prime ideal if whenever $x, y \in K$ and $xyP \subseteq P$, there exist $m \in \mathbb{N}$ such that $x^m \in R$ or $y^m P \subseteq P$ and an integral domain R is called a pseudo-almost valuation domain if every prime ideal is a pseudo-strongly prime ideal. Fuzzy ideals in a monoid or semigroup have been discussed in the literature, fuzzy (weakly) ideals in a semigroup have been introduced in [3]. Recently, generalized fuzzy prime ideal in an ordered semigroup has been introduced (redefined) in [10].

In [14, Page 182] author (with T. Shah) has introduced almost valuation monoid, pseudo-almost valuation monoid, almost pseudo-valuation monoid and discussed few of their characteristics.

In this note we introduce fuzzy ideal systems on monoids and subsequently, we discuss fuzzy r-ideal systems on fuzzy subsets, fuzzy ideals and farctionary fuzzy ideals of a monoid. We also present characterization of valuation monoids, almost valuation monoids, pseudo-valuation monoids, pseudo-almost valuation monoids and almost pseudo-valuation monoid through fuzzy r-ideals. Finally, we fuzzify the result [14, Theorem 1].

For basic definitions and terminologies of monoids please consult [7]. We refer [9], [13] and [16] for basic notations, terminologies and definitions of fuzzy discussions.

2. Fuzzy ideal systems

In this section we introduce fuzzy r-system on fuzzy ideals and also fuzzy r-system on fractionary fuzzy ideals. We begin with the following definition.

Definition 2.1. Consider H be a monoid and $P(H)$ is the power set of H. Let $H \subset P(H)$, a fuzzy ideal system r on a monoid $P(H)$ is a map on $P(H) \to P(H)$, defined by $\mu \to \mu_r$ (fuzzy subsets) such that for all fuzzy subsets (resp. fuzzy ideals) β, $\mu \in P(H)$ and $c \in H$ the following conditions hold:

(a) $\mu \cup \{0\} \subseteq \mu_r$
(b) $\mu \subseteq \beta_r$ implies $\mu_r \subseteq \beta_r$
(c) $c\mu_r \subset (c\mu)_r$.

A fuzzy ideal μ is called a fuzzy r-ideal if $\mu = \mu_r$ and is r -finitely generated if $\mu = \beta_r$ for a finitely generated fuzzy ideal β of H. From (a) it follows that for every fuzzy r -system we have $H_r = H$. If μ is a fuzzy r -ideal and X is a fuzzy subset of H then the set $(\mu : X) = \{h \in H \mid hX \subseteq \mu\}$ is a fuzzy r -ideal and $(\mu : X) = (\mu : X_r)$. A fuzzy ideal system r on H is said to be a finitary if for each fuzzy subset $X \in H$, $X_r = F_r$ where F ranges over the finite fuzzy subsets of X.

The simplest d-system on fuzzy subsets (resp. ideals) is the map from $P(H) \to P(H)$ given by $\mu \to \mu_d = \mu$. One of the ideal systems of interest is s-system, for fuzzy subset $\mu \subset H$, we define, $\mu_s = \{0\}$ when $\mu = \varnothing$ and μH when $\mu \neq \varnothing$. If r is a weak ideal system on fuzzy monoid H then every fuzzy r-ideal (resp. subset) is a fuzzy s-ideal (resp. subset).

Definition 2.2. A fuzzy subset μ of a monoid H is a fuzzy r-ideal of H, if for every $g, h \in H$, $\mu_r(g - h) \geq \mu_r(g) \wedge \mu_r(h)$ and $\mu_r(gh) \geq \mu_r(g) \vee \mu_r(h)$.

Basic operation between fuzzy r-ideals can be define as.

Definition 2.3. Let α_r, β_r are two fuzzy r-ideals of a monoid H, we define operation "$\circ$ "as.

$$(\alpha \circ \beta)_r(x) = \alpha_r(x) \circ \beta_r(x) = \vee\{\alpha_r(y) \wedge \beta_r(z) : y,\ z\ \in H\}$$

We define some terminologies for monoids which are useful for our forthcoming discussion.

A fuzzy r-ideal β_r of a monoid H is said to be fuzzy prime r-ideal if it is non-constant and for every two fuzzy r-ideals $(\mu \circ \upsilon)_r \subseteq \beta_r$ implies that either $\mu_r \subseteq \beta_r$ or $\upsilon_r \subseteq \beta_r$. A fuzzy r-ideal β_r of a monoid H is said to be fuzzy primary r-ideal if it is non-constant and for every two fuzzy r-ideals $(\mu \circ \upsilon)_r \subseteq \beta_r$ implies that either $\mu_r \subseteq \beta_r$ or $\upsilon_r \subseteq \sqrt{\beta_r}$. A fuzzy r subset μ_r of $G(H)$ is fuzzy H-submodule of $G(H)$ (a quotient groupoid of H) if $\mu_r(g' - h') \geq \mu_r(g') \wedge \mu_r(h')$, $\mu_r(gg') \geq \mu_r(g')$ and $\mu_r(0) = 1$, for every g', $h' \in G(H)$ and $g \in H$. For a subset W of a monoid H let $\chi_W^{(t)}$ be the fuzzy subset of $G(H)$ (a quotient groupoid of H) such that $\chi_W^{(t)}(h) = 1$ if $h \in W$ and $\chi_W^{(t)}(h) = t$ if $h \in G(H)\backslash W$, where $t \in [0, 1)$. For $g \in G(H)$ and $t \in [0, 1]$, we let g_t denote the fuzzy subset of $G(H)$ such that for every $h \in G(H)$, $g_t(h) = t$ if $h = g$ and $g_t(h) = 0$ otherwise. Let H be a monoid, a fuzzy H-submodule β_r of $G(H)$ (quotient groupoid of H) is called a fractionary fuzzy r-ideal of H if there exists $g \in H$, $g \neq 0$, and $g_1 \in G(H)$ such that $g_1 \circ\ \beta_r \subseteq \chi_H^{(t)}$ for some $t \in [0,\ 1)$.

In [5] star operations on fractionary fuzzy ideals have been introduced, we define r-ideal system on fractionary fuzzy ideals of monoid. We assume that $\beta_* = \{g' \in G(H) : \beta(g') = \beta(0)\}$.

We may define an ideal system on fractionary fuzzy r-ideals as.

Definition 2.4. Let H be a fuzzy monoid, an ideal system r on fractionary fuzzy ideal is the map $F(H) \to F(H)$, where $F(H)$ represent the fractionary subsets (resp. fractionary fuzzy ideals) of $G(H)$, defined by $\mu \to \mu_r$ such that for all fuzzy subsets (resp. fuzzy ideals) $\beta, \mu \in F(H)$ and $c \in G(H)$ the following conditions hold:

(a) $\mu \cup \{0\} \subseteq \mu_r$ and $(g_1 \circ \beta)_r = g_1 \circ \beta_r$
(b) $\mu \subseteq \beta_r$ implies $\mu_r \subseteq \beta_r$
(c) $cH \subseteq \{c\}_r$
(d) $c\mu_r = (c\mu)_r$
(e) $(\mu_r)_r = \mu_r$

If we restrict an ideal system r to be only fractionary fuzzy ideals then operation r is just like a star operation.

Following definition 2.4, a fractionary fuzzy r-ideal μ is called a fractionary fuzzy r-ideal if $\mu = \mu_r$ and is r -finitely generated if $\mu = \beta_r$ for a finitely generated fuzzy ideal β of H.

From definition 2.4(a), it follows that for every fuzzy r -system $H_r = H$ and every fuzzy principal ideal is a fuzzy r -ideal. If μ is a fractionary fuzzy r -ideal and X is a fractionary fuzzy subset of $F(H)$ then the set $(\mu : X) = \{h \in H \mid hX \subseteq \mu\}$ is a fuzzy r -ideal and $(\mu : X) = (\mu : X_r)$. A fuzzy ideal system r on H is said to be finitary if for each fuzzy subset $X \in H$, $X_r = F_r$ where F ranges over the finite fuzzy subsets of X.

Definition 2.5. Let β_r be a fractionary fuzzy r-ideal of a monoid H. Then $\beta_r|_H$ is a fuzzy ideal of H. If $\beta_r|_H$ is a prime(maximal) fuzzy r-ideal of H, then β_r is called a prime (maximal) fractional fuzzy r-ideal of H. If $\beta_r(h) = 0$ for all $h \in G(H)\backslash H$ then β_r is called an integral fractional fuzzy r-ideal of H. Thus, if β_r is a prime (maximal) integral fractional fuzzy r-ideal of H, then $Im(\beta_r) = \{0, t, 1\}$ for some $t \in [0, 1)$.

Proposition 2.6. *Let $\{\mu_i : i \in I\}$ be the collection of fuzzy r H-submodules of $G(H)$ and let $g \in G(H)$ and $d \neq 0$. Then*
(1) $g_1 \circ (\cap_i \mu_i) = \cap_i (g_1 \circ \mu_i)$
(2) $g_1 \circ (\cup_i \mu_i) = \cup_i (g_1 \circ \mu_i)$

Proof. (1) Let $g' \in G(H)$, consider $(g_1 \circ (\cap_i \mu_i))(g') = \wedge_i \beta_i(\frac{g'}{g}) = \wedge_i (g_1 \circ \mu_i)(g') = \cap_i (g_1 \circ \mu_i)(g')$. Similarly, we can prove (2), let $g' \in G(H)$, consider $(g_1 \circ (\cup_i \mu_i))(g') = \vee_i \beta_i(\frac{g'}{g}) = \vee_i (g_1 \circ \mu_i)(g') = \cup_i (g_1 \circ \mu_i)(g')$. □

Proposition 2.7. *Let α_r, β_r be a fractional fuzzy r-ideals of H. Then $\alpha_r + \beta_r$ and $\alpha_r \circ \beta_r$ are fractional fuzzy r-ideals of H.*

Proof. Since α_r and β_r are fractional fuzzy r-ideals of H, there exist $0 \neq g$, $h \in H$ such that $g_1 \circ \alpha_r \subseteq \chi_H^{(s)}$ and $h_1 \circ \beta_r \subseteq \chi_H^{(t)}$ for some s, $t \in [0, 1)$. So $(gh)_1 \circ \alpha_r = g_1 \circ h_1 \circ \alpha_r \subseteq h_1 \circ \chi_H^{(s)} \subseteq \chi_H^{(s)}$. Similarly, $(gh)_1 \circ \beta_r \subseteq \chi_H^{(t)}$. Thus $(gh)_1 \circ (\alpha_r + \beta_r) = (gh)_1 \circ \alpha_r + (gh)_1 \circ \beta_r \subseteq \chi_H^{(s)} + \chi_H^{(t)} \subseteq \chi_H^{(t \vee s)}$ and also $(gh)_1 \circ (\alpha_r \circ \beta_r) = (g_1 \circ \alpha_r) \circ (h_1 \circ \beta_r) \subseteq \chi_H^{(s)} \circ \chi_H^{(t)} \subseteq \chi_H^{(t \vee s)}$. So $\alpha_r + \beta_r$ and $\alpha_r \circ \beta_r$ are the fractional fuzzy r-ideals of H. □

3. Fuzzy ideals and monoids

In this section we characterized valuation monoid, almost valuation monoid, pseudo-valuation monoid, almost pseudo-valuation monoid and pseudo-almost valuation monoid by using fuzzy r-ideals. We initiate by recalling some definitions from [7] and [14].

A monoid H is called a valuation monoid if for all $a, b \in H$, either a/b or b/a [7, Page 167]. A monoid H said to be almost valuation monoid, if $G(H)$ be a quotient groupoid of a monoid H and for all $x \in G(H)$ either $x^n \in H$ or $x^{-n} \in H$. Similarly H is called pseudo-valuation monoid if $x \in G/H$ and $a \in H\backslash H^\times$(where $H^\times$ is a set of invertible elements of H) implies $x^{-1}a \in H$ [7, Page 182]. Following [14, Definition 1] H is said to be almost valuation monoid if $G(H)$ be a quotient groupoid of a monoid H and for all $x \in G(H)$ either $x^n \in H$ or $x^{-n} \in H$. An integral domain R is called almost valuation domain if it is almost valuation monoid [14]. Monoid H is called r-local, if H possesses exactly one r-maximal r-ideal [7, Definition 6.5].

An r-ideal $P \in I_r(H)$ is primary or a primary r-ideal if $P \neq H$, and $a, b \in H$, $ab \in P$ implies $a \in P$ or $b \in rad(P)$ [7, Page 61]. Let $G(H)$ be a quotient groupoid of H then r-ideal $P \in I_r(H)$ is strongly primary r-ideal if $a, b \in G(H)$ such that $ab \in P$ implies $a \in P$ or $b \in rad(P)$ [14, Definition 2 (a)]. If H is a monoid and $G(H)$ be its quotient groupoid then H is an almost pseudo-valuation monoid if every r-prime ideal P of H is strongly r-primary that is, P satisfies the following property x, $y \in G(H)$ such that $xy \in P$ and if $x \notin P$ implies some power of y is contained in P [14, Definition 2 (b)].

Fractionary fuzzy ideals of Dedekind domains have been discussed in [8]. In terms of fuzzy fractionary r-ideals we can define strongly prime and strongly primary fuzzy r-ideal of monoids. A prime integral fractionary fuzzy r-ideal β_r of a monoid H is said to be strongly prime r-ideal if any fractionary fuzzy ideals μ_r and υ_r of a monoid H, $(\mu \circ \upsilon)_r \subseteq \beta_r$ implies that either $\mu_r \subseteq \beta_r$ or $\upsilon_r \subseteq \beta_r$.

In terms of fuzzy ideals we can re-define valuation monoid and almost valuation monoid as.

Definition 3.1. (a) A monoid H is said to be a valuation monoid, if for all $g' \in G(H)$ either $(g')_1 \subseteq \chi_H^{(0)}$ or $(\frac{1}{g'})_1 \subseteq \chi_H^{(0)}$. Equivalently, a monoid H is said to be a valuation monoid, if β_r be a $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal of H, $g', g'' \in G(H)$ and $< g'_1 > \circ < g_1^{''} > \subseteq \beta_r \Rightarrow g'g'' \in \beta_{r*}$, either $g' \in H$ or $g'' \in H$ otherwise $g'^{-1} \in H$ or $g''^{-1} \in H$.

(b) A monoid H is said to be almost valuation monoid, if $G(H)$ be a quotient groupoid of a monoid H and for all $g' \in G(H)$ either $(g'^n)_1 \subseteq \chi_H^{(0)}$ or $(\frac{1}{g'^n})_1 \subseteq \chi_H^{(0)}$.

We re-define psuedo-valuation monoid in terms of fuzzy fractionary r-ideals as.

Definition 3.2. A monoid H with quotient groupoid $G(H)$ is said to be pseudo-valuation monoid, if every $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal of a monoid H is a strongly prime fuzzy r-ideal of H.

We fuzzify an important relation between valuation monoids and pseudo-valuation monoids.

Proposition 3.3. *Every valuation monoid is a fuzzy pseudo-valuation monoid.*

Proof. Let H be a valuation monoid and β_r be a $\{0,1\}$-valued prime integral fractionary r-ideal of H. Suppose g', $g'' \in G(H)$ and $< g'_1 > \circ < g_1^{''} > \subseteq \beta_r \Rightarrow xy \in \beta_{r*}$, thus g', $g'' \in H$. If g', $g'' \notin H$ then $\beta_r(g') = \beta_r(g^{''}) = 0$. Since H is a valuation monoid so by definition 3.1(a), we have g'^{-1}, $g''^{-1} \in H$. Clearly, $g'' = g'g''g'^{-1} \in \beta_{r*}$, $g' = g'g'g^{''-1} \in \beta_{r*}$, which is a contradiction. Either $g' \in H$ or $g'' \in H$, suppose $g' \notin H$ it implies $< g'_1 > \nsubseteq \beta_r$, so $g_1^{''} \in \beta_{r*}$. Thus $< g_1^{''} > \subseteq \beta_r$ it implies that β_r is a strongly prime fuzzy ideal of R. □

In literature (see[6, Theorem 2.3]) relation between $\{0,1\}$-valued fuzzy ideals β and β_* of integral domains have been discussed, we needed it for monoids to continue further discussion. Here, first we define strongly primary fuzzy r-ideal and then we introduce the relation between $\{0,1\}$-valued fuzzy r-ideals β_r and $(\beta_r)_*$ of monoids.

Definition 3.4. Prime integral fractionary fuzzy r-ideal β_r of a monoid H is said to be strongly primary fuzzy r-ideal if any fractionary fuzzy r-ideals μ and υ of a monoid H, $(\mu \circ \upsilon)_r \subseteq \beta_r$ implies that either $\mu_r \subseteq \beta_r$ or $\upsilon_r \subseteq \sqrt{\beta_r}$.

Proposition 3.5. *Let β_r be a $\{0,1\}$-valued integral fractionary fuzzy r-ideal of a monoid H. Then the following statements are equivalent.*

(1) *β_r is a strongly primary fuzzy r-ideal of a monoid H.*

(2) *$(\beta_r)_*$ is a strongly primary r-ideal of a monoid H.*

Proof. (1) $\Rightarrow$ (2) Let β_r is a strongly primary fuzzy r-ideal of a monoid H, and let g', $g'' \in G(H)$ (quotient groupoid of H) and $g' \cdot_r g'' \in (\beta_r)_*$. Then $< g'_1 > \circ$

$< g_1^{''} >\subseteq \beta_r$. Since β_r is a strongly primary fuzzy r-ideal of $H \Rightarrow < g_1' > \subseteq \beta_r$ or $< g_1^{''} >\subseteq \sqrt{\beta_r}$. So $g' \in (\beta_r)_*$ or $g_1^{''} \subseteq (\sqrt{\beta_r})_* = \sqrt{(\beta_r)_*} \Rightarrow (\beta_r)_*$ is a strongly primary r-ideal of a monoid H.

(2) $\Rightarrow$ (1) On contradictory suppose there exist fractionary fuzzy r-ideals μ_r, υ_r of a monoid H and $\mu_r \circ \upsilon_r \subseteq \beta_r$, $\mu_r \subsetneq \beta_r$ and $\upsilon_r \nsubseteq \sqrt{\beta_r}$. This implies that there exist g', $g'' \in G(H)$ such that $\mu_r(g') > \beta_r(g')$ and $\upsilon_r(g'') > (\sqrt{\beta_r})(g'')$, as β_r is a $\{0, 1\}$-valued ideal, we have $\beta_r(g') = 0$ and $(\sqrt{\beta_r})(g'') = 0 \Rightarrow g' \notin (\beta_r)_*$ also $g'' \notin (\sqrt{\beta_r})_* = \sqrt{(\beta_r)_*}$. Since $g' ._r g'' \notin (\beta_r)_*$, and so $\beta_r(g'g'') = 0$. But we have, $0 = \beta_r(g'g'') \geq (\mu \circ \upsilon)_r(g'g'') = (\mu_r \circ \upsilon_r)(g'g'') \geq \mu_r(g') \wedge \upsilon_r(g'') > 0$, which is contradiction. Thus $\mu_r \subseteq \beta_r$ or $\upsilon_r \subseteq \sqrt{\beta_r}$, and hence β_r is a strongly primary fuzzy r-ideal of a monoid H by definition 3.4. □

Proposition 3.6. *H is an almost pseudo-valuation monoid if and only if every $\{0,1\}$-valued prime integral fractionary r-ideal of H is a strongly primary fuzzy r-ideal of a monoid H.*

Proof. Let H is an almost pseudo-valuation monoid, consider β_r be a $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal of H. Since H is an almost pseudo-valuation monoid so $(\beta_r)_*$ is a strongly primary r-ideal of H. Hence β_r is a strongly primary fuzzy r-ideal of a monoid H by proposition 3.5. Conversely, suppose P_r be r-prime ideal of a monoid H and assume that g', $g'' \in G(H)$ (quotient groupoid of H) and $g' ._r g'' \in P_r$. Then $< g_1' > \circ < g_1'' >\subseteq \chi_{P_r}^{(0)}$. But $< g_1' >\subseteq \chi_{P_r}^{(0)}$ or $< g_1'' > \subseteq \sqrt{\chi_{P_r}^{(0)}}$. Thus $g' \in P_r$ or $g'' \in (\sqrt{\chi_{P_r}^{(0)}})_* = \sqrt{P_r}$. Thus H is a almost pseudo-valuation monoid by [14, Definition 2 (b)]. □

Remark 3.7. Proposition 3.6 gives us the characterization of almost pseudo-valuation monoids in terms of fractionary fuzzy r-ideals.

Prime ideal P of H is said to be a pseudo-strongly r-prime ideal if, whenever x, $y \in G$ (quotient groupoid of H) and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in H$ or $y^m P \subseteq P$[14, Definition 3 (a)]. Monoid H is said to be pseudo-almost valuation monoid if and only if for every nonzero $x \in G(H)$, there is a positive integer $n \geq 1$ such that either $x^n \in H$ or $ax^{-n} \in H$ for every nonunit $a \in H$[14, Definition 4]. Pseudo-strongly prime r-ideal of a monoid and pseudo-almost valuation monoid can be defined in terms of fuzzy r-ideals as.

Definition 3.8. (a) Any $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal β_r of a monoid H is said to be a fuzzy pseudo-strongly prime r-ideal if $g', g'' \in G(H)$ and $(g'g'')_1 \circ \beta_r$ then there exist $m \in N$ such that $(g'^m)_1 \subseteq \chi_H^{(0)}$ or $(g''^m)_1 \circ \beta_r \subseteq \beta_r$.

(b) A monoid H is said to be pseudo-almost valuation monoid, if each $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal β_r of a monoid H is a fuzzy pseudo-strongly prime r-ideal.

Proposition 3.9. *Every almost pseudo-valuation monoid is a pseudo-almost valuation monoid.*

Proof. Suppose β be a $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal of H and $g \in E(H) = \{g \in G(H) : g^n \notin H$ for each $n \geq 1\}$. Since H is an almost pseudo-valuation monoid so β is a strongly primary fuzzy r-ideal. Thus $(\frac{1}{g})_1 \circ \beta \subseteq \beta \Rightarrow \beta$ is a pseudo-strongly prime fuzzy ideal. Hence H is a pseudo-almost valuation monoid. □

Proposition 3.10. *Every almost valuation monoid is a pseudo-almost valuation monoid.*

Proof. Let H is an almost-valuation monoid. Consider β_r be an integral fractionary fuzzy (prime) r-ideal, if $(g'^n)_1 \subseteq \chi_H^{(0)}$ then we are done, otherwise, $(\frac{1}{g'^n})_1 \circ \beta_r \subseteq \beta_r$ by definition 3.1(b). Hence β_r is a pseudo strongly prime fuzzy ideal by definition 3.8(b), and H is a pseudo-almost valuation monoid. □

Finally we fuzzify [14, Theorem 1] to show the linkage between different results of this note. First we recall few terminologies, following [7, definition 6.4] r-ideal M is called r-maximal if $M \neq H$ and there is no r-ideal J such that $M \subseteq J \subseteq H$, and a monoid H is called r-local, if H possesses exactly one r-maximal r-ideal [7, definition 6.5].

Definition 3.11. A fuzzy r-ideal M is called r-maximal if $M \neq H$ and there is no fuzzy r-ideal J such that $M \subseteq J \subseteq H$. Similarly, a monoid H is called r-local if H possesses exactly one r-maximal fuzzy r-ideal.

Theorem 3.12. *Let r be a finitary fuzzy ideal system on H and $M = H\backslash H^{\times}$ then the following are equivalent:*

(1) *H is almost pseudo-valuation monoid.*

(2) *If $P_r \in r-spec(H)$ is a $\{0,1\}$-valued prime integral fractionary fuzzy r- ideal and g', $g'' \in G(H)$, then $g'g'' \in P_r$ implies $g' \in P_r$ or $(g'')^n \in P_r$.*

(3) *For all $P_r \in r - spec(H)$ is a $\{0,1\}$-valued prime integral fractionary fuzzy r-ideal and $g' \in G(H)\backslash H$, we have $(\frac{1}{(g')^n})_1 \subseteq (P_r : P_r)$.*

(4) *H is r-local and or all $g' \in G(H)\backslash H$, we have $(\frac{1}{(g')^n})_1 \subseteq (M_r : M_r)$.*

(5) *H is r-local and $(M : M)$ is a valuation monoid with maximal fuzzy primary s-ideal M.*

(6) *H is r-local and there exist a valuation monoid V for H such that $\sqrt{M}$ is a maximal fuzzy s-ideal of V.*

Proof. (1) $\Rightarrow$ (2) Let H be an almost pseudo-valuation monoid, consider P_r be a $\{0,1\}$-valued prime integral fractionary r-ideal of H and assume that g', $g'' \in G(H)$ (quotient groupoid of H) and $g'g'' \in P_r$. Then $< g_1' > \circ < g_1'' > \subseteq \chi_{P_r}^{(0)}$. But $< g_1' > \subseteq \chi_{P_r}^{(0)}$ or $< g_1'' > \subseteq \sqrt{\chi_{P_r}^{(0)}}$. Thus $g' \in P_r$ or $g'' \in (\sqrt{\chi_{P_r}^{(0)}})_* = \sqrt{P_r}$.

(2) $\Rightarrow$ (3) Let $P_r \in r - spec(H)$ is a $\{0,1\}$-valued prime integral fractionary fuzzy r- ideal and $g' \in G(H)\backslash H$ be given. If $p \in P_r$, then consider $p = (\frac{p}{g'^n})_1 \circ (g_1')^n \in P_r$ implies $(\frac{p}{g'^n})_1 \in P_r$. Consequently, $(\frac{p}{g'^n})_1 \circ P_r \subseteq P_r$, and therefore $(\frac{p}{g'^n})_1 \in (P : P)$.

(3) $\Rightarrow$ (4) We must prove that $P_r \subset \sqrt{Q_r}$ for all $P \in r - spec(H)$ and $Q \in r-\max$ (H). Let $P \neq Q$ and fix some element $q \in Q\backslash P$, if $p \in P$ then $(\frac{1}{p^n})_1 \circ q \notin H$ implies $p^n q^{-1} Q \subset \sqrt{Q}$ and hence $p^n = (p^n q)q \in \sqrt{Q}$.

(4) $\Rightarrow$ (5) If $g' \in G(H)\backslash(M : M) \subset G(H)\backslash H$, then $(\frac{1}{(g')^n})_1 \in (M : M)$, and therefore $(M : M)$ is a valuation monoid. Since $M(M : M) \subset M$, M is an s-ideal

of $(M : M)$. If $g' \in (M : M)\backslash(M : M)^{\times}$ then $(\frac{1}{g'^n})_1 \notin (M : M)$ implies $g'^n \in H$, and since $g' \notin H^{\times}$, we obtain $g'^n \in M$. Therefore M is the maximal primary fuzzy s-ideal of $(M : M)$.

(5) $\Rightarrow$ (6) It is very clear.

(6) $\Rightarrow$ (1) If $g' \in G(H)\backslash H$ and $a^n \in H\backslash H^{\times} = M$, then $(\frac{1}{g'})_1 \in V$, and consequently $(\frac{1}{g'})_1 \circ a^n \in M \subset H$. $\square$

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Waheed Ahmad Khan (sirwak2003@yahoo.com)
Department of Mathematics and Statistics, Caledonian College of Engineering,
P. O box. 2322, Seeb 111, Sultanate of Oman

Abdelghani Taouti (ganitaouti@yahoo.com.au)
Department of Mathematics and Statistics, Caledonian College of Engineering,
P. O box. 2322, Seeb 111, Sultanate of Oman