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Fuzzy soft semiregularization spaces

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Abstract. The purpose of this study is to introduce fuzzy soft semiregularization space induced by a fuzzy soft topological space. We also investigate some properties of fuzzy soft semiregularization spaces.

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1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic, and precise in character. But, in real life situation, the problems in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. For this reason, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets [28], intuitionistic fuzzy sets [4], rough sets [23],i.e., which we can use as mathematical tools for dealing with uncertainties. Moreover, Zadeh [29] outlined the generalized theory of uncertainty in a much broader perspective. But all these theories have their inherent difficulties as what were pointed out by Molodtsov in [21]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theories. Consequently, Molodtsov [21] initiated the concept of soft set theory as a new mathematical tool for dealing with vagueness and uncertainties which is free from the above difficulties.

Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum. Molodtsov [21] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration, theory of measurement, and so on. Maji et al. [19] gave first practical application of soft sets in decision making problems. They have also introduced the concept of fuzzy soft set, a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. Ahmad and Kharal [2, 15] also made further contributions to the properties of fuzzy soft sets and fuzzy soft mappings. Soft set and fuzzy soft set theories have a rich potential for applications in several directions, a few of which have been shown by some authors (see [7, 20, 21, 22, 24, 25]).

The algebraic structure of soft set and fuzzy soft set theories dealing with uncertainties has also been studied by some authors. Aktaş and Çağman [3] have introduced the notion of soft groups. Jun [11] applied soft sets to the theory of BCK/BCI-algebras, and introduced the concept of soft BCK/BCI algebras. Jun and Park [12] and Jun et al. [13, 14] reported the applications of soft sets in ideal theory of BCK/BCI-algebras and d-algebras. Feng et al. [10] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Sun et al. [27] presented the definition of soft modules and construct some basic properties using modules and Molodtsov's definition of soft sets. Aygünoğlu and Aygün [5] introduced the concept of fuzzy soft group and in the meantime, discussed some properties and structural characteristic of fuzzy soft group.

The topological structure of fuzzy soft set theory has been considered by Aygünoğlu, Çetkin and Aygün [6]. According to this definition a fuzzy soft topology τ which is a mapping from the parameter set E to $[0,1]^{\widetilde{(X,E)}}$ which satisfies the three certain conditions, i.e., a fuzzy soft topology τ is a fuzzy soft set on the family of fuzzy soft sets $\widetilde{(X,E)}$. Also, since the value of a fuzzy soft set f_A under the mapping τ_e gives the degree of openness of the fuzzy soft set with respect to the parameter $e \in E$, τ_e can be thought as a fuzzy soft topology in the sense of Šostak [26]. In this manner, Çetkin et al.[8, 9] investigated the fuzzy soft topological structures by using different parameter sets for the fuzzy soft set and the fuzzy soft topology.

In this study, we extend the concept of a fuzzy semiregular space [1, 16] to the concept of fuzzy soft semiregular space. We also investigate some properties of fuzzy soft semiregularization spaces.

2. Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I^X is the set of all fuzzy sets on X (where, $I = [0,1]$), $I_0 = (0,1]$ and for $\lambda \in [0,1]$, $\overline{\lambda}(x) = \lambda$, for all $x \in X$.

Definition 2.1 ([2, 18])**.** f_A is called a fuzzy soft set on X, where f is a mapping from E into I^X,i.e., $f_e \triangleq f(e)$ is a fuzzy set on X, for each $e \in A$ and $f_e = \overline{0}$, if $e \notin A$, where $\overline{0}$ is zero function on X. f_e, for each $e \in E$, is called an element of the fuzzy soft set f_A.

$\widetilde{(X,E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe ([18]).

Definition 2.2 ([18])**.** For two fuzzy soft sets f_A and g_B on X, we say that f_A is a fuzzy soft subset of g_B and write $f_A \sqsubseteq g_B$ if $f_e \leq g_e$, for each $e \in E$.

Definition 2.3 ([18])**.** Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

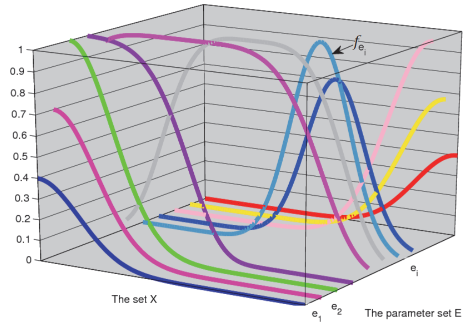

FIGURE 1. A fuzzy soft set f_E

Definition 2.4 ([18]). Union of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcup g_B$, where $C = A \cup B$ and $h_e = f_e \vee g_e$, for each $e \in E$. That is, $h_e = f_e \vee \overline{0} = f_e$ for each $e \in A - B$, $h_e = \overline{0} \vee g_e = g_e$ for each $e \in B - A$ and $h_e = f_e \vee g_e$, for each $e \in A \cap B$.

Definition 2.5 ([2, 18]). Intersection of two fuzzy soft sets f_A and g_B on X is the fuzzy soft set $h_C = f_A \sqcap g_B$, where $C = A \cap B$ and $h_e = f_e \wedge g_e$, for each $e \in E$.

Definition 2.6. The complement of a fuzzy soft set f_A is denoted by f_A^c, where $f^c : E \longrightarrow I^X$ is a mapping given by $f_e^c = \overline{1} - f_e$, for each $e \in E$.

Clearly $(f_A^c)^c = f_A$.

Definition 2.7 ([18]). (Null fuzzy soft set) A fuzzy soft set f_E on X is called a null fuzzy soft set and denoted by Φ, if $f_e = \overline{0}$, for each $e \in E$.

Definition 2.8. (Absolute fuzzy soft set) A fuzzy soft set f_E on X is called an absolute fuzzy soft set and denoted by $\widetilde{E}$, if $f_e = \overline{1}$, for each $e \in E$. Clearly $(\widetilde{E})^c = \Phi$ and $\Phi^c = \widetilde{E}$.

Definition 2.9. (λ-absolute fuzzy soft set) A fuzzy soft set f_E on X is called a λ-absolute fuzzy soft set and denoted by $\widetilde{E}^{\lambda}$, if $f_e = \overline{\lambda}$, for each $e \in E$. Clearly, $(\widetilde{E}^{\lambda})^c = \widetilde{E}^{1-\lambda}$.

Proposition 2.10 ([2]). *Let Δ be an index set and $f_A, g_B, h_C, (f_A)_i \triangleq (f_i)_{A_i}, (g_B)_i \triangleq (g_i)_{B_i} \in \widetilde{(X,E)}$, for all $i \in \Delta$. Then we have the following properties:*

(1) $f_A \sqcap f_A = f_A, \quad f_A \sqcup f_A = f_A$.
(2) $f_A \sqcap g_B = g_B \sqcap f_A, \quad f_A \sqcup g_B = g_B \sqcup f_A$.
(3) $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C, \quad f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap g_B) \sqcap h_C$.
(4) $f_A = f_A \sqcup (f_A \sqcap g_B), \quad f_A = f_A \sqcap (f_A \sqcup g_B)$.
(5) $f_A \sqcap \left(\bigsqcup_{i\in\Delta}(g_B)_i\right) = \bigsqcup_{i\in\Delta}(f_A \sqcap (g_B)_i)$.
(6) $f_A \sqcup (\sqcap_{i\in\Delta}(g_B)_i) = \sqcap_{i\in\Delta}(f_A \sqcup (g_B)_i)$.
(7) $\Phi \sqsubseteq f_A \sqsubseteq \widetilde{E}$.
(8) $(f_A^c)^c = f_A$.
(9) $(\sqcap_{i\in\Delta}(f_A)_i)^c = \bigsqcup_{i\in\Delta}(f_A)_i^c$.

(10) $\left(\bigsqcup_{i\in\Delta}(f_A)_i\right)^c = \sqcap_{i\in\Delta}(f_A)_i^c$.
(11) *If* $f_A \sqsubseteq g_B$, *then* $g_B^c \sqsubseteq f_A^c$.

3. Fuzzy Soft Topological Spaces

In this section we recall and give some results of fuzzy soft topological spaces which we need for the next section.

Definition 3.1 ([6]). A mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$.

(O1) $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$.

(O2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$, for all $f_A, g_B \in \widetilde{(X,E)}$.

(O3) $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta}\tau_e((f_A)_i)$, for all $(f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.

A fuzzy soft topology is called enriched (stratified) if it provides that

(O1)$'$ $\tau_e(\widetilde{E}^\lambda) = 1$.

Then the pair (X, τ_E) is called a fuzzy soft topological space (fsts, for short). The value $\tau_e(f_A)$ is interpreted as the degree of openness of a fuzzy soft set f_A with respect to parameter $e \in E$.

Let τ_E^1 and τ_E^2 be fuzzy soft topologies on X. We say that τ_E^1 is finer than τ_E^2 (τ_E^2 is coarser than τ_E^1), denoted by $\tau_E^2 \sqsubseteq \tau_E^1$, if $\tau_e^2(f_A) \leq \tau_e^1(f_A)$ for each $e \in E, f_A \in \widetilde{(X,E)}$.

Example 3.2 ([6]). Let $\mathcal{T}$ be a fuzzy topology on X in Šostak's sense, that is, $\mathcal{T}$ is a mapping from I^X to I. Take $E = I$ and define $\overline{\mathcal{T}} : E \longrightarrow I^X$ as $\overline{\mathcal{T}}(e) \triangleq \{\mu : \mathcal{T}(\mu) \geq e\}$ which is levelwise fuzzy topology of $\mathcal{T}$ in Chang's sense, for each $e \in I$. However, it is well known that each Chang's fuzzy topology can be considered as Šostak fuzzy topology by using fuzzifying method. Hence, $\mathcal{T}(e)$ satisfies (O1), (O2) and (O3).

According to this definition and by using the decomposition theorem of fuzzy sets [17], if we know the resulting fuzzy soft topology, then we can find the first fuzzy topology. Therefore, we can say that a fuzzy topology can be uniquely represented as a fuzzy soft topology.

Example 3.3. Let $E = \mathbb{N}$ be the set of natural numbers and $\tau : E \to [0,1]^{\widetilde{(X,E)}}$ be defined as follows:

$$\tau_e(f_A) = \begin{cases} 1, & \text{if } f_A = \Phi, \widetilde{E}, \\ \frac{1}{e}, & \text{otherwise.} \end{cases}, \quad \forall e \in E.$$

It is easy to testify that τ_E is a fuzzy soft topology on X.

Proposition 3.4 ([6]). *Let* $\{\tau_k\}_{k\in\Gamma}$ *be a family of fuzzy soft topologies on* X. *Then* $\tau = \bigwedge_{k\in\Gamma}\tau_k$ *is also a fuzzy soft topology on* X, *where* $\tau_e(f_A) = \bigwedge_{k\in\Gamma}(\tau_k)_e(f_A)$, *for each* $e \in E, f_A \in \widetilde{(X,E)}$.

Definition 3.5 ([6]). A mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft cotopology on X if it satisfies the following conditions for each $e \in E$:

(C1) $\eta_e(\Phi) = \eta_e(\widetilde{E}) = 1$.

(C2) $\eta_e(f_A \sqcup g_B) \geq \eta_e(f_A) \wedge \eta_e(g_B)$, for all $f_A, g_B \in \widetilde{(X,E)}$.
(C3) $\eta_e(\sqcap_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta} \eta_e((f_A)_i)$, for all $(f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.
The pair (X, η_E) is called a fuzzy soft cotopological space.

Let τ_E be a fuzzy soft topology on X, then the mapping $\eta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\eta_e(f_A) = \tau_e(f_A^c)$, for each $e \in E$ is a fuzzy soft cotopology on X. Let η_E be a fuzzy soft cotopology on X, then the mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ defined by $\tau_e(f_A) = \eta_e(f_A^c)$, for each $e \in E$, is a fuzzy soft topology on X.

Definition 3.6 ([6]). A mapping $\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft base on X if it satisfies the following conditions for each $e \in E$:
(B1) $\beta_e(\Phi) = \beta_e(\widetilde{E}) = 1$.
(B2) $\beta_e(f_A \sqcap g_B) \geq \beta_e(f_A) \wedge \beta_e(g_B)$, for all $f_A, g_B \in \widetilde{(X,E)}$.

Theorem 3.7 ([6]). *Let β_E be a fuzzy soft base on X. Define a map $\tau_\beta : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:*

$$(\tau_\beta)_e(f_A) = \bigvee \left\{ \bigwedge_{j\in\Lambda} \beta_e((f_A)_j) \mid f_A = \bigsqcup_{j\in\Lambda} (f_A)_j \right\}, \quad \forall e \in E.$$

Then $(\tau_\beta)_E$ is the coarsest fuzzy soft topology on X for which $(\tau_\beta)_e(f_A) \geq \beta_e(f_A)$, for all $e \in E, f_A \in \widetilde{(X,E)}$.

Definition 3.8. If β_E is a base on X, then $(\tau_\beta)_E$ is called the fuzzy soft topology generated by β_E. $(X, (\tau_\beta)_E)$ is called a fuzzy soft topological space generated by a base β_E on X.

Theorem 3.9 ([8]). *Let (X, τ_E) be a fuzzy soft topological space. Define a mapping $\mathcal{C}_\tau : E \times \widetilde{(X,E)} \times I_0 \longrightarrow \widetilde{(X,E)}$ as follows:*

$$\mathcal{C}_\tau(e, f_A, r) = \sqcap\{g_B \in \widetilde{(X,E)} \mid f_A \sqsubseteq g_B, \quad \tau_e(g_B^c) \geq r\}.$$

Then, for $e \in E, f_A, g_B \in \widetilde{(X,E)}$ and $r, s \in I_0$, the mapping $\mathcal{C}_\tau$ satisfies the following conditions:
(C1) $\mathcal{C}_\tau(e, \Phi, r) = \Phi$.
(C2) $f_A \sqsubseteq \mathcal{C}_\tau(e, f_A, r)$.
(C3) If $f_A \sqsubseteq g_B$, *then* $\mathcal{C}_\tau(e, f_A, r) \sqsubseteq \mathcal{C}_\tau(e, g_B, r)$.
(C4) If $r \leq s$, *then* $\mathcal{C}_\tau(e, f_A, r) \sqsubseteq \mathcal{C}_\tau(e, f_A, s)$.
(C5) $\mathcal{C}_\tau(e, f_A \sqcup g_B, r) = \mathcal{C}_\tau(e, f_A, r) \sqcup \mathcal{C}_\tau(e, g_B, r)$.
(C6) $\mathcal{C}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) = \mathcal{C}_\tau(e, f_A, r)$.

Theorem 3.10 ([8]). *Let (X, τ_E) be a fuzzy soft topological space. Define a mapping $\mathcal{I}_\tau : E \times \widetilde{(X,E)} \times I_0 \longrightarrow \widetilde{(X,E)}$ by:*

$$\mathcal{I}_\tau(e, f_A, r) = \bigsqcup\{g_B \in \widetilde{(X,E)} \mid g_B \sqsubseteq f_A \text{ and } \tau_e(g_B) \geq r\}.$$

Then, we have followings:
(1) $\mathcal{I}_\tau(e, f_A^c, r) = (\mathcal{C}_\tau(e, f_A, r))^c$.
(2) For $e \in E, f_A, g_B \in \widetilde{(X,E)}$ and $r, s \in I_0$, the mapping $\mathcal{I}_\tau$ satisfies the following conditions:

(I1) $\mathcal{I}_\tau(e, \widetilde{E}, r) = \widetilde{E}$.
(I2) $\mathcal{I}_\tau(e, f_A, r) \sqsubseteq f_A$.
(I3) If $f_A \sqsubseteq g_B$ *and* $r \leq s$, *then* $\mathcal{I}_\tau(e, f_A, s) \sqsubseteq \mathcal{I}_\tau(e, g_B, r)$.
(I4) $\mathcal{I}_\tau(e, f_A \sqcap g_B, r) = \mathcal{I}_\tau(e, f_A, r) \sqcap \mathcal{I}_\tau(e, g_B, r)$.
(I5) $\mathcal{I}_\tau(e, \mathcal{I}_\tau(e, f_A, r), r) = \mathcal{I}_\tau(e, f_A, r)$.

4. Fuzzy Soft Semiregularization Spaces

In this section, we study the relationships between fuzzy soft semiregularization spaces and fuzzy soft semiregular spaces.

Definition 4.1. Let (X, τ_E) be a fuzzy soft topological space.

(a) A fuzzy soft set f_A is said to be fuzzy soft regularly open if there exists $r_0 \in I_0$ such that $f_A = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$ for all $r \leq r_0$ and $e \in E$.

(b) A fuzzy soft set g_B is said to be fuzzy soft regularly closed if there exists $r_1 \in I_0$ such that $g_B = \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, g_B, r), r)$ for all $r \leq r_1$ and $e \in E$.

Lemma 4.2. *Let* (X, τ_E) *be a fuzzy soft topological space. Then we have the following statements.*

(1) A fuzzy soft set $f_A \in \widetilde{(X, E)}$ *is fuzzy soft regularly open if and only if* f_A^c *is fuzzy soft regularly closed.*

(2) If $f_A = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$, *for all* $e \in E, r \leq r_0$, *then* $f_A = \mathcal{I}_\tau(e, f_A, r)$ *for all* $e \in E, r \leq r_0$.

Proof. (1) We easily prove it from the following result: for each $e \in E, r \leq r_0$, $f_A = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$, we have

$$\begin{aligned} f_A^c &= (\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r))^c \\ &= \mathcal{C}_\tau(e, (\mathcal{C}_\tau(e, f_A, r))^c, r) \\ &= \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, f_A^c, r), r). \end{aligned}$$

(2) For all $e \in E, r \leq r_0$, we have $f_A = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$. Hence $\mathcal{I}_\tau(e, f_A, r) = \mathcal{I}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r)$ and $\mathcal{I}_\tau(e, f_A, r) = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) = f_A$, from (I5) of Theorem 3.10(2). □

Example 4.3. Let $X = \{x, y\}$ be a classical set and $E = \{e_1, e_2, e_3\}$ be the parameter set of X. Let $f_E, g_E \in \widetilde{(X, E)}$ be as follows:

$$f_e(x) = 0.3, f_e(y) = 0.4, g_e(x) = 0.6, g_e(y) = 0.2, \quad \forall e \in E.$$

We define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_e(h_E) = \begin{cases} 1, & \text{if } h_E = \Phi, \widetilde{E}, \\ \frac{1}{2} & \text{if } h_E = f_E, \\ \frac{2}{3} & \text{if } h_E = g_E, \\ \frac{2}{3} & \text{if } h_E = f_E \sqcap g_E, \\ \frac{1}{2} & \text{if } h_E = f_E \sqcup g_E, \\ 0, & \text{otherwise.} \end{cases}, \quad \forall e \in E.$$

Theorem 3.9 and Theorem 3.10, we obtain the following:
$\widetilde{E} = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, \widetilde{E}, r), r), \forall r \in I_0$,
$f_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_E, r), r), 0 < r \leq \frac{1}{2}$,

$f_E \sqcup g_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, g_E, r), r), 0 < r \leq \frac{1}{2}$,
$g_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, g_E, r), r), \frac{1}{2} < r \leq \frac{2}{3}$,
$f_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_E \sqcap g_E, r), r), 0 < r \leq \frac{1}{2}$,
$f_E \sqcap g_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_E \sqcap g_E, r), r), \frac{1}{2} < r \leq \frac{2}{3}$,
$f_E \sqcup g_E = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_E \sqcup g_E, r), r), 0 < r \leq \frac{1}{2}$.

Hence f_E and $f_E \sqcup g_E$ are fuzzy soft regularly open. But g_E and $f_E \sqcap g_E$ are not fuzzy soft regularly open. We have $g_E = \mathcal{I}_\tau(e, g_E, r), 0 < r \leq \frac{2}{3}, e \in E$ but $g_E \neq \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, g_E, r), r), 0 < r \leq \frac{2}{3}$. Hence the converse of Lemma 4.2 (2) is not true.

In the following theorem, we construct the fuzzy soft topology generated by fuzzy soft regularly open sets.

Theorem 4.4. *Let (X, τ_E) be a fuzzy soft topological space and Θ_τ be a family of all fuzzy soft regularly open sets. Define a mapping $\beta_\tau : E \longrightarrow [0,1]^{\Theta_\tau}$ by*

$$(\beta_\tau)_e(f_A) = \bigvee\{r \in I_0 \mid f_A = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)\}$$

Then $(\beta_\tau)_E$ is a fuzzy soft base on X such that $\tau_{\beta_\tau} \sqsubseteq \tau$.

Proof. (B1) It immediately follows from the definition of the mapping β_τ.

(B2) Suppose there exist $(f_A)_1, (f_A)_2 \in \Theta_\tau$ and $e \in E$ such that

$$(\beta_\tau)_e((f_A)_1 \sqcap (f_A)_2) < (\beta_\tau)_e((f_A)_1) \wedge (\beta_\tau)_e((f_A)_2).$$

From the definition of $(\beta_\tau)_e$, there exist $r_1, r_2 \in I_0$ with for each $0 < r \leq r_1, r_2$,

$$(f_A)_i = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_i, r), r)$$

such that

$$(\beta_\tau)_e((f_A)_1 \sqcap (f_A)_2) < r_1 \wedge r_2 \leq (\beta_\tau)_e((f_A)_1) \wedge (\beta_\tau)_e((f_A)_2).$$

Put $r_0 = r_1 \wedge r_2$. Since $\mathcal{I}_\tau(e, (f_A)_i, r) = (f_A)_i$ for all $0 < r \leq r_1, r_2$ from Lemma 4.2(2), we have for each $0 < r \leq r_0$

$$\begin{array}{rcl} \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_1 \sqcap (f_A)_2, r), r) & \sqsupseteq & \mathcal{I}_\tau(e, (f_A)_1 \sqcap (f_A)_2, r) \\ & = & \mathcal{I}_\tau(e, (f_A)_1, r) \sqcap \mathcal{I}_\tau(e, (f_A)_2, r) \\ & = & (f_A)_1 \sqcap (f_A)_2. \qquad (A) \end{array}$$

On the other hand, since $(f_A)_i = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_i, r), r)$, for each $0 < r \leq r_0$, we have

$$\begin{array}{rcl} \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_1 \sqcap (f_A)_2, r), r) & \sqsubseteq & \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_1, r) \sqcap \mathcal{C}_\tau(e, (f_A)_2, r), r) \\ & = & \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_1, r), r) \sqcap \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_2, r), r) \\ & = & (f_A)_1 \sqcap (f_A)_2. \qquad (B) \end{array}$$

From (A) and (B), we have for each $0 < r \leq r_0$,

$$(f_A)_1 \sqcap (f_A)_2 = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_1 \sqcap (f_A)_2, r), r).$$

Thus $(\beta_\tau)_e((f_A)_1 \sqcap (f_A)_2) \geq r_0$, for each $e \in E$. It is a contradiction. Hence,

$$(\beta_\tau)_e((f_A)_1 \sqcap (f_A)_2) \geq (\beta_\tau)_e((f_A)_1) \wedge (\beta_\tau)_e((f_A)_2), \quad \forall e \in E, (f_A)_1, (f_A)_2 \in \Theta_\tau.$$

Finally, we will show that $(\tau_{\beta_\tau})_E \sqsubseteq \tau_E$. Suppose there exist $f_A \in \widetilde{(X, E)}, e \in E$ and $r_1 \in I_0$ such that

$$(\tau_{\beta_\tau})_e(f_A) > r_1 > \tau_e(f_A).$$

From Theorem 3.7, there exists a family $\{(f_A)_i \in \Theta_\tau \mid f_A = \bigsqcup_{i\in\Gamma}(f_A)_i\}$ such that

$$(\tau_{\beta_\tau})_e(f_A) \geq \bigwedge_{i\in\Gamma}(\beta_\tau)_e((f_A)_i) > r_1 > \tau_e(f_A).$$

For each $i \in \Gamma$, since $(\beta_\tau)_e((f_A)_i) > r_1$, there exists $r_i \in I_0$ with for each $0 < r \leq r_i$,

$$(f_A)_i = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_i, r), r)$$

such that $(\beta_\tau)_e((f_A)_i) \geq r_i > r_1$.

On the other hand, for $i \in \Gamma$, since $\mathcal{I}_\tau(e, (f_A)_i, r_i) = (f_A)_i$ from Lemma 4.2 (2), we have $\tau_e((f_A)_i) \geq r_i$. Thus,

$$\tau_e(f_A) \geq \bigwedge_{i\in\Gamma}\tau_e((f_A)_i) \geq \bigwedge_{i\in\Gamma} r_i \geq r_1.$$

It is a contradiction. Hence, $(\tau_{\beta_\tau})_E \sqsubseteq \tau_E$. □

Example 4.2 In Example 4.1, we have $\Theta_\tau = \{\widetilde{E}, f_E, f_E \sqcup g_E\}$ and $(\tau_{\beta_\tau})_E$ as follows:

$$(\tau_{\beta_\tau})_e(h_C) = \begin{cases} 1, & \text{if } h_C = \Phi, \widetilde{E}, \\ \frac{1}{2} & \text{if } h_C = f_E, \\ \frac{1}{2} & \text{if } h_C = f_E \sqcup g_E, \\ 0, & \text{otherwise}. \end{cases}, \quad \forall e \in E.$$

Moreover, we have $(\tau_{\beta_\tau})_E \sqsubseteq \tau_E$.

We simply write $(\tau_s)_E$ instead of $(\tau_{\beta_\tau})_E$.

Definition 4.5. A fuzzy soft topological space $(X, (\tau_s)_E)$ is said to be the fuzzy soft semiregularization space (fssrs, for short) of (X, τ_E). A fuzzy soft topological space (X, τ_E) is said to be fuzzy soft semiregular if $\tau_E = (\tau_s)_E$.

Lemma 4.6. *Let $(X, (\tau_s)_E)$ be the fssrs of a fsts (X, τ_E). Then we have the following properties:*

(1) $\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r), r)$, for all $f_A \in \widetilde{(X, E)}$, $e \in E$ and $r \in I_0$.

(2) If $h_C = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$ for $0 < r \leq r_0$ and $e \in E$, then $(\tau_s)_e \geq r_0$.

(3) If $g_B = \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, f_A, r), r)$ for $0 < r \leq r_0$ and $e \in E$, then $g_B = \mathcal{C}_{\tau_s}(e, \mathcal{I}_\tau(e, f_A, r), r)$ for $0 < r \leq r_0$.

(4) If $h_C = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$ and $\mathcal{C}_{\tau_s}(e, f_A, r) = \mathcal{C}_\tau(e, f_A, r)$ for $0 < r \leq r_0$ and $e \in E$, then $h_C = \mathcal{I}_{\tau_s}(e, \mathcal{C}_{\tau_s}(e, f_A, r), r)$ for $0 < r \leq r_0$ and $e \in E$.

Proof. (1) Since $\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) \sqsubseteq \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r)$ from (C2) of Theorem 3.9, we have

$$\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) \sqsubseteq \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r), r). \text{ (D)}$$

Conversely, since $\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) \sqsubseteq \mathcal{C}_\tau(e, f_A, r)$, we have

$$\mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r) \sqsubseteq \mathcal{C}_\tau(e, f_A, r)$$

$$\Rightarrow \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r), r) \sqsubseteq \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r). \text{ (E)}$$

Thus, by (D) and (E), we have

$$\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r) = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r), r), r).$$

(2) From (1), put $h_C = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$ for $0 < r \leq r_0, e \in E$. Then, for $0 < r \leq r_0$ and $e \in E$, $h_C = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, h_C, r), r)$. Hence, h_C is fuzzy soft regularly open. Thus $(\tau_s)_e(h_C) \geq r_0$, for $e \in E$.

(3) Since by Theorem 3.10(1), for $0 < r \leq r_0$ and $e \in E$,

$$g_B^c = (\mathcal{C}_\tau(e, \mathcal{I}_\tau(e, f_A, r), r))^c = \mathcal{I}_\tau(e, (\mathcal{I}_\tau(e, f_A, r))^c, r) = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A^c, r), r),$$

by (2), $(\tau_s)_e(g_B^c) \geq r_0$. It implies that for $0 < r \leq r_0$ and $e \in E$,

$$\mathcal{C}_{\tau_s}(e, \mathcal{I}_\tau(e, f_A, r), r) = \sqcap\{h_C \mid \mathcal{I}_\tau(e, f_A, r) \sqsubseteq h_C \text{ and } (\tau_s)_e(h_C^c) \geq r\} \sqsubseteq g_B.$$

On the other hand, we have for $0 < r \leq r_0$ and $e \in E$,

$$\begin{aligned} & \mathcal{C}_{\tau_s}(e, \mathcal{I}_\tau(e, f_A, r), r) = \sqcap\{h_C \mid \mathcal{I}_\tau(e, f_A, r) \sqsubseteq h_C \text{ and } (\tau_s)_e(h_C^c) \geq r\} \\ \sqsupseteq\ & \sqcap\{h_C \mid \mathcal{I}_\tau(e, f_A, r) \sqsubseteq h_C \text{ and } \tau_e(h_C^c) \geq r\} \qquad ((\tau_s)_E \sqsubseteq \tau_E) \\ =\ & \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, f_A, r), r) = g_B. \end{aligned}$$

(4) Let $h_C = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r)$ for $0 < r \leq r_0$ and $e \in E$. Then for $0 < r \leq r_0$ and $e \in E$,

$h_C^c = (\mathcal{I}_\tau(e, \mathcal{C}_\tau(e, f_A, r), r))^c = \mathcal{C}_\tau(e, (\mathcal{C}_\tau(e, f_A, r))^c, r) = \mathcal{C}_\tau(e, \mathcal{I}_\tau(e, f_A^c, r), r) = \mathcal{C}_{\tau_s}(e, \mathcal{I}_\tau(e, f_A^c, r), r)$. (*by*(3))

It implies, for $0 < r \leq r_0$ and $e \in E$,

$h_C = (\mathcal{C}_{\tau_s}(e, \mathcal{I}_\tau(e, f_A^c, r), r))^c = \mathcal{I}_{\tau_s}(e, (\mathcal{I}_\tau(e, f_A^c, r))^c, r) = \mathcal{I}_{\tau_s}(e, \mathcal{C}_\tau(e, f_A, r), r) = \mathcal{I}_{\tau_s}(e, \mathcal{C}_{\tau_s}(e, f_A, r), r)$,

since $\mathcal{C}_{\tau_s}(e, f_A, r) = \mathcal{C}_\tau(e, f_A, r)$. □

Theorem 4.7. *Let $(X, (\tau_s)_E)$ be the fssrs of a fsts (X, τ_E) and $\beta_\tau : E \longrightarrow [0,1]^{\theta_\tau}$ be a base of the fssrs $(X, (\tau_s)_E)$. If $\mathcal{C}_{\tau_s}(e, (f_A)_i, r) = \mathcal{C}_\tau(e, (f_A)_i, r)$ for $0 < r \leq r_i$ such that $(\beta_\tau)_e((f_A)_i) = r_i$ for each $(f_A)_i \in \Theta_\tau, e \in E$. Then $(X, (\tau_s)_E)$ is fuzzy semiregular.*

Proof. We only show that $(\tau_s)_E \sqsubseteq ((\tau_s)_s)_E$ because $(\tau_s)_E \sqsupseteq ((\tau_s)_s)_E$ from Theorem 4.4. Suppose there exist $f_A \in \widetilde{(X, E)}, e \in E$ and $r_1 \in I_0$ such that

$$(\tau_s)_e(f_A) > r_1 > ((\tau_s)_s)_e(f_A).$$

From Theorem 3.7, there exists a family $\{(f_A)_i \in \theta_\tau \mid f_A = \bigsqcup_{i \in \Gamma} (f_A)_i\}$ such that

$$(\tau_s)_e(f_A) \geq \bigwedge_{i \in \Gamma} (\beta_\tau)_e((f_A)_i) > r_1 > ((\tau_s)_s)_e(f_A).$$

For each $i \in \Gamma$, since $(\beta_\tau)_e((f_A)_i) > r_1$, there exist $r_i \in I_0$ with for each $0 < r \leq r_i$ and $e \in E$,

$$(f_A)_i = \mathcal{I}_\tau(e, \mathcal{C}_\tau(e, (f_A)_i, r), r)$$

such that

$$(\beta_\tau)_e((f_A)_i) \geq r_i > r_1.$$

Since $\mathcal{C}_{\tau_s}(e,(f_A)_i,r) = \mathcal{C}_\tau(e,(f_A)_i,r)$ for $0 < r \leq r_i$ and $e \in E$, we have

$$(f_A)_i = \mathcal{I}_\tau(e,\mathcal{C}_\tau(e,(f_A)_i,r),r) = \mathcal{I}_{\tau_s}(e,\mathcal{C}_{\tau_s}(e,(f_A)_i,r),r). \quad \textit{by} \quad \text{(Lemma 4.6(4))}$$

It implies $(f_A)_i \in \Theta_{\tau_s}$ with $(\beta_{\tau_s})_e((f_A)_i) \geq r_i$. Thus,

$$((\tau_s)_s)_e(f_A) \geq \bigwedge_{i\in\Gamma} (\beta_{\tau_s})_e((f_A)_i) \geq \bigwedge_{i\in\Gamma} r_i \geq r_1.$$

It is a contradiction. Hence, $(\tau_s)_E \sqsubseteq ((\tau_s)_s)_E$. □

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