Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 4, (April 2014), pp. 653–660
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

The Laplace decomposition method for solving n-th order fuzzy differential equations

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Received 24 July 2013; Revised 15 August 2013; Accepted 21 August 2013

Abstract. In this paper, the Laplace decomposition method is employed to obtain approximate analytical solution of the n-th order fuzzy differential equations with fuzzy initial conditions. Some illustrative examples are presented to show the efficiency of the present method in comparison with the exact solution.

2010 AMS Classification: 34A07

Keywords: Laplace decomposition method, Fuzzy number, n-th order fuzzy differential equation.

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1. Introduction

Theory of fuzzy sets has been developed, and applied in a variety of fields [7, 18, 20, 23] and the fuzzy differential equations (FDEs) is just one of the branches of fuzzy set theory which has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh[8]; it was followed up by Dubois and Prade [10], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [22] and Goetschel and Voxman [10]. Kandel and Byatt [15] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [16, 17], Seikkala [24], He and Yi [11], Kloeden [19] and by other researchers (see [6, 9, 14]).

The numerical methods for solving fuzzy differential equations are introduced in [2, 3]. Buckley and Feuring [5] introduced two analytical methods for solving n-th order linear differential equations with fuzzy initial conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, in that they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function.

Recently the Homotopy perturbation and the variational iteration methods has been applied for solving n-th order fuzzy differential equations with fuzzy initial conditions [1, 12, 13]. In this study, the Laplace decomposition method (LDM) is applied to n-th order fuzzy differential equations. The rest of this paper is structured as follows:
In section 2, some basic definitions and results which will be used later are brought. In section 3, we shall propose fuzzy Laplace decomposition method for solving fuzzy differential equation . Then the proposed method is implemented to two numerical-analytical examples in section 4 and finally, conclusion is drawn in section 5.

2. Preliminaries

In this section the most basic notations used in fuzzy calculus are introduced [10, 16, 21]. We start with defining a fuzzy number.

Definition 2.1. A fuzzy number is a fuzzy set $u : R^1 \longrightarrow I = [0, 1]$ which satisfies
(i) u is upper semi-continuous.
(ii) $u(x) = 0$ outside some interval $[c, d]$.
(iii) There are real numbers $a, b : c \leq a \leq b \leq d$ for which
1. $u(x)$ is monotonic increasing on $[c, a]$,
2. $u(x)$ is monotonic decreasing on $[b, d]$,
3. $u(x) = 1$, $a \leq x \leq b$.

The set of all fuzzy numbers (as given by Definition 1) is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva[16].

Definition 2.2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$;$0 \leq r \leq 1$ which satisfying the following requirements:
i. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
ii. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
iii. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k > 0$ we define addition $(u + v)$ and multiplication by k as

$$(\underline{u+v})\,(r) = \underline{u}\,(r) + \underline{v}\,(r)\,, \quad (\overline{u+v})\,(r) = \overline{u}\,(r) + \overline{v}\,(r)\,, \tag{2.1}$$

$$(\underline{ku})\,(r) = k\underline{u}(r), \qquad (\overline{ku})\,(r) = k\overline{u}(r). \tag{2.2}$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (2.1) and (2.2) is denoted by E^1 and is a convex cone. It can be shown that Eqs. (2.1) and (2.2) are equivalent to the addition and multiplication as defined by using the α-cut approach[10] and the extension principles[21]. We will next define the fuzzy function notation and a metric D in E^1[10].

Definition 2.3 ([4])**.** For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u, v) = max\ \{sup_{0\leq r\leq 1}|\underline{u}(r) - \underline{v}(r)|,\ sup_{0\leq r\leq 1}|\overline{u}(r) - \overline{v}(r)|\} \tag{2.3}$$

is the distance between u and v.

3. Laplace decomposition method for n-th order FDEs

The purpose of this section is to discuss the use of LDM for solving n-th order fuzzy differential equations. We consider general inhomogeneous nonlinear equation with initial conditions is given below:

$$Lu(t) + Ru(t) + Nu(t) = h(t), \tag{3.1}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the non-linear terms and $h(t)$ is the source term. First we explain the main idea of LDM:
The methodology consists of applying Laplace transform on both sides of (3.1)

$$\mathfrak{L}[Lu(t)] + \mathfrak{L}[Ru(t)] + \mathfrak{L}[Nu(t)] = \mathfrak{L}[h(t)]. \tag{3.2}$$

Using the differential property of Laplace transform and initial conditions we get

$$\begin{aligned} s^n\mathfrak{L}[u(t)] &- s^{n-1}u(0) - s^{n-2}u'(0) - ... - u^{n-1}(0) \\ &+ \mathfrak{L}[Ru(t)] + \mathfrak{L}[Nu(t)] = \mathfrak{L}[h(t)], \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathfrak{L}[u(t)] &= \frac{u(0)}{s} + \frac{u'(0)}{s^2} + ... + \frac{u^{n-1}(0)}{s^n} \\ &- \frac{1}{s^n}\mathfrak{L}[Ru(t)] - \frac{1}{s^n}\mathfrak{L}[Nu(t)] + \frac{1}{s^n}\mathfrak{L}[h(t)]. \end{aligned} \tag{3.4}$$

The next step is representing solutions as an infinite series given below

$$u(t) = \sum_{i=0}^{\infty} u_i(t), \tag{3.5}$$

and the nonlinear operator is decomposed as

$$Nu(t) = \sum_{i=0}^{\infty} A_i, \tag{3.6}$$

where A_n is Adomian polynomials [25] of $u_0, u_1, ...u_n$ and can be calculated by below formula:

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}[N(\sum_{\alpha=0}^{\infty} \lambda^\alpha u_\alpha)]_{\lambda=0}, \qquad i = 0, 1, 2, ..., \tag{3.7}$$

putting Eq.(3.5)and Eq.(3.6) in Eq.(3.4) we will get

$$\begin{aligned} \mathfrak{L}[\sum_{i=0}^{\infty} u_i(t)] &= \frac{u(0)}{s} + \frac{u'(0)}{s^2} + ... + \frac{u^{n-1}(0)}{s^n} \\ &- \frac{1}{s^n}\mathfrak{L}[Ru(t)] - \frac{1}{s^n}\mathfrak{L}[\sum_{i=0}^{\infty} A_i] + \frac{1}{s^n}\mathfrak{L}[h(t)], \end{aligned} \tag{3.8}$$

on comparing both side of Eq.(3.8) and by using standard ADM we have:

$$\mathfrak{L}[u_0(t)] = \frac{u(0)}{s} + \frac{u'(0)}{s^2} + \cdots + \frac{u^{n-1}(0)}{s^n} + \frac{1}{s^n}\mathfrak{L}[h(t)] = k(s), \tag{3.9}$$

$$\mathfrak{L}[u_1(t)] = -\frac{1}{s^n}\mathfrak{L}[Ru_0(t)] - \frac{1}{s^n}\mathfrak{L}[A_0], \tag{3.10}$$

$$\mathfrak{L}[u_2(t)] = -\frac{1}{s^n}\mathfrak{L}[Ru_1(t)] - \frac{1}{s^n}\mathfrak{L}[A_1]. \tag{3.11}$$

In general, the recursive relation is given by

$$\mathfrak{L}[u_{i+1}(t)] = -\frac{1}{s^n}\mathfrak{L}[Ru_i(t)] - \frac{1}{s^n}\mathfrak{L}[A_i] \qquad i \geq 0. \tag{3.12}$$

Applying inverse Laplace transform to Eq.(3.9)-Eq.(3.12), our required recursive relation is given by

$$u_0(t) = G(t), \tag{3.13}$$

$$u_{i+1}(t) = -L^{-1}[\frac{1}{s^n}\mathfrak{L}[Ru_i(t)] + \frac{1}{s^n}\mathfrak{L}[A_i]] \qquad i \geq 0, \tag{3.14}$$

where $G(t)$ represents the term arising from source term and prescribed initial conditions.

4. Numerical examples

To give a clear overview of the method and show the ability of the method, we present the following examples.

Example 4.1 ([4]). Consider the following second-order fuzzy linear differential equation

$$\begin{cases} y'' - 4y' + 4y = 0, \quad t \in [0,1], \\ \widetilde{y}(0) = (2+r, 4-r), \quad \widetilde{y}'(0) = (5+r, 7-r), \end{cases} \tag{4.1}$$

The exact solution is as follows:

$$\begin{aligned} \underline{Y}(t,r) &= (2+r)e^{2t} + (1-r)te^{2t}, \\ \overline{Y}(t,r) &= (4-r)e^{2t} + (r-1)te^{2t}, \end{aligned} \tag{4.2}$$

By applying the Laplace transform on both side of (4.2) we have:

$$\begin{aligned} s^2\mathfrak{L}[\underline{y}(t)] - \underline{y}(0)s - \underline{y}'(0) - 4(s\mathfrak{L}[\underline{y}(t)] - \underline{y}(0)) + 4\mathfrak{L}[\underline{y}(t)] &= 0, \\ s^2\mathfrak{L}[\overline{y}(t)] - \overline{y}(0)s - \overline{y}'(0) - 4(s\mathfrak{L}[\overline{y}(t)] - \overline{y}(0)) + 4\mathfrak{L}[\overline{y}(t)] &= 0. \end{aligned} \tag{4.3}$$

Substituting the initial conditions (4.1), leads

$$\begin{aligned} \mathfrak{L}[\underline{y}(t)] &= \frac{(2+r)s + (5+r) - 4(2+r)}{(s^2-4s)} - \frac{4}{(s^2-4s)}\mathfrak{L}[\underline{y}(t)], \\ \mathfrak{L}[\overline{y}(t)] &= \frac{(4-r)s + (7-r) - 4(4-r)}{(s^2-4s)} - \frac{4}{(s^2-4s)}\mathfrak{L}[\overline{y}(t)]. \end{aligned} \tag{4.4}$$

In view of (3.5), we decomposed $\underline{y}(t)$ and $\overline{y}(t)$ in following form:

$$\underline{y}(t) = \sum_{i=0}^{\infty} \underline{y}_i(t), \qquad \overline{y}(t) = \sum_{i=0}^{\infty} \overline{y}_i(t), \tag{4.5}$$

By substituting Eq.(4.5) in Eq.(4.4) we have:

$$\begin{aligned} \underline{y}_0 &= \frac{3(1+r)}{4} + \frac{(5+r)}{4} e^{4t}, \\ \underline{y}_{n+1} &= \mathfrak{L}^{-1}[-\frac{4}{(s^2-4s)} \mathfrak{L}[\underline{y}_n]], \quad n \geq 0, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \overline{y}_0 &= \frac{3(3-r)}{4} + \frac{(7-r)}{4} e^{4t}, \\ \overline{y}_{n+1} &= \mathfrak{L}^{-1}[-\frac{4}{(s^2-4s)} \mathfrak{L}[\overline{y}_n]], \quad n \geq 0. \end{aligned} \tag{4.7}$$

Using the above recursive relation we can find other components of solution.

$$\begin{aligned} \underline{Y}(t,r) &\cong \sum_{i=0}^{5} \underline{y}_i(t) \\ &= (2+r) + (5+r)t + 6t^2 - \frac{2}{3}(-7+r)t^3 + (\frac{8}{3} - \frac{2}{3}r)t^4 + (\frac{6}{5} - \frac{2}{5}r)t^5, \\ \overline{Y}(t,r) &\cong \sum_{i=0}^{5} \overline{y}_i(t) \\ &= (4-r) + (7-r)t + 6t^2 + \frac{2}{3}(5+r)t^3 + (\frac{4}{3} + \frac{2}{3}r)t^4 + (\frac{2}{5} + \frac{2}{5}r)t^5. \end{aligned}$$

In tables 1 and 2, we compare results with the numerical solutions [4]

Table 1. *For* $\underline{y}$ *(* $t = 0.01$*)*

r	error(LDM)	error [4]
0.0	0.445866e-12	0.461836e-3
0.1	0.427658e-12	0.476055e-3
0.2	0.410338e-12	0.490273e-3
0.3	0.392131e-12	0.504491e-3
0.4	0.374367e-12	0.518710e-3
0.5	0.35616e-12	0.532928e-3
0.6	0.33884e-12	0.547146e-3
0.7	0.321076e-12	0.561365e-3
0.8	0.303313e-12	0.575584e-3
0.9	0.285105e-12	0.589801e-3
1.0	0.267342e-12	0.604020e-4

Table 2. *For* $\overline{y}$ *(* $t = 0.01$*)*

r	error(LDM)	error [4]
0.0	0.888178e-13	0.746204e-3
0.1	0.107025e-12	0.731985e-3
0.2	0.124345e-12	0.717767e-3
0.3	0.142553e-12	0.703549e-3
0.4	0.16076e-12	0.689330e-3
0.5	0.178524e-12	0.675112e-3
0.6	0.195843e-12	0.660894e-3
0.7	0.213607e-12	0.646675e-3
0.8	0.231815e-12	0.632456e-3
0.9	0.249578e-12	0.618239e-3
1.0	0.267342e-12	0.604020e-3

Example 4.2 ([1]). Consider the following fourth-order fuzzy linear differential equation

$$\left\{\begin{array}{l} y^{(4)}(t)-y=0, \qquad t\in[0,1] \\ \widetilde{y}(0)=(r-1,1-r), \quad \widetilde{y}'(0)=(r-1,1-r), \\ \widetilde{y}''(0)=(r-1,1-r), \quad \widetilde{y}'''(0)=(r-1,1-r). \end{array}\right. \tag{4.8}$$

with the exact fuzzy solution:

$$\begin{aligned} \underline{Y}(t,r) &= (r-1)e^t, \\ \overline{Y}(t,r) &= (1-r)e^t. \end{aligned} \tag{4.9}$$

Applying the Laplace transform we have:

$$\begin{aligned} s^4\mathfrak{L}[\underline{y}(t)]-s^3\underline{y}(0)-s^2\underline{y}^{'}(0)-s\underline{y}^{''}(0)-\underline{y}^{'''}(0)-\mathfrak{L}[\underline{y}(t)] &= 0, \\ s^4\mathfrak{L}[\overline{y}(t)]-s^3\overline{y}(0)-s^2\overline{y}^{'}(0)-s\overline{y}^{''}(0)-\overline{y}^{'''}(0)-\mathfrak{L}[\overline{y}(t)] &= 0. \end{aligned} \tag{4.10}$$

Using the initial conditions Eq. (4.8), we have:

$$\begin{aligned} \mathfrak{L}[\underline{y}(t)] &= \frac{r-1}{s}+\frac{r-1}{s^2}+\frac{r-1}{s^3}+\frac{r-1}{s^4}+\frac{1}{s^4}\mathfrak{L}[\underline{y}(t)], \\ \mathfrak{L}[\overline{y}(t)] &= \frac{1-r}{s}+\frac{1-r}{s^2}+\frac{1-r}{s^3}+\frac{1-r}{s^4}+\frac{1}{s^4}\mathfrak{L}[\overline{y}(t)]. \end{aligned} \tag{4.11}$$

Now we get an infinite series given below:

$$\underline{y}(t)=\sum_{i=0}^{\infty}\underline{y}_i(t), \qquad \overline{y}(t)=\sum_{i=0}^{\infty}\overline{y}_i(t), \tag{4.12}$$

By substituting Eq.(4.12) in Eq.(4.11) we have:

$$\begin{aligned} \underline{y}_0 &= (r-1)+(r-1)t+\frac{(r-1)}{2!}t^2+\frac{(r-1)}{3!}t^3, \\ \underline{y}_{n+1} &= \mathfrak{L}^{-1}[\ \frac{1}{s^4}\mathfrak{L}[\underline{y}_n]\ ], \quad n\geq 0, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \overline{y}_0 &= (1-r)+(1-r)t+\frac{(1-r)}{2!}t^2+\frac{(1-r)}{3!}t^3, \\ \overline{y}_{n+1} &= \mathfrak{L}^{-1}[\ \frac{1}{s^4}\mathfrak{L}[\overline{y}_n]\ ], \quad n\geq 0. \end{aligned} \tag{4.14}$$

So, we obtain:

$$\begin{aligned} \underline{Y}(t,r) &= (r-1)e^t, \\ \overline{Y}(t,r) &= (1-r)e^t. \end{aligned} \tag{4.15}$$

Which is the exact solution.

5. Conclusions

In this paper we have used Laplace decomposition method for solving the n-th order linear differential equation having fuzzy initial conditions. The method is a powerful mathematical tool to solving n-th order fuzzy differential equations. Convergency of (LDM) has been discussed for n-th order fuzzy differential equation. Numerical examples were used to illustrate the nature and performance of this analytical-numerical method. *Mathematica* has been used for computations in this paper.

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