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# Soft linear functionals in soft normed linear spaces

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Abstract. In the present paper an idea of soft linear functional over soft linear spaces has been introduced and some basic properties of such operators are studied. Hahn-Banach theorem along with its various consequences and Uniform Boundedness Principle theorem are extended in soft set settings. Weak convergence, strong convergences are defined and their properties are studied. Lastly an analogue of open mapping theorem and closed graph theorem are furnished in soft set settings.

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Keywords: Soft sets, Soft linear space, Soft linear functional, Continuous soft linear functional, Hahn-Banach theorem, Uniform Boundedness Principle theorem, Open mapping theorem, Closed graph theorem, Weak convergence, Strong convergence.

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## 1. Introduction

Molodtsov [21] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Following his work Maji et al. ([18], [19]) introduced several operations on soft sets and applied soft sets to decision making problems. Chen et al. [4] presented a new definition of soft set parametrization reduction and some works in this line have been found in ([22], [17], [25]). Soft group was introduced by Aktas and Cagman [1] and soft BCK/BCI – algebras and its application in ideal theory was investigated by Jun ([15], [16]). Feng et al. [12] worked on soft semirings, soft ideals and idealistic soft semirings. Some works on semigroups and soft ideals over a semi group are found in ([2], [23]). The idea of a soft topological space was first given by M. Shabir, M. Naz [24] and consequently H. Hazra et al. [14] and Cagman et al. [3] introduced new definitions of soft toplogy. Mappings between soft sets were described by P. Majumdar, S. K. Samanta [20]. Feng et al. [13] worked on soft sets combined with fuzzy sets and rough sets. Recently we have introduced soft real sets, soft real numbers, soft complex sets, soft complex numbers in ([5], [6]). Two different concepts of soft metric have been presented in ([7], [8]). 'Soft linear (vector) space' and 'soft norm' on an absolute 'soft vector space' have been introduced in [9]. An idea of 'soft inner product' has been given in [10]. In [11] we proposed an idea of 'soft linear operator' on 'soft linear spaces' and studied various properties of such operators.

In this paper we have extended the four fundamental theorems of functional analysis in soft set settings. Firstly we introduced a notion of soft linear functional over soft linear spaces and some of their properties are studied. In section 2, some preliminary results are given. In section 3, a notion of 'soft linear functional' over a 'soft linear space' is given and some properties of such operators are studied. In section 4, Hahn-Banach theorem, its various consequences and Uniform Boundedness Principle theorem are established in soft set settings. Weak convergence, strong convergences are defined and their properties are studied in section 5. An analogue of open mapping theorem and closed graph theorems are furnished in soft set settings in section 6. Section 7 concludes the paper.

## 2. Preliminaries

**Definition 2.1** ([21])**.** Let $U$ be an universe and $E$ be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of $U$ and $A$ be a non-empty subset of $E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over $U$ is a parametrized family of subsets of the universe $U$. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon$ − approximate elements of the soft set $(F, A)$.

**Definition 2.2** ([13])**.** For two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$, we say that $(F, A)$ is a soft subset of $(G, B)$ if

(1) $A \subseteq B$ and
(2) for all $e \in A$, $F(e) \subseteq G(e)$. We write $(F, A)\widetilde{\subset}(G, B)$.

$(F, A)$ is said to be a soft superset of $(G, B)$, if $(G, B)$ is a soft subset of $(F, A)$. We denote it by $(F, A)\widetilde{\supset}(G, B)$.

**Definition 2.3** ([13])**.** Two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ are said to be equal if $(F, A)$ is a soft subset of $(G, B)$ and $(G, B)$ is a soft subset of $(F, A)$.

**Definition 2.4** ([19])**.** The union of two soft sets $(F, A)$ and $(G, B)$ over the common universe $U$ is the soft set
$(H, C)$, where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We express it as $(F, A)\,\widetilde{\cup}\,(G, B) = (H, C)$.

The following definition of intersection of two soft sets is given as that of the bi-intersection in [12].

**Definition 2.5** ([12]). The intersection of two soft sets $(F, A)$ and $(G, B)$ over the common universe $U$ is the soft set $(H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Let $X$ be an initial universal set and $A$ be the non-empty set of parameters. In the above definitions the set of parameters may vary from soft set to soft set, but in our considerations, throught this paper all soft sets have the same set of parameters $A$. The above definitions are also valid for these type of soft sets as a particular case of those definitions.

**Definition 2.6** ([13]). The complement of a soft set $(F, A)$ is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

**Definition 2.7** ([19]). A soft set $(F, A)$ over $U$ is said to be an *absolute* soft set denoted by $\check{U}$ if for all $\varepsilon \in A$, $F(\varepsilon) = U$. A soft set $(F, A)$ over $U$ is said to be a *null* soft set denoted by $\Phi$ if for all $\varepsilon \in A$, $F(\varepsilon) = \varnothing$.

**Definition 2.8** ([5]). Let $X$ *be* a non-empty set and $A$ be a non-empty parameter set. Then a function $\varepsilon : A \to X$ is said to be a soft element of $X$. A soft element $\varepsilon$ of $X$ is said to belongs to a soft set $B$ of $X$, which is denoted by $\varepsilon \widetilde{\in} B$, if $\varepsilon(e) \in A(e)$, $\forall e \in A$. Thus for a soft set A of $X$ with respect to the index set $A$, we have $B(e) = \{\varepsilon(e), \varepsilon \widetilde{\in} B\}$, $e \in A$.

It is to be noted that every singleton soft set (a soft set $(F, A)$ for which $F(e)$ is a singleton set, $\forall e \in A$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in A$.

**Definition 2.9** ([5]). Let $R$ be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of $R$ and $A$ taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a *soft real set*. It is denoted by $(F, A)$. If specifically $(F, A)$ is a singleton soft set, then after identifying $(F, A)$ with the corresponding soft element, it will be called a *soft real number*.

The set of all soft real numbers is denoted by $\mathbb{R}(A)$ and the set of all non-negative soft real numbers by $\mathbb{R}(A)^*$.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\overline{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.

**Definition 2.10** ([6]). Let $C$ be the set of complex numbers and $\wp(C)$ be the collection of all non-empty bounded subsets of the set of complex numbers. $A$ be a set of parameters. Then a mapping

$$F : A \to \wp(C)$$

is called a soft complex set. It is denoted by $(F, A)$.

If in particular $(F, A)$ is a singleton soft set , then identifying $(F, A)$ with the corresponding soft element, it will be called a *soft complex number*.

The set of all soft complex numbers is denoted by $\mathbb{C}(A)$.

**Definition 2.11** ([7]). For two soft real numbers $\tilde{r}, \tilde{s}$ we define

(i). $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$.
(ii). $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$.
(iii). $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$.
(iv). $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

**Definition 2.12** ([6]). Let $(F, A)$, $(G, A) \in \mathbb{C}(A)$. Then the sum, difference, product and division are defined by
$(F+G)(\lambda) = z + w; z \in F(\lambda),\ w \in G(\lambda), \forall \lambda \in A;$
$(F-G)(\lambda) = z - w; z \in F(\lambda),\ w \in G(\lambda), \forall \lambda \in A;$
$(FG)(\lambda) = zw; z \in F(\lambda),\ w \in G(\lambda), \forall \lambda \in A;$
$(F/G)(\lambda) = z/w; z \in F(\lambda),\ w \in G(\lambda),\ \forall \lambda \in A;$ provided $G(\lambda) \neq \varnothing$, $\forall \lambda \in A$.

**Definition 2.13** ([6]). Let $(F, A)$ be a soft complex number. Then the modulus of $(F, A)$ is denoted by $(|F|, A)$ and is defined by $|F|(\lambda) = |z|; z \in F(\lambda)$, $\forall \lambda \in A$, where $z$ is an ordinary complex number.
Since the modulus of each ordinary complex number is a non-negative real number and by definition of soft real numbers it follows that $(|F|, A)$ is a non-negative soft real number for every soft complex number $(F, A)$.

Let $X$ be a non-empty set. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. Let $\mathcal{S}(\check{X})$ be the collection all soft sets $(F, A)$ over $X$ for which $F(\lambda) \neq \varnothing$, for all $\lambda \in A$ together with the null soft set $\Phi$.
Let $(F, A)(\neq \Phi) \in \mathcal{S}(\check{X})$, then the collection of all soft elements of $(F, A)$ will be denoted by $SE(F, A)$. For a collection $\mathfrak{B}$ of soft elements of $\check{X}$, the soft set generated by $\mathfrak{B}$ is denoted by $SS(\mathfrak{B})$.

**Definition 2.14** ([7]). A mapping $d : SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)^*$, is said to be a *soft metric* on the soft set $\check{X}$ if $d$ satisfies the following conditions:
(M1). $d(\tilde{x}, \tilde{y}) \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M2). $d(\tilde{x}, \tilde{y}) = \overline{0}$, if and only if $\tilde{x} = \tilde{y}$.
(M3). $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M4). For all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}, d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$
The soft set $\check{X}$ with a soft metric $d$ on $\check{X}$ is said to be a *soft metric space* and is denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$.

**Theorem 2.15.** *(Decomposition theorem) [[7]] If a soft metric $d$ satisfies the condition:*
*(M5). For $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda) : \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\}$ is a singleton set, and if for $\lambda \in A$, $d_\lambda : X \times X \to \mathbb{R}^+$ is defined by $d_\lambda(\xi, \eta) = d(\tilde{x}, \tilde{y})(\lambda)$, where $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta$. Then $d_\lambda$ is a metric on $X$.*

**Definition 2.16.** Let $(\check{X}, d)$ be a soft metric space, $\tilde{r}$ be a non-negative soft real number and $\tilde{a} \tilde{\in} \check{X}$. By an *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$, we mean the collection of soft elements of $\check{X}$ satisfying $d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}$.
The *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$ is denoted by $B(\tilde{a}, \tilde{r})$.
Thus $B(\tilde{a}, \tilde{r}) = \{\tilde{x} \tilde{\in} \check{X}; d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}\} \subset SE(\check{X})$.
$SS(B(\tilde{a}, \tilde{r}))$ will be called a *soft open ball* with centre $\tilde{a}$ and radius $\tilde{r}$.

**Definition 2.17** ([7]). Let $\mathfrak{B}$ be a collection of soft elements of $\check{X}$ in a soft metric space $(\check{X}, d)$. Then a soft element $\tilde{a}$ is said to be an *interior element* of $\mathfrak{B}$ if there exists a positive soft real number $\tilde{r}$ such that $\tilde{a} \in B(\tilde{a}, \tilde{r}) \subset \mathfrak{B}$.

**Definition 2.18** ([7]). Let $(Y, A)$ be a soft subset in a soft metric space $(\check{X}, d)$. Then a soft element $\tilde{a}$ is said to be an *interior element* of $(Y, A)$ if there exists a positive soft real number $\tilde{r}$ such that $\tilde{a} \in B(\tilde{a}, \tilde{r}) \subset SE(Y, A)$.

**Definition 2.19** ([7]). Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a non-null collection of soft elements of $\check{X}$. Then $\mathfrak{B}$ is said to be '*open in* $\check{X}$ *with respect to* $d$' or ' *open in* $(\check{X}, d)$' if all elements of $\mathfrak{B}$ are interior elements of $\mathfrak{B}$.

**Definition 2.20** ([7]). Let $(\check{X}, d)$ be a soft metric space and $(Y, A)$ be a non-null soft subset $\in \mathcal{S}\left(\check{X}\right)$ in $(\check{X}, d)$. Then $(Y, A)$ is said to be '*soft open in* $\check{X}$ *with respect to* $d$' if there is a collection $\mathfrak{B}$ of soft elements of $(Y, A)$ such that $\mathfrak{B}$ is open with respect to d and $(Y, A) = SS(\mathfrak{B})$.

**Definition 2.21** ([7]). Let $(\check{X}, d)$ be a soft metric space. A soft set $(Y, A) \in \mathcal{S}(\check{X})$, is said to be '*soft closed in* $\check{\check{X}}$ *with respect to* $d$' if its complement $(Y, A)^c$ is a member of $\mathcal{S}(\check{X})$ and is soft open in $(\check{X}, d)$.

**Proposition 2.22.** *[[7]] Let $\left(\check{X}, d\right)$ be a soft metric space satisfying (M5). Then $(F, A)$ is soft open with respect to $d$ if and only if $(F, A)(\lambda)$ is open in $(X, d_\lambda)$, for each $\lambda \in A$.*

We now prove the following proposition which will be required to establish closed graph theorem in soft set settings.

**Proposition 2.23.** *Let $\left(\check{X}, d\right)$ be a soft metric space satisfying (M5). Then $(F, A)$ is soft closed with respect to $d$ if and only if $(F, A)^c(\lambda)$ is closed in $(X, d_\lambda)$, for each $\lambda \in A$.*

*Proof.* $(F, A)$ is soft closed $\Leftrightarrow (F, A)^c$ is soft open $\Leftrightarrow (F, A)^c(\lambda)$ is open in $(X, d_\lambda)$, for each $\lambda \in A \Leftrightarrow F^c(\lambda)$ is open in $(X, d_\lambda)$, for each $\lambda \in A \Leftrightarrow F(\lambda)$ is closed in $(X, d_\lambda)$, for each $\lambda \in A \Leftrightarrow (F, A)(\lambda)$ is closed in $(X, d_\lambda)$, for each $\lambda \in A$. □

**Definition 2.24** ([9]). Let $V$ be a vector space over a field $K$ and let $A$ be a parameter set. Let $G$ be a soft set over $(V, A)$. Now $G$ is said to be a soft vector space or soft linear space of $V$ over $K$ if $G(\lambda)$ is a vector subspace of $V$, $\forall \lambda \in A$.

**Definition 2.25** ([9]). Let $G$ be a soft vector space of $V$ over $K$. Then a soft element of $G$ is said to be a soft vector of $G$. In a similar manner a soft element of the soft set $(K, A)$ is said to be a soft scalar, K being the scalar field.

**Definition 2.26** ([9]). Let $\tilde{x}, \tilde{y}$ be soft vectors of $G$ and $\tilde{k}$ be a soft scalar. Then the addition $\tilde{x} + \tilde{y}$ of $\tilde{x}, \tilde{y}$ and scalar multiplication $\tilde{k}.\, \tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by $(\tilde{x} + \tilde{y})(\lambda) = \tilde{x}(\lambda) + \tilde{y}(\lambda)$, $\left(\tilde{k}.\, \tilde{x}\right)(\lambda) = \tilde{k}(\lambda).\tilde{x}(\lambda)$, $\forall \lambda \in A$. Obviously, $\tilde{x} + \tilde{y}$, $\tilde{k}.\, \tilde{x}$ are soft vectors of $G$.

**Definition 2.27** ([9]). Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $\|.\| : SE(\check{X}) \to \mathbb{R}(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$ if $\|.\|$ satisfies the following conditions:

(N1). $\|\tilde{x}\| \tilde{\geq} \overline{0}$, for all $\tilde{x} \tilde{\in} \check{X}$;
(N2). $\|\tilde{x}\| = \overline{0}$ if and only if $\tilde{x} = \Theta$;
(N3). $\|\tilde{\alpha}.\tilde{x}\| = |\tilde{\alpha}| \|\tilde{x}\|$ for all $\tilde{x} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$;
(N4). For all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$.
The soft vector space $\check{X}$ with a soft norm $\|.\|$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, \|.\|, A)$ or $(\check{X}, \|.\|)$. (N1), (N2), (N3) and (N4) are said to be soft norm axioms.

**Example 2.28** ([9]). Let $\mathbb{R}(A)$ be the set of all soft real numbers. We define $\|.\| : \mathbb{R}(A) \to \mathbb{R}(A)^*$ by $\|\tilde{x}\| = |\tilde{x}|$, for all $\tilde{x} \tilde{\in} \mathbb{R}(A)$, where $|\tilde{x}|$ denotes the modulus of soft real numbers. Then $\|.\|$ satisfy all the soft norm axioms. So, $\|.\|$ is a soft norm on $\mathbb{R}(A)$ and since $SS(\mathbb{R}(A)) = \check{\mathbb{R}}$, $(\check{\mathbb{R}}, \|.\|, A)$ or $(\check{\mathbb{R}}, \|.\|)$ is a soft normed linear space. With the same modulus soft norm as above, it can be easily verified that $SS(\mathbb{C}(A)) = \check{\mathbb{C}}$ is also a soft normed linear space.

**Proposition 2.29** ([9]). *Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Let us define $d : \check{X} \times \check{X} \to \mathbb{R}(A)^*$ by $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\|$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$. Then $d$ is a soft metric [7] on $\check{X}$ satisfying*

(i). $d(\tilde{x} + \tilde{a}, \ \tilde{y} + \tilde{a}) = d(\tilde{x}, \tilde{y})$;
(ii). $d(\tilde{\alpha}.\tilde{x}, \ \tilde{\alpha}.\tilde{y}) = |\tilde{\alpha}| d(\tilde{x}, \tilde{y})$, *for all* $\tilde{x}, \ \tilde{y} \tilde{\in} \check{X}$ *and for every soft scalar* $\tilde{\alpha}$.

**Theorem 2.30** ([9]). *Suppose a soft norm $\|.\|$ satisfies the condition*
*(N5). For $\xi \in X$, and $\lambda \in A$, $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set.*
*Then for each $\lambda \in A$, the mapping $\|.\|_\lambda : X \to R^+$ defined by $\|\xi\|_\lambda = \|\tilde{x}\|(\lambda)$, for all $\xi \in X$ and $\tilde{x} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi$, is a norm on $X$.*

**Proposition 2.31.** *Let $(\check{X}, \|.\|, A)$ be a soft normed linear spaces satisfying (N5), then the induced soft metric $d : \check{X} \times \check{X} \to \mathbb{R}(A)^*$ by $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\|$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$; satisfies (M5).*

*Proof.* By Proposition 2.29, it follows that $d$ is a soft metric on $\check{X}$.
Let $(\xi, \eta) \in X \times X$, and $\lambda \in A$, choose $\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}' \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi$ $= \tilde{x}'(\lambda), \tilde{y}(\lambda) = \eta = \tilde{y}'(\lambda)$.
Then $d(\tilde{x}, \tilde{y})(\lambda) = \|\tilde{x} - \tilde{y}\|(\lambda) = \|\tilde{x}(\lambda) - \tilde{y}(\lambda)\|_\lambda = \|\xi - \eta\|_\lambda = \left\|\tilde{x}'(\lambda) - \tilde{y}'(\lambda)\right\|_\lambda$ $= \left\|\tilde{x}' - \tilde{y}'\right\|(\lambda) = d\left(\tilde{x}', \tilde{y}'\right)(\lambda)$
Therefore, for $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda) : \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\}$ is a singleton set. So, $d$ satisfies (M5). □

**Definition 2.32** ([9]). A sequence of soft elements $\{\tilde{x}_n\}$ in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be convergent and converges to a soft element $\tilde{x}$ if $\|\tilde{x}_n - \tilde{x}\| \to \overline{0}$ as $n \to \infty$. This means for every $\tilde{\varepsilon} \tilde{>} \overline{0}$, chosen arbitrarily, there exists a natural number $N = N(\tilde{\varepsilon})$, such that $\overline{0} \tilde{\leq} \|\tilde{x}_n - \tilde{x}\| \tilde{<} \tilde{\varepsilon}$, whenever $n > N$.
i.e., $n > N \implies \tilde{x}_n \in B(\tilde{x}, \ \tilde{\varepsilon})$. We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

**Definition 2.33** ([9]). A sequence $\{\tilde{x}_n\}$ of soft elements in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be a Cauchy sequence in $\check{X}$ if corresponding to every $\tilde{\varepsilon} \tilde{>} \overline{0}$, there exists $m \in N$ such that $\|\tilde{x}_i - \tilde{x}_j\| \tilde{\leq} \tilde{\varepsilon}$, $\forall \ i, j \geq m$ i.e., $\|\tilde{x}_i - \tilde{x}_j\| \to \overline{0}$ as $i, j \to \infty$.

**Definition 2.34** ([9]). Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Then $\check{X}$ is said to be complete if every Cauchy sequence in $\check{X}$ converges to a soft element of $\check{X}$ . Every complete soft normed linear space is called a soft Banach Space.

**Theorem 2.35** ([9]). *Every Cauchy sequence in $\mathbb{R}(A)$, where $A$ is a finite set of parameters, is convergent, i.e., the set of all soft real numbers with its usual modulus soft norm with respect to a finite set of parameters, is a soft Banach space.*

**Definition 2.36** ([9]). Let $\{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$ be a set of soft vectors of a soft vector space $G$ such that $\widetilde{\alpha}_i(\lambda) \neq \theta$ for any $\lambda \in A$ and $i = 1, 2, .., n$. Then $\{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$ is said to be linearly independent in $G$ if for any set of soft scalars $\tilde{c}_1,\ \tilde{c}_2,\ \ldots, \tilde{c}_n$, $\tilde{c}_1.\ \widetilde{\alpha}_1 + \tilde{c}_2.\ \widetilde{\alpha}_2 + .. + \tilde{c}_n.\ \widetilde{\alpha}_n = \Theta$ implies $\tilde{c}_1 =\ \tilde{c}_2 =\ \cdots = \tilde{c}_n = \overline{0}$.

**Proposition 2.37** ([9]). *A set $S = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$ of soft vectors in a soft vector space $G$ over $V$ is linearly independent if and only if the sets $S(\lambda) = \{\widetilde{\alpha}_1(\lambda), \widetilde{\alpha}_2(\lambda), \ldots, \widetilde{\alpha}_n(\lambda)\}$ are linearly independent in $V$, $\forall \lambda \in A$.*

**Definition 2.38** ([9]). A soft linear space $\check{X}$ is said to be of finite dimensional if there is a finite set of linearly independent soft vectors in $\check{X}$ which also generates $\check{X}$, i.e., any soft element of $\check{X}$ can be expressed as a linear combination of those linearly independent soft vectors.
The set of those linearly independent soft vectors is said to be the basis for $\check{X}$ and the number of soft vectors of the basis is called the dimension of $\check{X}$.

**Lemma 2.39** ([9]). *Let $\tilde{x}_1, \tilde{x}_2, \ldots.., \tilde{x}_n$ be a linearly independent set of soft vectors in a soft linear space $\check{X}$, satisfying (N5). Then there is a soft real number $\tilde{c}\tilde{>}\overline{0}$ such that for every set of soft scalars $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots.., \widetilde{\alpha}_n$ we have*

$$\|\widetilde{\alpha}_1\tilde{x}_1 + \widetilde{\alpha}_2\tilde{x}_2 + \cdots + \widetilde{\alpha}_n\tilde{x}_n\| \,\widetilde{\geq}\, \tilde{c}\left(|\widetilde{\alpha}_1| + |\widetilde{\alpha}_2| + \cdots + |\widetilde{\alpha}_n|\right).$$

**Definition 2.40** ([9]). A soft subset $(Y, A)$ with $Y(\lambda) \neq \varnothing, \forall \lambda \in A$, in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be bounded if there exists a soft real number $\tilde{k}$ such that $\|\tilde{x}\| \,\widetilde{\leq}\, \tilde{k}$, $\forall \tilde{x} \in (Y, A)$.

**Definition 2.41** ([9]). A sequence of soft real numbers $\{\tilde{s}_n\}$ is said to be convergent if for arbitrary $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists a natural number $N$ such that for all $n \geq N$, $|\tilde{s} - \tilde{s}_n| \,\tilde{<}\, \widetilde{\varepsilon}$. We denote it by $\lim_{n\to\infty} \tilde{s}_n = \tilde{s}$.

**Definition 2.42** ([11]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be an operator. Then $T$ is said to be soft linear if
(L1). $T$ is additive, i.e., $T(\tilde{x}_1 + \tilde{x}_2) = T(\tilde{x}_1) + T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2 \widetilde{\in} \check{X}$,
(L2). $T$ is homogeneous, i.e. for every soft scalar $\tilde{c}$, $T(\tilde{c}\tilde{x}) = \tilde{c}T(\tilde{x})$, for every soft element $\tilde{x} \widetilde{\in} \check{X}$.
The properties (L1) and (L2) can be put in a combined form $T(\tilde{c}_1\tilde{x}_1 + \tilde{c}_2\tilde{x}_2) = \tilde{c}_1 T(\tilde{x}_1) + \tilde{c}_2 T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2 \widetilde{\in} \check{X}$ and every soft scalars $\tilde{c}_1,\ \tilde{c}_2$.

**Definition 2.43** ([11]). The operator $T : SE(\check{X}) \to SE(\check{Y})$ is said to be continuous at $\tilde{x}_0 \widetilde{\in} \check{X}$ if for every sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ with $\tilde{x}_n \to \tilde{x}_0$ as $n \to \infty$, we have $T(\tilde{x}_n) \to T(\tilde{x}_0)$ as $n \to \infty$ i.e., $\|\tilde{x}_n - \tilde{x}_0\| \to \overline{0}$ as $n \to \infty$ implies $\|T(\tilde{x}_n) - T(\tilde{x}_0)\| \to \overline{0}$ as $n \to \infty$. If $T$ is continuous at each soft element of $\check{X}$, then $T$ is said to be a continuous operator.

**Definition 2.44** ([11])**.** Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where $\check{X}, \check{Y}$ are soft normed linear spaces. The operator $T$ is called bounded if there exists some positive soft real number $\tilde{M}$ such that for all $\tilde{x}\tilde{\in}\check{X}$, $\|T(\tilde{x})\| \tilde{\leq} \tilde{M}\,\|\tilde{x}\|$.

**Theorem 2.45** ([11])**.** *Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where $\check{X}, \check{Y}$ are soft normed linear spaces. If $T$ is bounded then $T$ is continuous.*

**Theorem 2.46** ([11])**.** *(Decomposition Theorem). Suppose a soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$, where $\check{X}, \check{Y}$ are soft vector spaces, satisfies the condition (L3). For $\xi \in X$, and $\lambda \in A$, $\left\{T(\tilde{x})(\lambda) : \tilde{x}\ \tilde{\in}\check{X} \text{ such that } \tilde{x}(\lambda) = \xi\right\}$ is a singleton set.*
*Then for each $\lambda \in A$, the mapping $T_\lambda : X \to Y$ defined by $T_\lambda(\xi) = T(\tilde{x})(\lambda)$, for all $\xi \in X$ and $\tilde{x}\tilde{\in}\check{X}$ such that $\tilde{x}(\lambda) = \xi$, is a linear operator.*

**Theorem 2.47** ([11])**.** *Let $\check{X}, \check{Y}$ be soft normed linear spaces which satisfy (N5) and $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator satisfying (L3). If $T$ is continuous then $T$ is bounded.*

**Theorem 2.48** ([11])**.** *Let $\check{X}, \check{Y}$ be soft normed linear spaces which satisfy (N5) and $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator satisfying (L3). If $\check{X}$ is of finite dimension, then $T$ is bounded and hence continuous.*

**Definition 2.49** ([11])**.** Let $T$ be a bounded soft linear operator from $SE(\check{X})$ into $SE\left(\check{Y}\right)$. Then the norm of the operator $T$ denoted by $\|T\|$, is a soft real number defined as the following:
For each $\lambda \in A$, $\|T\|(\lambda) = \inf\{t \in R; \|T(\tilde{x})\|(\lambda) \leq t.\,\|\tilde{x}\|(\lambda)\text{, for each } \tilde{x}\ \tilde{\in}\check{X}\}$.

**Theorem 2.50** ([11])**.** *Let $\check{X}, \check{Y}$ be soft normed linear spaces which satisfy (N5) and $T$ satisfy (L3). Then for each $\lambda \in A$, $\|T\|(\lambda) = \|T_\lambda\|_\lambda$, where $\|T_\lambda\|_\lambda$ is the norm of the linear operator $T_\lambda : X \to Y$.*

**Theorem 2.51** ([11])**.** $\|T(\tilde{x})\| \tilde{\leq} \|T\|\,\|\tilde{x}\|$, for all $\tilde{x}\tilde{\in}\check{X}$.

**Theorem 2.52** ([11])**.** *Let $\check{X}, \check{Y}$ be soft normed linear spaces which satisfy (N5) and $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator satisfying (L3). Then*

(i). $\|T\|(\lambda) = \sup\left\{\|T(\tilde{x})\|(\lambda) : \|\tilde{x}\| \tilde{\leq} \bar{1}\right\} = \|T_\lambda\|_\lambda$, *for each* $\lambda \in A$;

(ii). $\|T\|(\lambda) = \sup\left\{\|T(\tilde{x})\|(\lambda) : \|\tilde{x}\| = \bar{1}\right\} = \|T_\lambda\|_\lambda$, *for each* $\lambda \in A$;

(iii). $\|T\|(\lambda) = \sup\left\{\frac{\|T(\tilde{x})\|}{\|\tilde{x}\|}(\lambda) : \|\tilde{x}\|(\mu) \neq 0, \text{ for all } \mu \in A\right\} = \|T_\lambda\|_\lambda$, *for each* $\lambda \in A$.

**Theorem 2.53** ([11])**.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces which satisfy (N5). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a continuous soft linear operator satisfying (L3). Then $T_\lambda$ is continuous on $X$ for each $\lambda \in A$.*

**Theorem 2.54** ([11])**.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces which satisfy (N5) over a finite set of parameters $A$. Let $\{T_\lambda; \lambda \in A\}$ be a family of continuous linear operators such that $T_\lambda : X \to Y$ for each $\lambda$. Then the operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $T(\lambda) = T_\lambda$, $\forall\lambda \in A$; is a continuous soft linear operator satisfying (L3).*

We now prove the following results which will be useful in this paper.

**Theorem 2.55.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces which satisfy (N5). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a bounded soft linear operator satisfying (L3). Then $T_\lambda$ is bounded on $X$ for each $\lambda \in A$.*

*Proof.* Let $\lambda \in A$. Since $T$ is bounded, there exists a positive soft real number $\tilde{M}$ such that for all $\tilde{x}\tilde{\in}\check{X}, \|T(\tilde{x})\| \tilde{\leq} \tilde{M}\,\|\tilde{x}\|$ . i.e., $\|T(\tilde{x})\|(\lambda) \leq \left(\tilde{M}\,\|\tilde{x}\|\right)(\lambda) = \tilde{M}(\lambda) . \|\tilde{x}\|(\lambda)$ , $\forall \tilde{x}\tilde{\in}\check{X} \|T_\lambda(\tilde{x}(\lambda))\|_\lambda \leq \tilde{M}(\lambda) . \|(\tilde{x}(\lambda))\|_\lambda$, $\forall \tilde{x}(\lambda) \in X$, i.e., $\|T_\lambda(x)\|_\lambda \leq \tilde{M}(\lambda) . \|x\|_\lambda$, $\forall x \in X$. This shows that $T_\lambda$ is bounded on $X$. Since $\lambda \in A$ is arbitrary, $T_\lambda$ is bounded on $X$ for each $\lambda \in A$. □

**Theorem 2.56.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces which satisfy (N5). Let $\{T_\lambda; \lambda \in A\}$ be a family of bounded linear operators such that $T_\lambda : X \to Y$ for each $\lambda$. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$; is a bounded soft linear operator satisfying (L3).*

*Proof.* By Theorem 2.54, it is obvious that $T$ is soft linear. For each $\lambda$, since $T_\lambda : X \to Y$ is bounded, there exists a positive real number $M_\lambda$ such that for all $x \in X, \|T_\lambda(x)\|_\lambda \leq M_\lambda . \|x\|_\lambda$.
Let us consider a positive soft real number $\tilde{M}$ such that $\tilde{M}(\lambda) = M_\lambda$, $\forall \lambda \in A$.
Then we have for all $\tilde{x}\tilde{\in}\check{X}, \|T(\tilde{x})\|(\lambda) = \|T_\lambda(\tilde{x}(\lambda))\|_\lambda \leq M_\lambda . \|(\tilde{x}(\lambda))\|_\lambda$
$= \tilde{M}(\lambda) . \|\tilde{x}\|(\lambda) = \left(\tilde{M}\,\|\tilde{x}\|\right)(\lambda), \forall \lambda \in A.$
i.e., for all $\tilde{x}\tilde{\in}\check{X}, \|T(\tilde{x})\| \tilde{\leq} \tilde{M}\,\|\tilde{x}\|$ . Therefore, the soft linear operator $T$ is bounded. Obviously $T$ satisfies (L3). □

We now re-established the Theorem 2.54, without finiteness restriction on the parameter set.

**Theorem 2.57.** *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces which satisfy (N5). Let $\{T_\lambda; \lambda \in A\}$ be a family of continuous linear operators such that $T_\lambda : X \to Y$ for each $\lambda$. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$; is a continuous soft linear operator satisfying (L3).*

*Proof.* For each $\lambda$, since $T_\lambda : X \to Y$ is continuous it is bounded. By Theorem 2.56, the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, $\forall \lambda \in A$; is a bounded soft linear operator satisfying (L3). Since by Theorem 2.45, for a soft linear operator boundedness implies continuity, it follows that $T$ is a continuous soft linear operator satisfying (L3). □

With the help of this theorem, we have the following results of soft linear operators without finiteness restriction on the parameter set.

**Definition 2.58.** (Soft linear space of operators) Let $\check{X}, \check{Y}$ be soft normed linear spaces satisfying (N5). Consider the set $W$ of all continuous soft linear operators $S$, $T$ etc. which satisfy (L3) each mapping $SE(\check{X})$ into $SE\left(\check{Y}\right)$. Then using Theorem 2.53, it follows that for each $\lambda \in A, S_\lambda$, $T_\lambda$ etc. are continuous soft linear operators from $X$ to $Y$. Let $W(\lambda) = \{T_\lambda(= T(\lambda)); T \in W\}$, for all $\lambda \in A$. Also using 2.53 and Theorem 2.57, it follows that,$W(\lambda)$ is the collection of all continuous linear operators from $X$ to $Y$. By the property of crisp linear operators it follows that $W(\lambda)$ forms a vector space for each $\lambda \in A$ with respect to the usual operations of addition and scalar multiplication of linear operators. It also follows that $W(\lambda)$ is identical with the set of all continuous linear operators from $X$ to $Y$ for all $\lambda \in A$. Thus the absolute soft set generated by $W(\lambda)$ form an absolute soft vector space. Hence $W$ can be interpreted as to form an absolute soft vector space. We shall denote this absolute soft linear (vector) space by $L(\check{X}, \check{Y})$.

**Proposition 2.59.** *Each element of $SE\left(L\left(\check{X}, \check{Y}\right)\right)$ can be identified uniquely with a member of $W$ i.e., to a continuous soft linear operator $T : SE(\check{X}) \to SE\left(\check{Y}\right)$.*

**Theorem 2.60.** *$L(\check{X}, \check{Y})$ is a soft normed linear space where for $\tilde{f} \in SE\left(L\left(\check{X}, \check{Y}\right)\right)$, we can identify $\tilde{f}$ to a unique $T \in W$ and $\left\|\tilde{f}\right\|$ is defined by $\left\|\tilde{f}\right\|(\lambda) = \|T\|(\lambda) = \sup\left\{\|T(\tilde{x})\|(\lambda) : \|\tilde{x}\| \tilde{\leq} \bar{1}\right\}$, for each $\lambda \in A$.*

**Definition 2.61.** Let $S_n,\ S \in W$. Then $\|S_n - S\|(\lambda)$
$= \sup\{\|(S_n - S)(\tilde{x})\|(\lambda) : \|\tilde{x}\| \tilde{\leq} \bar{1}\}$
$= \sup\{\|(S_n(\tilde{x}) - S(\tilde{x}))\|(\lambda) : \|\tilde{x}\| \tilde{\leq} \bar{1}\}$, for each $\lambda \in A$.
If $\|S_n - S\| \to \bar{0}$ as $n \to \infty$, then we say that the sequence of operators $\{S_n\}$ converges in norm to the operator $S$ and we write $S_n \to S$ (in norm).

**Definition 2.62.** Let $\tilde{f}_n, \tilde{f} \in L\left(\check{X}, \check{Y}\right)$, then $\tilde{f}_n, \tilde{f}$ can be identified uniquely to $S_n,\ S \in W$. We define $\tilde{f}_n \to \tilde{f}$ (in norm) if $S_n \to S$ (in norm).

**Theorem 2.63.** *Let $\check{X}, \check{Y}$ be soft normed linear spaces which satisfy (N5) over a finite parameter set $A$. If $\check{Y}$ is a soft Banach space, then $L\left(\check{X}, \check{Y}\right)$ is also a soft Banach space with respect to the above identification.*

**Definition 2.64** ([11])**.** An operator $T : SE(\check{X}) \to SE\left(\check{Y}\right)$ is called injective or one-to-one if $T(\tilde{x}_1) = T(\tilde{x}_2)$ implies $\tilde{x}_1 = \tilde{x}_2$. It is called surjective or onto if $R(T) = SE(\check{Y})$. The operator $T$ is bijective if $T$ is both injective and surjective.

## 3. Soft linear functionals

**Definition 3.1.** A soft linear functional $f$ is a soft linear operator such that $f : SE(\check{X}) \to K$ where $\check{X}$ is a soft linear space and $K = \mathbb{R}(A)$ if $\check{X}$ is a real soft linear space and $K = \mathbb{C}(A)$ if $\check{X}$ is a complex soft linear space.

It follows that the difference between a soft linear operator and a soft linear functional is that in the case of soft linear functional, the range is the set of all soft real numbers or the set of all soft complex numbers. Since $SS(\mathbb{R}(A)) = \check{\mathbb{R}}$ and $SS(\mathbb{C}(A)) = \check{\mathbb{C}}$ are soft normed linear spaces, the definitions and theorems for soft linear operators over soft normed linear spaces remain true for soft linear functionals. We state, without proof, the following theorems for soft linear functionals, where in each case we shall assume that $\check{X}$ is a soft normed linear space and $f$ is a soft linear functional on $\check{X}$ as defined in Definition 3.1.

**Example 3.2.** Consider the absolute soft set generated by $C[a, b]$ and let us denote it by $\tilde{C}[a, b]$ i.e.,$(\tilde{C}[a, b])(\lambda) = C[a, b], \forall \lambda \in A$. Then $\tilde{C}[a, b]$ is an absolute soft vector space. For $\tilde{x} \tilde{\in} \tilde{C}[a, b]$ let us define $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_\lambda = \sup_{a \leq t \leq b} |\tilde{x}(\lambda)(t)|$, $\forall \lambda \in A$.

Then it can be easily verified that $\|.\|$ is a soft norm on $\tilde{C}[a,b]$ and consequently $\tilde{C}[a,b]$ is a soft normed linear space.
Let $\tilde{y}_0$ be a fixed element of $\tilde{C}[a,b]$. For $\tilde{x}\tilde{\in}\tilde{C}[a,b]$, let us define

$$[f(\tilde{x})](\lambda) = \int_a^b \tilde{x}(\lambda)(t).\tilde{y}_0(\lambda)(t)dt,\ \forall\lambda \in A.$$

Then for $\tilde{x}\tilde{\in}\tilde{C}[a,b]$, $[f(\tilde{x})](\lambda)$ is a real number for each $\lambda \in A$ and hence $f(\tilde{x})$ is a soft real number. We also have,
$[f(\tilde{x}_1+\tilde{x}_2)](\lambda) = \int_a^b (\tilde{x}_1+\tilde{x}_2)(\lambda)(t).\tilde{y}_0(\lambda)(t)dt = \int_a^b (\tilde{x}_1(\lambda)+\tilde{x}_2(\lambda))(t).\tilde{y}_0(\lambda)(t)dt$
$= \int_a^b \tilde{x}_1(\lambda)(t).\tilde{y}_0(\lambda)(t)dt + \int_a^b \tilde{x}_2(\lambda)(t).\tilde{y}_0(\lambda)(t)dt = [f(\tilde{x}_1)](\lambda) + [f(\tilde{x}_2)](\lambda) = [f(\tilde{x}_1)+f(\tilde{x}_2)](\lambda),\ \forall\lambda \in A.$
i.e., $f(\tilde{x}_1+\tilde{x}_2) = f(\tilde{x}_1)+f(\tilde{x}_2)$, and for any soft scalar $\widetilde{\alpha}$,
$[f(\widetilde{\alpha}\tilde{x})](\lambda) = \int_a^b (\widetilde{\alpha}\tilde{x})(\lambda)(t).\tilde{y}_0(\lambda)(t)dt = \widetilde{\alpha}(\lambda)\int_a^b \tilde{x}(\lambda)(t).\tilde{y}_0(\lambda)(t)dt$
$= [\widetilde{\alpha}f(\tilde{x})](\lambda),\ \forall\lambda \in A.$ i.e., $f(\widetilde{\alpha}\tilde{x}) = \widetilde{\alpha}f(\tilde{x})$
So, $f(\tilde{x})$ is a soft linear functional on $\tilde{C}[a,b]$.

**Theorem 3.3.** *If a soft linear functional $f$ is continuous at some soft element $\tilde{x}_0\tilde{\in}\check{X}$ then $f$ is continuous at every soft element of $\check{X}$.*

**Definition 3.4.** The soft linear functional $f$ is called bounded if there exists some positive soft real number $\tilde{M}$ such that for all $\tilde{x}\tilde{\in}\check{X}, \|f(\tilde{x})\| \tilde{\leq} \tilde{M}\|\tilde{x}\|$.

**Theorem 3.5.** *A soft linear functional $f$ is continuous if it is bounded.*

**Theorem 3.6.** *Let $\check{X}$ be a soft normed linear space which satisfy (N5) and $f : SE(\check{X}) \to K$ be a soft linear functional satisfying (L3). If $f$ is continuous then $f$ is bounded.*

**Theorem 3.7.** *Let $\check{X}$ be a soft normed linear space which satisfy (N5) and $f : SE(\check{X}) \to K$ be a soft linear functional. If $\check{X}$ is of finite dimension, then $f$ is bounded and hence continuous.*

**Definition 3.8.** Let$f$ be a bounded soft linear functional. Then the norm of the functional $f$ denoted by $\|f\|$, is a soft real number defined as the following:
For each $\lambda \in A, \|f\|(\lambda) = \inf\{t \in R; \|f(\tilde{x})\|(\lambda) \leq t.\|\tilde{x}\|(\lambda)$, for all $\tilde{x}\tilde{\in}\check{X}\}$.

**Theorem 3.9.** *Let $\check{X}$ be a soft normed linear space which satisfy (N5) and $f$ satisfy (L3), then for each $\lambda \in A, \|f\|(\lambda) = \|f_\lambda\|_\lambda$, where $\|f_\lambda\|_\lambda$ is the norm of the linear functional $f_\lambda$ on $X$.*

**Example 3.10.** Consider the absolute soft set generated by $C[0,1]$ and let us denote it by $\tilde{C}[0,1]$. Then $\tilde{C}[0,1]$ is an absolute soft vector space. For $\tilde{x}\tilde{\in}\tilde{C}[0,1]$ let us define $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_\lambda = \sup_{0\leq t\leq 1}|\tilde{x}(\lambda)(t)|,\ \forall\lambda \in A.$
Then it can be easily verified that $\|.\|$ is a soft norm on $\tilde{C}[0,1]$ and consequently $\tilde{C}[0,1]$ is a soft normed linear space. Also $\tilde{C}[0,1]$ satisfies (N5).
For $\tilde{x}\tilde{\in}\tilde{C}[0,1]$, let us define

$$[f(\tilde{x})](\lambda) = \int_0^1 \tilde{x}(\lambda)(t)dt,\ \forall\lambda \in A.$$

Then in a similar procedure as in Example 3.2, it can be easily verified that $f(\tilde{x})$ is a soft linear functional on $\tilde{C}[0,1]$. Also $f(\tilde{x})$ satisfies (L3).
Also $|f(\tilde{x})|(\lambda) = \left|\int_0^1 \tilde{x}(\lambda)(t)dt\right| \leq \underset{0\leq t\leq 1}{sup} |\tilde{x}(\lambda)(t)| = \|\tilde{x}(\lambda)\|_\lambda = \|\tilde{x}\|(\lambda), \forall\lambda \in A.$
i.e., $|f(\tilde{x})| \tilde{\leq} \|\tilde{x}\|$, for every $\tilde{x}\tilde{\in}\tilde{C}[0,\ 1]$. Hence $f$ is bounded and therefore continuous soft linear functional on $\tilde{C}[0,1]$.
Since $\tilde{C}[0,1]$ satisfies (N5) and $f(\tilde{x})$ satisfies (L3), it follows by Theorem 3.9 that, $\|f\|(\lambda) = \|f_\lambda\|_\lambda = 1$, for all $\lambda \in A$. So, $\|f\| = \bar{1}$.

**Theorem 3.11.** $\|f(\tilde{x})\| \tilde{\leq} \|f\| \|\tilde{x}\|$, for all $\tilde{x}\tilde{\in}\check{X}$.

**Theorem 3.12.** *Let $\check{X}$ be a soft normed linear space which satisfy (N5) and $f : SE(\check{X}) \to K$ be a soft linear functional on $\check{X}$ satisfying (L3). Then*

(i). $\|f\|(\lambda) = \sup\left\{\|f(\tilde{x})\|(\lambda) : \|\tilde{x}\| \tilde{\leq} \bar{1}\right\} = \|f_\lambda\|_\lambda$, *for each* $\lambda \in A$;
(ii). $\|f\|(\lambda) = \sup\left\{\|f(\tilde{x})\|(\lambda) : \|\tilde{x}\| = \bar{1}\right\} = \|f_\lambda\|_\lambda$, *for each* $\lambda \in A$;
(iii). $\|f\|(\lambda) = \sup\left\{\frac{\|f(\tilde{x})\|}{\|\tilde{x}\|}(\lambda) : \|\tilde{x}\|(\mu) \neq 0,\ \text{for all } \mu \in A\right\} = \|f_\lambda\|_\lambda$, *for each* $\lambda \in A$.

**Theorem 3.13.** *Let $\check{X}$ be a soft normed linear space satisfying (N5). Let $f : SE(\check{X}) \to K$ be a continuous soft linear functional on $\check{X}$ satisfying (L3). Then $f_\lambda$ is continuous linear on $X$ for each $\lambda \in A$.*

**Theorem 3.14.** *Let $\check{X}$ be a soft normed linear space satisfying (N5). Let $\{f_\lambda; \lambda \in A\}$ be a family of continuous linear functionals such that $f_\lambda : X \to \mathbb{R}$ or $\mathbb{C}$ for each $\lambda$. Then the functional $f : SE\left(\check{X}\right) \to K(= \mathbb{R}(A)$ or $\mathbb{C}(A))$ defined by $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, $\forall\lambda \in A$ and $\forall\tilde{x}\tilde{\in}\check{X}$, is a continuous soft linear functional satisfying (L3).*

**Definition 3.15.** (Conjugate spaces) Let $\check{X}$ be a soft normed linear space satisfying (N5). Let $W$ be the collection of all continuous soft linear functionals $f$ which satisfy (L3) each mapping $SE(\check{X}) \to K$. Let $W(\lambda) = \{f_\lambda(= f(\lambda)); f \in W\}$, for all $\lambda \in A$. Then as the case of soft linear operators we can easily verify that the absolute soft set generated by $W(\lambda), \lambda \in A$, form an absolute soft vector space and also a soft normed linear space. This soft normed linear space is called the *conjugate space* of $\check{X}$. This is denoted by $\check{X}^*$.

**Remark 3.16.** The conjugate space $\check{\mathrm{X}}^*$ of $\check{\mathrm{X}}$ is not a soft Banach space in general, since completeness of $\check{\mathrm{X}}^*$ cannot be obtained without finiteness restriction on the parameter set. However, if $\check{\mathrm{X}}$ be a soft normed linear space satisfying (N5) over a finite parameter set then conjugate space $\check{\mathrm{X}}^*$ becomes a soft Banach space.

**Proposition 3.17.** *The conjugate space $\check{X}^*$ of $\check{X}$ is a soft normed linear space satisfying (N5).*

*Proof.* By Definition 3.15, it follows that the conjugate space $\check{\mathrm{X}}^*$ of $\check{X}$ is a soft normed linear space.
To show that $\check{X}^*$ satisfy (N5), we have to prove that for any crisp linear functional $u$ on $X$ and for any $\lambda \in A$,
$\{\left\|\tilde{f}\right\|(\lambda) : \tilde{f}\tilde{\in}\check{X}^*$ *with* $\tilde{f}(\lambda) = u\}$ is a singleton set.

Let $\mu \in A$ and $w$ be any crisp linear functional on $X$.
Let $\tilde{f},\ \tilde{g}\tilde{\in}\check{X}^*$ such that $\tilde{f}\left(\mu\right) = \tilde{g}\left(\mu\right) = w$. Then we can identify $\tilde{f},\ \tilde{g}$ to unique continuous soft linear functionals $f, g$ respectively both of which satisfies (L3), also $\tilde{f}\left(\lambda\right) = f\left(\lambda\right), \tilde{g}\left(\lambda\right) = g\left(\lambda\right)$ for all $\lambda \in A$ and $\left\|\tilde{f}\right\|\left(\lambda\right) = \left\|f\right\|\left(\lambda\right),\ \left\|\tilde{g}\right\|\left(\lambda\right) = \left\|g\right\|\left(\lambda\right);$ for all $\lambda \in A$. So in particular, $f\left(\mu\right) = \tilde{f}\left(\mu\right) = \tilde{g}\left(\mu\right) = g\left(\mu\right) = w$ *and* $\left\|\tilde{f}\right\|\left(\mu\right) = \left\|f\right\|\left(\mu\right),\ \left\|\tilde{g}\right\|\left(\mu\right) = \left\|g\right\|\left(\mu\right).$
Since $\check{X}$ satisfy (N5) and $f, g$ satisfies (L3), we get by Theorem 3.9,

$$\left\|\tilde{f}\right\|\left(\mu\right) = \left\|f\right\|\left(\mu\right) = \left\|f\left(\mu\right)\right\|_{\mu} = \left\|w\right\|_{\mu} = \left\|g\left(\mu\right)\right\|_{\mu} = \left\|g\right\|\left(\mu\right) = \left\|\tilde{g}\right\|\left(\mu\right).$$

$\Rightarrow \{\left\|\tilde{f}\right\|\left(\mu\right) : \tilde{f}\tilde{\in}\check{X}^*\ \textit{with}\ \tilde{f}\left(\mu\right) = w\}$ is a singleton set.
$\Rightarrow \check{X}^*$ satisfy (N5). □

## 4. Hahn-Banach theorem and its consequences

**Definition 4.1.** Let $\check{X}$ be a soft normed linear space and $G$ be a soft subspace of $\check{X}$. Suppose that $f$ be a continuous soft linear functional defined only for soft elements $\tilde{x}$ in $G$. Let $\left\|f\right\|_G\left(\lambda\right) = \sup\left\{\left|f\left(\tilde{x}\right)\right|\left(\lambda\right) : \tilde{x}\tilde{\in}G, \left\|\tilde{x}\right\| \tilde{\leq} \tilde{1}\right\}$, for each $\lambda \in A$.
A continuous soft linear functional $F$ defined on $\check{X}$ is called an extension of $f$ onto the whole space $\check{X}$ if $f\left(\tilde{x}\right) = F(\tilde{x})$ for all $\tilde{x}\tilde{\in}G$.

**Theorem 4.2.** *Let $\check{X}$ be a real soft normed linear space satisfying (N5). Let $f$ be a continuous soft linear functional on a soft subspace $G$ of $\check{X}$ satisfying (L3). Then there exists a continuous soft linear functional $F$ defined on $\check{X}$ satisfying (L3), such that*

(i). *$f\left(\tilde{x}\right) = F(\tilde{x})$ for all $\tilde{x}\tilde{\in}G$; and*
(ii). $\left\|f\right\|_G = \left\|F\right\|_{\check{X}} = \left\|F\right\|.$

*Proof.* Since $\check{X}$ is a soft normed linear space satisfying (N5) and $f$ is a continuous soft linear functional over $G$ satisfying (L3), then by Theorem 3.9 and Theorem 3.13, it follows that for each $\lambda \in A, f_\lambda$ is a continuous linear functional on the crisp subspace $G(\lambda)$ of $X$ and $\left\|f\right\|\left(\lambda\right) = \left\|f_\lambda\right\|_\lambda$, where $\left\|f_\lambda\right\|_\lambda$ is the norm of $f_\lambda$ in $G\left(\lambda\right)$. By the Hahn-Banach Theorem in crisp sense there exists a continuous linear functional $F_\lambda(x)$ defined on $X$ such that

(i). $f_\lambda\left(x\right) = F_\lambda(x)$ for all $x \in G$; and
(ii). $\left\|f_\lambda\right\|_{G(\lambda)} = \left\|F_\lambda\right\|_X = \left\|F_\lambda\right\|_\lambda$.

Since each $F_\lambda$ is continuous, by Theorem 3.14, it follows that the soft linear functional $F(\tilde{x})$ defined on $\check{X}$ such that $\left(F\left(\tilde{x}\right)\right)\left(\lambda\right) = F_\lambda(\tilde{x}\left(\lambda\right))$, for each $\lambda \in A$, and $\forall\tilde{x}\tilde{\in}\check{X}$; is a continuous soft linear functional on $\check{X}$ satisfying (L3). Again we have,
(i). $f\left(\tilde{x}\right)\left(\lambda\right) = f_\lambda\left(\left(\tilde{x}\right)\left(\lambda\right)\right) = F_\lambda(\left(\tilde{x}\right)\left(\lambda\right)) = F(\tilde{x})\left(\lambda\right)$ for each $\lambda \in A$, for all $\tilde{x}\tilde{\in}G$; i.e., $f\left(\tilde{x}\right) = F(\tilde{x})$ for all $\tilde{x}\tilde{\in}G$; and
(ii). $\left\|f\right\|_G\left(\lambda\right) = \left\|f_\lambda\right\|_{G(\lambda)} = \left\|F_\lambda\right\|_X = \left\|F\right\|_{\check{X}}\left(\lambda\right)$ and $\left\|f\right\|_G\left(\lambda\right) = \left\|f_\lambda\right\|_{G(\lambda)} = \left\|F_\lambda\right\|_\lambda = \left\|F\right\|\left(\lambda\right)$, for each $\lambda \in A$; i.e.,

$$\left\|f\right\|_G = \left\|F\right\|_{\check{X}} = \left\|F\right\|.$$

□

**Theorem 4.3.** *Let $\check{X}$ be a real soft normed linear space which satisfy (N5). Let $\tilde{x}_0$ be an arbitrary soft element of $\check{X}$ such that $\tilde{x}_0(\lambda) \neq \theta$, for any $\lambda \in A$ and let $\tilde{M}$ be an arbitrary positive soft real number. Then there exists a continuous soft linear functional $f$ defined on $\check{X}$ such that $\|f\| = \tilde{M}$ and $f(\tilde{x}_0) = \|f\| \, . \, \|\tilde{x}_0\|$.*

*Proof.* We have for each $\lambda \in A$, $(X, \|.\|_\lambda)$ is a normed linear space, $\tilde{x}_0(\lambda) \neq \theta, \tilde{M}(\lambda)$ is an arbitrary positive real number. Then there exists a continuous linear functional $f_\lambda$ defined on $X$ such that $\|f_\lambda\|_\lambda = \tilde{M}(\lambda)$ and $f_\lambda((\tilde{x}_0)(\lambda)) = \|f_\lambda\|_\lambda . \|(\tilde{x}_0)(\lambda)\|_\lambda$.
Let us consider a functional $f$ on $\check{X}$ such that $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, and $\forall \tilde{x} \tilde{\in} \check{X}$. Then $f$ is a continuous soft linear functional on $\check{X}$.
Again, $\|f\|(\lambda) = \|f_\lambda\|_\lambda = \tilde{M}(\lambda)$, for each $\lambda \in A$
and $(\|f\| \, . \, \|\tilde{x}_0\|)(\lambda) = \|f_\lambda\|_\lambda . \|(\tilde{x}_0)(\lambda)\|_\lambda = f_\lambda((\tilde{x}_0)(\lambda)) = (f(\tilde{x}_0))(\lambda)$, for each $\lambda \in A$.
Hence $\|f\| = \tilde{M}$ and $f(\tilde{x}_0) = \|f\| \, . \, \|\tilde{x}_0\|$. □

**Definition 4.4.** Let $\check{X}$ be a real soft normed linear space satisfying (N5). A continuous soft linear functional $f$ on $\check{X}$ is said to be a non-zero continuous soft linear functional if it satisfies (L3) and $f_\lambda$ are non-zero for each $\lambda \in A$ i.e., for each $\lambda \in A$, there exists $x_\lambda \in X$ such that $f_\lambda(x_\lambda) \neq 0$.

**Proposition 4.5.** *Let $\check{X}$ be a real soft normed linear space which satisfy (N5). Then for every $\tilde{x} \tilde{\in} \check{X}$,*
$\left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda) \, ; f \text{ } is \text{ } non-zero \text{ } on \text{ } \check{X}\right\} = \{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \text{ } is \text{ } non-zero \text{ } on \text{ } X\}$, *for each $\lambda \in A$.*

*Proof.* Let $t \in \left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda) \, ; f \text{ } is \text{ } non-zero \text{ } on \text{ } \check{X}\right\}$. Then there is a non-zero continuous soft linear functional $f$ on $\check{X}$ such that $t = \left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda)$. By Definition 4.4, $f$ satisfies (L3) and $f_\lambda$ are non-zero for each $\lambda \in A$.
$\therefore \, t = \left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda) = \frac{|f(\tilde{x})|(\lambda)}{\|f\|(\lambda)} = \frac{|f_\lambda(\tilde{x}(\lambda))|}{\|f_\lambda\|_\lambda}$ and $f_\lambda$ is a non-zero continuous linear functional on $X$.

$$\therefore t \in \{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \text{ } is \text{ } non-zero \text{ } on \text{ } \check{X}\} \tag{4.1}$$

Again let $s \in \{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \text{ } is \text{ } non-zero \text{ } on \text{ } X\}$. Then there is a non-zero continuous linear functional $u$ on $X$ such that $s = \frac{|u(\tilde{x}(\lambda))|}{\|u\|_\lambda}$. Let us consider a functional $g$ on $\check{X}$ defined by $(g(\tilde{x}))(\lambda) = u(\tilde{x}(\lambda))$, $\forall \lambda \in A$ and $\tilde{x} \tilde{\in} \check{X}$.
Then by Theorem 3.14 and Definition 4.4, $g$ is a non-zero continuous soft linear functional on $\check{X}$ and $g_\lambda = u, \forall \lambda \in A$.
We also have $s = \frac{|u(\tilde{x}(\lambda))|}{\|u\|_\lambda} = \frac{|g_\lambda(\tilde{x}(\lambda))|}{\|g_\lambda\|_\lambda} = \frac{|g(\tilde{x})|(\lambda)}{\|g\|(\lambda)} = \left(\frac{|g(\tilde{x})|}{\|g\|}\right)(\lambda)$.

$$\therefore \; s \in \left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda) \, ; f \text{ } is \text{ } non-zero \text{ } on \text{ } \check{X}\right\} \tag{4.2}$$

From 4.1 and 4.2, it follows that $\left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda) \, ; f \text{ } is \text{ } non-zero \text{ } on \text{ } \check{X}\right\}$

$= \{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \ is \ non-zero \ on \ X\}$, for each $\lambda \in A$. ☐

**Theorem 4.6.** *Let $\check{X}$ be a real soft normed linear space which satisfy (N5). Then for every $\tilde{x}\tilde{\in}\check{X}, \|\tilde{x}\|(\lambda) = \sup\{(|f(\tilde{x})| / \|f\|)(\lambda)\}$, for each $\lambda \in A$; where the supremum is taken over all non-zero continuous soft linear functionals $f$.*

*Proof.* For every $\tilde{x}\tilde{\in}\check{X}, \tilde{x}(\lambda) \in X$ and we have $\|\tilde{x}(\lambda)\|_\lambda = \sup\{|w(\tilde{x}(\lambda))|/\|w\|_\lambda\}$,for each $\lambda \in A$; where the supremum is taken over all non-zero continuous linear functionals $w$ over $X$.
For all non-zero continuous soft linear functionals $f$ on $\check{X}$, we have by Proposition 4.5,
$\left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda); f \ is \ non-zero \ on \ \check{X}\right\} = \{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \ is \ non-zero \ on \ X\}$, for each $\lambda \in A$.
$\therefore \sup\left\{\left(\frac{|f(\tilde{x})|}{\|f\|}\right)(\lambda); f \ is \ non-zero \ on \ \check{X}\right\}$
$= \sup \left\{\frac{|w(\tilde{x}(\lambda))|}{\|w\|_\lambda}; w \ is \ non-zero \ on \ X\right\} = \|\tilde{x}(\lambda)\|_\lambda = \|\tilde{x}\|(\lambda)$, for each $\lambda \in A$. ☐

**Theorem 4.7.** *Let $\check{X}$ be a real soft normed linear space which satisfy (N5). Let $G$ be a soft subspace of $\check{X}$ and let $\tilde{y}_0\tilde{\in}\check{X}$ be such that $\tilde{y}_0(\lambda) \notin G(\lambda)$, $\forall\lambda \in A$. Let $\tilde{d}\tilde{>}\overline{0}$ be such that $\tilde{d}(\lambda) = \inf\{\|\tilde{y}_0 - \tilde{x}\|(\lambda); \tilde{x}\tilde{\in}G\}$ , for each $\lambda \in A$. Then there exists a continuous soft linear functional $f$ defined on $\check{X}$ such that*

(i). $f(\tilde{x}) = \overline{0}$ *for* $\tilde{x}\tilde{\in}G$,
(ii). $f(\tilde{y}_0) = \overline{1}$ *and*
(iii). $\|f\| = \frac{\overline{1}}{\tilde{d}}$.

*Proof.* For each $\lambda \in A, \tilde{y}_0(\lambda) \in X - G(\lambda)$; $\tilde{d}(\lambda) = \inf\{\|\tilde{y}_0 - \tilde{x}\|(\lambda); \tilde{x}\tilde{\in}G\} = \inf\{\|\tilde{y}_0(\lambda) - \tilde{x}(\lambda)\|_\lambda; \tilde{x}(\lambda) \in G(\lambda)\} = \inf\{\|\tilde{y}_0(\lambda) - x\|_\lambda; x \in G(\lambda)\}$ and $G(\lambda)$ is a subspace of $X$. Then there exists a continuous linear functional $f_\lambda$ defined on $X$ such that
(i). $f_\lambda(x) = 0$ for $x \in G(\lambda)$, (ii). $f_\lambda((\tilde{y}_0)(\lambda)) = 1$ and (iii). $\|f_\lambda\|_\lambda = \frac{1}{\tilde{d}(\lambda)}$.
Let us consider a functional $f$ on $\check{X}$ such that $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, and $\forall\tilde{x}\tilde{\in}\check{X}$. Then $f$ is a continuous soft linear functional on $\check{X}$.
Also, (i). for $\tilde{x}\tilde{\in}G$, $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda)) = 0$, for each $\lambda \in A \Longrightarrow f(\tilde{x}) = \overline{0}$ for $\tilde{x}\tilde{\in}G$;
(ii). $f(\tilde{y}_0)(\lambda) = f_\lambda((\tilde{y}_0)(\lambda)) = 1$ for each $\lambda \in A \Longrightarrow f(\tilde{y}_0) = \overline{1}$; and
(iii). $\|f\|(\lambda) = \|f_\lambda\|_\lambda = \frac{1}{\tilde{d}(\lambda)}$ for each $\lambda \in A \Longrightarrow \|f\| = \frac{\overline{1}}{\tilde{d}}$. ☐

**Theorem 4.8.** *Let $\check{X}$ be a real soft normed linear space which satisfy (N5). Let $G$ be a soft subspace of $\check{X}$ and let $\tilde{x}_0\tilde{\in}\check{X}$ be such that $\tilde{x}_0(\lambda) \notin G(\lambda)$, $\forall\lambda \in A$. Suppose for some soft real number $\tilde{h}, \tilde{h}(\lambda) = \inf\{\|\tilde{x}_0 - \tilde{x}\|(\lambda); \tilde{x}\tilde{\in}G\} > 0$, for each $\lambda \in A$. Then there exists a continuous soft linear functional $f$ defined on $\check{X}$ such that*

(i). $f(\tilde{x}_0) = \tilde{h}$,
(ii). $\|f\| = \overline{1}$ *and*
(iii). $f(\tilde{x}) = \overline{0}$ *for* $\tilde{x}\tilde{\in}G$.

*Proof.* For each $\lambda \in A, \tilde{x}_0(\lambda) \in X - G(\lambda)$; $\tilde{h}(\lambda) = \inf\{\|\tilde{x}_0 - \tilde{x}\|(\lambda); \tilde{x}\tilde{\in}G\} = \inf\{\|\tilde{x}_0(\lambda) - \tilde{x}(\lambda)\|_\lambda; \tilde{x}(\lambda) \in G(\lambda)\}$ and $G(\lambda)$ is a subspace of $X$. Then there exists a continuous linear functional $f_\lambda$ defined on $X$ such that
(i) $f_\lambda((\tilde{x}_0)(\lambda)) = \tilde{h}(\lambda)$, (ii) $\|f_\lambda\|_\lambda = 1$ and (iii) $f_\lambda(x) = 0$ for $x \in G(\lambda)$.
Let us consider a functional $f$ on $\check{X}$ such that $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, and $\forall \tilde{x}\tilde{\in}\check{X}$. Then $f$ is a continuous soft linear functional on $\check{X}$.
Also, (i). $f(\tilde{x}_0)(\lambda) = f_\lambda((\tilde{x}_0)(\lambda)) = \tilde{h}(\lambda)$ for each $\lambda \in A \Longrightarrow f(\tilde{x}_0) = \tilde{h}$;
(ii). $\|f\|(\lambda) = \|f_\lambda\|_\lambda = 1$ for each $\lambda \in A \Longrightarrow \|f\| = \overline{1}$; and
(iii). for $\tilde{x}\tilde{\in}G$, $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda)) = 0$, for each $\lambda \in A \Longrightarrow f(\tilde{x}) = \overline{0}$ for $\tilde{x}\tilde{\in}G$. ☐

**Definition 4.9.** (Second Conjugate Spaces) Let $\check{X}$ be a soft normed linear space satisfying (N5). Then the conjugate space $\check{X}^*$ is also a soft normed linear space satisfying (N5). We can therefore construct successively the spaces $(\check{X}^*)^* = \check{X}^{**}$, $(\check{X}^{**})^* = \check{X}^{***}$ and so on.
Each of these spaces is a soft normed linear space. The space $\check{X}^{**}$ is called the second conjugate space of $\check{X}$. The space $\check{X}^{**}$ is, therefore, can be identified with the space of all continuous soft linear functionals defined on $\check{X}^*$.
If $\tilde{x}\tilde{\in}\check{X}$ is fixed and $\tilde{f}\tilde{\in}\check{X}^*$ is variable, then for each $\tilde{f}\tilde{\in}\check{X}^*$, by Proposition 2.59, we can identify $\tilde{f}$ to a unique continuous soft linear functional $f$ which satisfy (L3) and $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, and $\forall \tilde{x}\tilde{\in}\check{X}$. Then for different $\tilde{f}\tilde{\in}\check{X}^*$, identifying $\tilde{f}$ with $f$, we obtain different values of $f(\tilde{x})$. Therefore, the expression $f(\tilde{x})$ where $\tilde{x}$ is fixed and $\tilde{f}$ is variable, represents a certain functional $F_{\tilde{x}}$, say over $\check{X}^*$. So, we write

$$F_{\tilde{x}}(f) = f(\tilde{x}) \tag{4.3}$$

where $\tilde{x}$ is fixed and $\tilde{f}$ is variable.
We show that $F_{\tilde{x}}$ is a continuous soft linear functional over $\check{X}^*$.
We have $F_{\tilde{x}}(f_1 + f_2) = (f_1 + f_2)(\tilde{x}) = f_1(\tilde{x}) + f_2(\tilde{x}) = F_{\tilde{x}}(f_1) + F_{\tilde{x}}(f_2)$,
$F_{\tilde{x}}(\tilde{c}.f) = (\tilde{c}.f)(\tilde{x}) = \tilde{c}.f(\tilde{x}) = \tilde{c}.F_{\tilde{x}}(f)$, where $\tilde{c}$ is a soft scalar.
And $|F_{\tilde{x}}(f)| = |f(\tilde{x})| \tilde{\leq} \|f\| \|\tilde{x}\|$ for every $f$.
Therefore, $F_{\tilde{x}}$ is soft linear and bounded and so is continuous soft linear functional. So, for every fixed $\tilde{x}\tilde{\in}\check{X}$ there corresponds a unique continuous soft linear functional $F_{\tilde{x}}$ given by 4.3.
We now prove that $F_{\tilde{x}} \in SE(\check{X}^{**})$.
In the construction of first Conjugate Space $(\check{X}^*)$, the underlying vector space is $W(\lambda)$ which is identical with the vector space of all continuous linear functionals on $X$ with respect to usual addition and scalar multiplication of linear functionals i.e., the vector space $X^*$ for all $\lambda \in A$. The corresponding absolute soft vector space is $\check{X}^*$.
Similarly in the construction of second Conjugate Space $(\check{X}^{**})$, according to the Definition 4.9, the underlying vector space is identical with the set of all continuous linear functionals on $X^*$ i.e., the vector space $X^{**}$ for all $\lambda \in A$. The corresponding absolute soft vector space is $\check{X}^{**}$.
Now we have by Equation 4.3 of Definition 4.9,

$F_{\tilde{x}}(f) = f(\tilde{x})$, where $\tilde{x}$ is fixed and $f$ is variable.
Therefore, for $\lambda \in A, [F_{\tilde{x}}(f)](\lambda) = [f(\tilde{x})](\lambda)$

$$= f_\lambda(\tilde{x}(\lambda))$$

where $f_\lambda$ are continuous linear functionals on $X$.
Let for $\lambda \in A, {F_x}^\lambda$ be defined by ${F_x}^\lambda(\varphi) = \varphi(x), \forall \varphi \in X^*$. Then${F_x}^\lambda \in X^{**}$ and $[F_{\tilde{x}}(f)](\lambda) = f_\lambda(\tilde{x}(\lambda)) = {F_{\tilde{x}(\lambda)}}^\lambda(f_\lambda)$, $\forall \lambda \in A$.
i.e., $F_{\tilde{x}}(f)$ can be considered as the parameterized family $\{{F_{\tilde{x}(\lambda)}}^\lambda; \lambda \in A\}$ of members of $X^{**}$, i.e., a soft element of $\check{X}^{**}$ i.e., a soft element of $\check{X}^{**}$.
Therefore, $F_{\tilde{x}} \in SE(\check{X}^{**})$.
Let us define a mapping $C : SE(\check{X}) \to SE(\check{X}^{**})$ by $C(\tilde{x}) = F_{\tilde{x}}$.
We now verify that $C$ is a bijective soft linear operator between $SE(\check{X})$ and the range of $C$, which is a subset of $SE(\check{X}^{**})$.
If $\tilde{a}$, $\tilde{b}$ be soft scalars, then $F_{\tilde{a}\tilde{x}+\tilde{b}\tilde{y}}(f) = f\left(\tilde{a}\tilde{x} + \tilde{b}\tilde{y}\right) = \tilde{a}f(\tilde{x}) + \tilde{b}f(\tilde{y}) = \tilde{a}F_{\tilde{x}}(f) + \tilde{b}F_{\tilde{y}}(f) = (\tilde{a}F_{\tilde{x}} + \tilde{b}F_{\tilde{y}})(f)$, for every $f$.
So, $C\left(\tilde{a}\tilde{x} + \tilde{b}\tilde{y}\right) = F_{\tilde{a}\tilde{x}+\tilde{b}\tilde{y}} = \tilde{a}F_{\tilde{x}} + \tilde{b}F_{\tilde{y}} = \tilde{a}C(\tilde{x}) + \tilde{b}C(\tilde{y})$.
Therefore, $C$ is a soft linear operator. By Theorem 4.6 and Theorem 3.12,
$\|F_{\tilde{x}}\|(\lambda) = \sup\left\{\frac{|F_{\tilde{x}}(f)|}{\|f\|} : \|f\|(\mu) \neq 0, \forall \mu \in A\right\}, \forall \lambda \in A$
$= \sup\left\{\frac{|f(\tilde{x})|}{\|f\|} : \|f\|(\mu) \neq 0, \forall \mu \in A\right\}, \forall \lambda \in A = \|\tilde{x}\|(\lambda), \forall \lambda \in A$.

$$\therefore \|F_{\tilde{x}}\| = \|\tilde{x}\| \tag{4.4}$$

Now $F_{\tilde{x}-\tilde{y}}(f) = f(\tilde{x} - \tilde{y}) = f(\tilde{x}) - f(\tilde{y}) = F_{\tilde{x}}(f) - F_{\tilde{y}}(f) = (F_{\tilde{x}} - F_{\tilde{y}})(f)$ and this implies $F_{\tilde{x}-\tilde{y}} = F_{\tilde{x}} - F_{\tilde{y}}$, and so by 4.4, $\|F_{\tilde{x}} - F_{\tilde{y}}\| = \|F_{\tilde{x}-\tilde{y}}\| = \|\tilde{x} - \tilde{y}\|$.
and this shows that if $\tilde{x} \neq \tilde{y}$ then $F_{\tilde{x}} \neq F_{\tilde{y}}$, which shows that $C$ is injective.
Keeping in view that $\|F_{\tilde{x}}\| = \|\tilde{x}\|$, we see that $C$ is bijective.

**Definition 4.10.** Let $\check{X}$ be a soft normed linear space satisfying (N5) over a finite parameter set $A$. Then the conjugate space $\check{X}^*$ is also a soft Banach space satisfying (N5). Similarly the spaces $\check{X}^{**}$, $\check{X}^{***}$, .. are also so. Then using the above identification and since every element of $SE(\check{X}^{**})$ has its identification with a unique continuous soft linear functional on $\check{X}^*$, in case range of $C$ is all of $SE(\check{X}^{**})$, we shall say that $\check{X}$ is reflexive.

**Theorem 4.11.** *(Uniform Boundedness Principle) Let $\check{X}$ be a soft Banach space and $\check{Y}$ be a soft normed linear space both of which satisfy (N5). Let $\{T_i\}$ be a non-empty sequence of continuous soft linear operators such that $T_i : SE(\check{X}) \to SE(\check{Y})$ and $T_i$ satisfy (L3) for each $i$. If the sequence $\{T_i(\tilde{x})\}$ is bounded in $\check{Y}$ for each $\tilde{x}\tilde{\in}\check{X}$, then $\{\|T_i\|\}$ is a bounded sequence of soft real numbers.*

*Proof.* For each $\lambda \in A, \check{X}(\lambda) = X$, $\check{Y}(\lambda) = Y$ are respectively a Banach space and a normed linear space and $\{T_i(\lambda) = T_{i\lambda}\}$ is a non-empty sequence of continuous linear operators such that $T_{i\lambda} : X \to Y$ for each $i$. Also the sequence $\{(T_i(\tilde{x}))(\lambda) = T_{i\lambda}((\tilde{x})(\lambda))\}$ is bounded in $Y$ for each $\tilde{x}(\lambda) \in X$, then by Uniform Boundedness Principle of normed linear spaces the sequence $\{\|T_i\|(\lambda) = \|T_{i\lambda}\|_\lambda\}$ is a bounded sequence of real numbers. Then for each $\lambda \in A$, there is positive real number $k_\lambda$ such

that $\|T_{i\lambda}(x)\|_\lambda \leq k_\lambda$, for each $x \in X$. Let us consider a soft real number $\tilde{k}$ such that $\tilde{k}(\lambda) = k_\lambda$, $\forall \lambda \in A$. Then for each $\tilde{x}\tilde{\in}\check{X}$, $\|T_i(\tilde{x})\|(\lambda) = \|T_{i\lambda}((\tilde{x})(\lambda))\|_\lambda \leq k_\lambda = \tilde{k}(\lambda)$, for each $\tilde{x}(\lambda) \in X$; i.e., $\|T_i(\tilde{x})\| \tilde{\leq} \tilde{k}$, for each $\tilde{x}\tilde{\in}\check{X}$.
$\therefore \{\|T_i\|\}$ is a bounded sequence of soft real numbers. □

Since $SS(\mathbb{R}(A)) = \check{\mathbb{R}}$ or $SS(\mathbb{C}(A)) = \check{\mathbb{C}}$ is a soft normed linear space, the following proposition is a direct consequence of the above theorem.

**Proposition 4.12.** *Let $\check{X}$ be a soft Banach space satisfying (N5). Let $\{f_i\}$ be a non-empty sequence of continuous soft linear functionals on $\check{X}$ such that $f_i$ satisfies (L3) for each $i$ and $\{f_i(\tilde{x})\}$ is bounded for each $\tilde{x}\tilde{\in}\check{X}$, then $\{\|f_i\|\}$ is a bounded sequence of soft real numbers.*

**Definition 4.13.** A soft subset $(Y, A)$ of a soft normed linear space $\check{X}$ is said to be a bounded soft subset if there exists a soft real number $\tilde{k}$ such that $\|\tilde{x}\| \tilde{\leq} \tilde{k}$, for each $\tilde{x}\tilde{\in}(Y, A)$.

**Theorem 4.14.** *Let $(Y, A)$ with $Y(\lambda) \neq \varnothing$, $\forall \lambda \in A$, be a soft subset of a soft normed linear space $\check{X}$ satisfying (N5). Let the set $\{f(\tilde{x}); \tilde{x}\tilde{\in}(Y, A)\}$ be bounded for each $f \in \check{X}^*$, then $(Y, A)$ is a bounded soft subset of $\check{X}$.*

*Proof.* For each $\lambda \in A$, $Y(\lambda) \neq \varnothing$ is a subset of the normed linear space $X$ and $f_\lambda$ is a continuous linear functional defined on $X$. Also the set $\{f_\lambda(x); x \in Y(\lambda)\}$ is bounded for each $f_\lambda \in X^*$. Then using the properties of crisp norm linear spaces, we find that $Y(\lambda)$ is a bounded subset of $X$. Thus for each $\lambda \in A$, there is a positive real number $k_\lambda$ such that $\|x\|_\lambda \leq k_\lambda$, for each $x \in Y(\lambda)$. Let us consider a soft real number $\tilde{k}$ such that $\tilde{k}(\lambda) = k_\lambda$, $\forall \lambda \in A$. Then for each $\tilde{x}\tilde{\in}(Y, A)$, $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_\lambda \leq k_\lambda = \tilde{k}(\lambda)$, for each $\tilde{x}(\lambda) \in X$; i.e., $\|\tilde{x}\| \tilde{\leq} \tilde{k}$, for each $\tilde{x}\tilde{\in}(Y, A)$. Therefore, $(Y, A)$ is a bounded soft subset of $\check{X}$. □

## 5. Weak convergence of sequence of soft elements

**Definition 5.1.** Let $\check{X}$ be a soft normed linear space satisfying (N5). Suppose that $\tilde{x}_n,\ \tilde{x}_0\tilde{\in}\check{X}$. The sequence $\{\tilde{x}_n\}$ of soft elements is said to *converge weakly* to $\tilde{x}_0$ if for all $f \in \check{X}^*$, $f(\tilde{x}_n) \to f(\tilde{x}_0)$ as $n \to \infty$. We write $\tilde{x}_n \to^{wk} \tilde{x}_0$ and we say that $\tilde{x}_0$ is a *weak limit* of the sequence $\{\tilde{x}_n\}$.
It is clear that if $\tilde{x}_n \to^{wk} \tilde{x}_0$ and $\tilde{y}_n \to^{wk} \tilde{y}_0$ then $\tilde{x}_n + \tilde{y}_n \to^{wk} \tilde{x}_0 + \tilde{y}_0$ and if $\tilde{c}$ be any soft scalar and $\tilde{x}_n \to^{wk} \tilde{x}_0$ then $\tilde{c}\tilde{x}_n \to^{wk} \tilde{c}\tilde{x}_0$.

**Theorem 5.2.** *Let $\check{X}$ be a soft normed linear space satisfying (N5). A sequence in $\check{X}$ cannot converge weakly to two different limits.*

*Proof.* Suppose that $\tilde{x}_n \to^{wk} \tilde{x}_0$ and $\tilde{x}_n \to^{wk} \tilde{y}_0$, then for arbitrary $f \in \check{X}^*$, $f(\tilde{x}_n) \to f(\tilde{x}_0)$ and $f(\tilde{x}_n) \to f(\tilde{y}_0)$. So, $f(\tilde{x}_0) = f(\tilde{y}_0)$ i.e., $f(\tilde{x}_0 - \tilde{y}_0) = \overline{0}$
Let $\lambda \in A$ be arbitrary. Then we have, $f(\tilde{x}_0 - \tilde{y}_0)(\lambda) = \overline{0}(\lambda) = 0$ i.e.,

$$f_\lambda(\tilde{x}_0(\lambda) - \tilde{y}_0(\lambda)) = 0 \tag{5.1}$$

Then by a consequence of Hahn-Banach theorem in crisp sense, we choose a $u \in X^*$ such that $\|u\|_\lambda = 1$ and $u(\tilde{x}_0(\lambda) - \tilde{y}_0(\lambda)) = \|\tilde{x}_0(\lambda) - \tilde{y}_0(\lambda)\|_\lambda$.

Replacing $f_\lambda$ by $u$ in 5.1, we obtain $0 = u(\tilde{x}_0(\lambda) - \tilde{y}_0(\lambda)) = \|\tilde{x}_0(\lambda) - \tilde{y}_0(\lambda)\|_\lambda$ and so $\tilde{x}_0(\lambda) = \tilde{y}_0(\lambda)$.
Since $\lambda \in A$ is arbitrary, $\tilde{x}_0(\lambda) = \tilde{y}_0(\lambda), \forall \lambda \in A$ i.e., $\tilde{x}_0 = \tilde{y}_0$. Hence $\{\tilde{x}_n\}$ cannot converge weakly to two different limits. □

**Definition 5.3.** We say that a sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ *converges strongly* or *converges in norm* to $\tilde{x}_0 \tilde{\in} \check{X}$ if $\|\tilde{x}_n - \tilde{x}_0\| \to \bar{0}$ as $n \to \infty$.

**Theorem 5.4.** *Let $\check{X}$ be a soft normed linear space satisfying (N5). Then for any sequence of soft elements in $\check{X}$, strong convergence implies weak convergence.*

*Proof.* Let $\{\tilde{x}_n\}$ converges strongly to $\tilde{x}_0$ i.e., $\|\tilde{x}_n - \tilde{x}_0\| \to \bar{0}$ as $n \to \infty$. For arbitrary $f \in \check{X}^*$, we obtain
$|f(\tilde{x}_n) - f(\tilde{x}_0)| = |f(\tilde{x}_n - \tilde{x}_0)| \tilde{\leq} \|f\| \, \|\tilde{x}_n - \tilde{x}_0\| \to \bar{0}$ as $n \to \infty$. So, $\tilde{x}_n \to^{wk} \tilde{x}_0$. □

**Theorem 5.5.** *Let $\check{X}$ be a finite dimensional soft normed linear space which satisfy (N5). Then strong convergence and weak convergence coincide.*

*Proof.* By Theorem 5.4, strong convergence implies weak convergence. We prove that in $\check{X}$, weak convergence implies strong convergence. Let $\{\tilde{x}_n\}$ be a sequence of soft elements of $\check{X}$ such that $\tilde{x}_n \to^{wk} \tilde{x}_0$. As $\check{X}$ is of finite dimensional, there exists a finite number of linearly independent soft elements $\tilde{e}_1, \tilde{e}_2, \ldots.., \tilde{e}_k$ in $\check{X}$ such that every $\tilde{x} \tilde{\in} \check{X}$ can be represented as $\tilde{x} = \tilde{\xi}_1 \tilde{e}_1 + \tilde{\xi}_2 \tilde{e}_2 + \cdots + \tilde{\xi}_k \tilde{e}_k$, where $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_k$ are soft scalars. Therefore we can write $\tilde{x}_n = \tilde{\xi}_1^{(n)} \tilde{e}_1 + \tilde{\xi}_2^{(n)} \tilde{e}_2 + \cdots + \tilde{\xi}_k^{(n)} \tilde{e}_k$, $n = 1, 2, 3, \ldots$ and $\tilde{x}_0 = \tilde{\xi}_1^{(0)} \tilde{e}_1 + \tilde{\xi}_2^{(0)} \tilde{e}_2 + \cdots + \tilde{\xi}_k^{(0)} \tilde{e}_k$.
We now define soft functionals $f_1, f_2, \ldots, f_k$ over $\check{X}$ as follows. If $\tilde{x} = \tilde{\xi}_1 \tilde{e}_1 + \tilde{\xi}_2 \tilde{e}_2 + \cdots + \tilde{\xi}_k \tilde{e}_k \tilde{\in} \check{X}$, then $f_i(\tilde{x}) = \tilde{\xi}_i$, $i = 1, 2, \ldots, k$. Clearly each $f_i$ is a linear functional. By Theorem 3.7, each $f_i$ is bounded and so continuous.
Now, $f_i(\tilde{x}_n) = \tilde{\xi}_i^{(n)}$ and $f_i(\tilde{x}_0) = \tilde{\xi}_i^{(0)}$. Since $\tilde{x}_n \to^{wk} \tilde{x}_0$, $f_i(\tilde{x}_n) \to f_i(\tilde{x}_0)$ i.e., $\tilde{\xi}_i^{(n)} \to \tilde{\xi}_i^{(0)}$ as $n \to \infty$ for $i = 1, 2, \ldots, k$. Let us consider a positive soft real number $\tilde{M}$ such that for each $\lambda \in A$, $\tilde{M}(\lambda) = \max\{\|\tilde{e}_i\|(\lambda)\}$, $i = 1, 2, .., k$.Let $\tilde{\varepsilon} \tilde{>} \bar{0}$ be arbitrary, then there exists a positive integer $n_0$ such that $|\tilde{\xi}_i^{(n)} - \tilde{\xi}_i^{(0)}| \tilde{<} \frac{\tilde{\varepsilon}}{\tilde{M}.k}$ for $n \geq n_0$ and $i = 1, 2, \ldots, k$.
Then for $n \geq n_0$, $\|\tilde{x}_n - \tilde{x}_0\| = \left\| \sum_{i=1}^k (\tilde{\xi}_i^{(n)} - \tilde{\xi}_i^{(0)}) \tilde{e}_i \right\| \tilde{\leq} \sum_{i=1}^k \left| \tilde{\xi}_i^{(n)} - \tilde{\xi}_i^{(0)} \right| .\tilde{M}$
$\tilde{<} \tilde{M}. \frac{\tilde{\varepsilon}}{\tilde{M}.k}.k = \tilde{\varepsilon}$.
Consequently, $\{\tilde{x}_n\}$ converges strongly to $\tilde{x}_0$. □

**Theorem 5.6.** *If a sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ converges weakly then the sequence of norms $\{\|\tilde{x}_n\|\}$ is bounded.*

*Proof.* For $\tilde{f} \tilde{\in} \check{X}^*$, then can be identified uniquely to a continuous soft linear functional $f$ and we have $\{f(\tilde{x}_n)\}$ is a convergent sequence of soft real numbers. So let $|f(\tilde{x}_n)| \tilde{\leq} \tilde{c}(f)$ for all $n$ where $\tilde{c}(f)$ is a soft real number depending only on $f$. Using the relation 5.1 of Definition 4.9, we can write
$F_{\tilde{x}_n}(f) = f(\tilde{x}_n)$, $n = 1, 2, \ldots$ So, $|F_{\tilde{x}_n}(f)| = |f(\tilde{x}_n)| \tilde{\leq} \tilde{c}(f)$ for $n = 1, 2, \ldots$
This shows that the sequence $\{F_{\tilde{x}_n}(f)\}$ is bounded.

Since $\check{X}^*$ is a soft Banach space, the principle of uniform boundedness (Theorem 4.11) implies that $\|F_{\tilde{x}_n}\|$ is bounded. By relation 4.4 of Definition 4.9, $\|F_{\tilde{x}_n}\| = \|\tilde{x}_n\|$ and so the sequence $\{\|\tilde{x}_n\|\}$ is bounded. □

## 6. Open mapping theorem and closed graph theorem

**Definition 6.1.** Let $\check{X}$ and $\check{Y}$ be soft metric spaces and $f$ be a mapping of $SE(\check{X})$ into $SE(\check{Y})$. Then $f$ is called an *open mapping* if whenever $\tilde{G}$ is soft open in $\check{X}$, $SS(f(\tilde{G}))$ is soft open in $\check{Y}$, where $f\left(\tilde{G}\right) = \{f\left(\tilde{x}\right), \tilde{x}\tilde{\in}\tilde{G}\}$.

**Theorem 6.2.** *(Open mapping theorem) If $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5) and $T : SE(\check{X}) \to SE(\check{Y})$ is a continuous soft linear operator which satisfy (L3), then $T$ is an open mapping.*

*Proof.* Since $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5), $X, Y$ are Banach spaces. By Theorem 2.53, it follows that, for each $\lambda \in A, T_\lambda : X \to Y$ is a continuous linear operator. Then by the open mapping theorem in crisp sense, it follows that for each $\lambda \in A, T_\lambda : X \to Y$ is an open mapping.
Since $\check{X}$ and $\check{Y}$ are soft normed linear spaces which satisfy (N5), the induced soft metrics satisfy (M5).[by Prop. 2.31] Let $\tilde{G}$ is soft open in $\check{X}$, then by Proposition 2.22, $\tilde{G}(\lambda)$ is open in $X$, $\forall\lambda \in A$. Since $T_\lambda : X \to Y$ is an open mapping, $T_\lambda(\tilde{G}(\lambda))$ is open in $Y$. We also have $\left(T\left(\tilde{G}\right)\right)(\lambda) = T_\lambda\left(\tilde{G}\left(\lambda\right)\right), \forall\lambda \in A$. Thus $SS\left(T\left(\tilde{G}\right)\right)(\lambda)$ is open in $Y$, $\forall\lambda \in A$. Hence by Proposition 2.22, $SS\left(T\left(\tilde{G}\right)\right)$ is soft open in $\check{Y}$. Therefore, $T$ is an open mapping. □

**Definition 6.3.** Suppose that $(F, A)$ and $(G, A)$ are soft sets, $(H, A)$ be a soft subset of $(F, A)$ and $T$ is a soft linear operator from $SE(H, A)$ to $SE(G, A)$. Then the collection of ordered pairs $G_T = \{[\tilde{x}, T\left(\tilde{x}\right)], \tilde{x}\tilde{\in}(H, A)\}$ is called the *graph* of the soft linear operator $T$.

Let $\check{X}$ and $\check{Y}$ be two absolute soft vector spaces. Let $\check{X} \times \check{Y}$ denote the soft set generated by the collection of all ordered pairs of soft elements $[\tilde{x}, \tilde{y}]$ where $\tilde{x}\tilde{\in}\check{X}$ and $\tilde{y}\tilde{\in}\check{Y}$ and $[\tilde{x}, \tilde{y}]\left(\lambda\right) = [\tilde{x}\left(\lambda\right), \tilde{y}\left(\lambda\right)]$ for each $\lambda \in A$. Then for each $\lambda \in A$, $\left(\check{X} \times \check{Y}\right)(\lambda) = X \times Y$ and hence is a vector space. Thus $\check{X} \times \check{Y}$ is an absolute soft vector space.

If $\check{X}$ and $\check{Y}$ are soft normed linear spaces, then we can also introduce a soft norm in $\check{X} \times \check{Y}$. If $[\tilde{x}, \tilde{y}]\tilde{\in}\check{X} \times \check{Y}$ then we define $\|[\tilde{x}, \tilde{y}]\| = \|\tilde{x}\| + \|\tilde{y}\|$. It can be easily verified that $\|.\|$ satisfy all the conditions (N1) - N(4) of soft norm. Hence $\check{X} \times \check{Y}$ becomes a soft normed linear space.

**Definition 6.4.** Let $\check{X}$ and $\check{Y}$ be soft normed linear spaces and $\tilde{D}$ be a soft subspace of $\check{X}$. Then the soft linear operator $T : SE(\tilde{D}) \to SE(\check{Y})$ is called *closed* if the relations $\tilde{x}_n\tilde{\in}\tilde{D}$, $\tilde{x}_n \to \tilde{x}$, $T(\tilde{x}_n) \to \tilde{y}$ imply that $\tilde{x}\tilde{\in}\tilde{D}$ and $T\left(\tilde{x}\right) = \tilde{y}$.

**Theorem 6.5.** *If $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5) and over a finite set of parameters $A$; $T : SE(\check{X}) \to SE(\check{Y})$ is a soft linear operator satisfying (L3). Then $T$ is a closed if and only if $T_\lambda : X \to Y$ is closed for each $\lambda \in A$.*

*Proof.* Let $T$ be closed. Let $\lambda \in A$ be arbitrary and $x_n \in X$, $x_n \to x$, $T_\lambda(x_n) \to y$. Let us consider a sequence $\tilde{x}_n$ of soft elements of $\check{X}$ such that $\tilde{x}_n(\lambda) = x_n$, $\forall \lambda \in A$. We also consider soft elements $\tilde{x}$, $\tilde{y}$ such that $\tilde{x}(\lambda) = x, \tilde{y}(\lambda) = y$, $\forall \lambda \in A$. Since $x_n \to x$, $T_\lambda(x_n) \to y$ and the parameter set $A$ is finite, by Definition 2.32, it follows that, $\tilde{x}_n \to \tilde{x}$, $T(\tilde{x}_n) \to \tilde{y}$. Again since $T$ is closed, it follows that $\tilde{x} \tilde{\in} \check{X}$ and $T(\tilde{x}) = \tilde{y}$. So $x \in X, T_\lambda(x) = T(\tilde{x})(\lambda) = \tilde{y}(\lambda) = y$ i.e., $T_\lambda(x) = y$. Thus $T_\lambda : X \to Y$ is closed. Since $\lambda \in A$ is arbitrary, it follows that $T_\lambda : X \to Y$ is closed for each $\lambda \in A$.

Conversely, let $T_\lambda : X \to Y$ be closed for each $\lambda \in A$. Let $\tilde{x}_n \tilde{\in} \check{X}$ be such that $\tilde{x}_n \to \tilde{x}$, $T(\tilde{x}_n) \to \tilde{y}$. Then $\tilde{x}_n(\lambda) \in X$, $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$, $(T(\tilde{x}_n))(\lambda) \to \tilde{y}(\lambda), \forall \lambda \in A$ i.e., $\tilde{x}_n(\lambda) \in X$, $\tilde{x}_n(\lambda) \to \tilde{x}(\lambda)$, $T_\lambda(\tilde{x}_n(\lambda)) \to \tilde{y}(\lambda), \forall \lambda \in A$. Since each $T_\lambda$ is closed, $\tilde{x}(\lambda) \in X$ and $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda)) = \tilde{y}(\lambda), \forall \lambda \in A$ i.e., $\tilde{x} \tilde{\in} \check{X}$ and $T(\tilde{x}) = \tilde{y}$. So, $T$ is closed. $\square$

**Theorem 6.6.** *If $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5) and over a finite set of parameters $A$; $T : SE(\check{X}) \to SE(\check{Y})$ is a soft linear operator satisfying (L3). Let $G_T$ be the graph of $T$ as defined in Definition 6.3. Let $SS(G_T)$ denote the soft set generated by $G_T$. Then $SS(G_T)$ is soft closed if and only if $T$ is closed.*

*Proof.* Since $\check{X}$ and $\check{Y}$ are soft normed linear spaces which satisfy (N5), the induced soft metrics satisfy (M5). [by Prop. 2.31]
We have $G_T = \{[\tilde{x}, T(\tilde{x})], \tilde{x} \tilde{\in} \check{X}\}$ i.e., $SS(G_T)(\lambda) = SS\left(\{[\tilde{x}, T(\tilde{x})], \tilde{x} \tilde{\in} \check{X}\}\right)(\lambda), \forall \lambda \in A = \left\{[\tilde{x}(\lambda), T(\tilde{x})(\lambda)], \tilde{x}(\lambda) \in \check{X}(\lambda)\right\}, \forall \lambda \in A = \{[\tilde{x}(\lambda), T_\lambda(\tilde{x}(\lambda))], \tilde{x}(\lambda) \in X\}$, $\forall \lambda \in A = \{[x, T_\lambda(x)], x \in X\}, \forall \lambda \in A = G_{T_\lambda}, \forall \lambda \in A$.
Now $SS(G_T)$ is soft closed $\Leftrightarrow SS(G_T)(\lambda)$ is closed for each $\lambda \in A \Leftrightarrow G_{T_\lambda}$ is closed for each $\lambda \in A \Leftrightarrow T_\lambda : X \to Y$ is closed for each $\lambda \in A \Leftrightarrow T$ is closed. $\square$

**Theorem 6.7.** *(Closed graph theorem) If $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5) and over a finite set of parameters $A$; $T : SE(\check{X}) \to SE(\check{Y})$ is a soft linear operator satisfying (L3). Then $T$ is a continuous soft linear operator if $SS(G_T)$ is soft closed (i.e., $T$ is closed).*

*Proof.* Since $\check{X}$ and $\check{Y}$ are soft Banach spaces which satisfy (N5), $X, Y$ are Banach spaces. By Theorem 2.46, it follows that, for each $\lambda \in A, T_\lambda : X \to Y$ is a linear operator. Now by Theorem 6.6, $SS(G_T)$ is soft closed implies $T$ is closed and by Theorem 6.5, if follows that $T_\lambda : X \to Y$ is closed for each $\lambda \in A$. Then by the closed graph theorem in crisp sense, it follows that for each $\lambda \in A, T_\lambda : X \to Y$ is continuous. Since each $T_\lambda : X \to Y$ is continuous by Theorem 2.57, it follows that, the soft linear operator $T$ is continuous. $\square$

## 7. Conclusions

In this paper we have introduced a concept of soft linear functional on a soft linear space. Four fundamental theorems of functional analysis have been established in soft set settings. Weak convergence and strong convergence of sequence of soft elements are also studied. There is an ample scope for further research on soft normed linear space and soft linear functionals.

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