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Coupled coincidence point results in partially ordered fuzzy metric spaces

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Abstract. Coupled coincidence and fixed point problems have been in the focus of the research interest for last few years. The problem was introduced in fuzzy metric spaces only recently in 2011. In this paper we work out a coupled coincidence point theorem for a compatible pair of mappings in fuzzy metric spaces. The space endowed with a partial ordering. We use a combination of analytic and order theoretic concepts in our theorem. The result is illustrated with an example.

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1. Introduction and Preliminaries

Fuzzy concepts, after its introduction by Zadeh [23] in 1965, have made quick headways in almost all branches of pure and applied mathematics. The flexibility in fuzzy concepts allows the fuzzifications of different mathematical structures in more than one ways. Metric space has also been fuzzified following several approaches, some of these can be found in [6, 11, 12]. We consider here the definition of fuzzy metric space suggested by George and Veeramani [6] which is a modification of the definition given in [12], done for topological reasons. Fuzzy fixed point theory has mostly developed on this fuzzy metric space. This is probably because the space has certain salient features necessary for a successful development of a metric fixed point theory, one of these being that the topology on this space is Hausdorff topology. Some references on fuzzy fixed point problems discussed in this space are noted in [2, 5, 6, 7, 15, 16, 18]. In the following we first describe this space to the extent of our requirement in this paper.

Definition 1.1 ([22]). A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a t-norm if the following properties are satisfied:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Generic examples of $t-$norm are $a *_1 b = \min\{a,b\}$, $a *_2 b = \frac{ab}{max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$, $a *_4 b = max\{a+b-1, 0\}$.

The following is the definition given by George and Veeramani[6].

Definition 1.2 ([6]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:
(i)$M(x,y,t) > 0$,
(ii) $M(x,y,t) = 1$ if and only if $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ and
(v) $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ is continuous.

Let $(X, M, *)$ be a GV-fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x,r,t)$ with center $x \in X$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable [6].
A metric space (X,d) can be considered as a fuzzy metric space $(X, M, *)$ with $a * b = \min\{a,b\}$ and M defined as $M(x,y,t) = \frac{t}{t+d(x,y)}$.
Amongst other inequivalently defined fuzzy metric spaces, we will only consider this space and hence will refer to it simply as a fuzzy metric space.

Example 1.3 ([6]). Let $X = \mathbb{R}$. Let $a * b = a.b$ for all $a, b \in [0,\infty)$. For each $t \in (0,\infty)$, let

$$M(x,y,t) = e^{-\frac{|x-y|}{t}},$$

for all $x, y \in X$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.4 ([6]). Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma, which was originally proved for the fuzzy metric space introduced by Kramosil and Mishilek [12] is also true in the present case.

Lemma 1.5 ([7]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.*

Lemma 1.6 ([19]). *M is a continuous function on $X^2 \times (0, \infty)$.*

The concept of coupled fixed point was introduced by Guo et al [8]. Bhaskar et al [1] proved a coupled contraction mapping theorem in partially ordered metric spaces. Coupled coincidence point results were proved by Lakshmikantham et al [13] for two commuting mappings and by Chaudhury et al [3] for compatible pair of mappings. There are several results in this direction of research in metric spaces. Some of these are noted in [14, 17, 20]. It is our purpose in this paper to prove a coupled coincidence point theorem for two mappings in complete fuzzy metric spaces.
Let $(X, \preceq)$ be a partially ordered set and F be a self map on X. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$[1].

Definition 1.7 ([1]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its first argument and is non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for fixed $y \in X$ and if for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for fixed $x \in X$.

Definition 1.8 ([13]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for all $y \in X$ and if for all $y_1, y_2 \in X$, $gy_1 \preceq gy_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Definition 1.9 ([1]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \;\; F(y, x) = y.$$

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.

Definition 1.10 ([13]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y), \;\; gy = F(y, x).$$

Definition 1.11 ([3]). Let (X, d) be a metric space. The mappings F and g where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

In fuzzy metric spaces coupled fixed point results were first successfully proved by Zhu et al [24]. After that coupled coincidence point and coupled fixed point results in this space have appeared in works of Hu [9], Choudhury et al [4], Jain et al [10]. In particular, compatibility was defined by Hu [9] as the fuzzy counterpart of the concept introduced in Choudhury et al [3].

Definition 1.12 ([4, 9])**.** Let $(X, M, *)$ be a fuzzy metric space. The mappings F and g where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if for all $t > 0$

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

2. Major section

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete fuzzy metric space where $a * b \geq a.b$ for all $a, b \in [0, 1]$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings that F has the mixed g-monotone property and satisfying the following conditions:*
(i)$F(X \times X) \subseteq gX$,
(ii) g is continuous and monotonic increasing,
(iii) (g, F) is a compatible pair,
(iv) $M(F(x, y), F(u, v), t) \geq \gamma(M(gx, gu, t) * M(gy, gv, t))$, (2.1)
for all $x, y, u, v \in X$, $t > 0$ with $gx \preceq gu$ and $gy \succeq gv$, where $\gamma : [0, 1] \to [0, 1]$ is a continuous function such that $\gamma(a) > \sqrt{a}$ for each $0 \leq a \leq 1$. Also suppose that X has the following properties:

(a) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$, (2.2)

(b) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$. (2.3)

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exists $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point in X.

Proof. Let x_0, y_0 be two points in X be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. We define the sequence $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$gx_1 = F(x_0, y_0) \text{ and } gy_1 = F(y_0, x_0)$$
$$gx_2 = F(x_1, y_1) \text{ and } gy_2 = F(y_1, x_1)$$

and, in general, for all $n \geq 0$,

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n). \tag{2.4}$$

This construction is possible by the condition $F(X \times X) \subseteq gX$.
Next, we prove that for all $n \geq 0$,

$$gx_n \preceq gx_{n+1} \tag{2.5}$$

and

$$gy_n \succeq gy_{n+1}. \tag{2.6}$$

Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Therefore (2.5) and (2.6) hold for $n = 0$.

Let (2.5) and (2.6) hold for some $n = m$. As F has the mixed g-monotone property and $gx_m \preceq gx_{m+1}$ and $gy_m \succeq gy_{m+1}$, from (2.4), we get

$$gx_{m+1} = F(x_m, y_m) \preceq F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \preceq F(y_m, x_m) = gy_{m+1}. \tag{2.7}$$

Also, for the same reason, we have

$$gx_{m+2} = F(x_{m+1}, y_{m+1}) \succeq F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \succeq F(y_{m+1}, x_{m+1}) = gy_{m+2}. \tag{2.8}$$

Then from (2.7) and (2.8),

$$gx_{m+1} \preceq gx_{m+2} \text{ and } gy_{m+1} \succeq gy_{m+2}.$$

Then, by induction, it follows that (2.5) and (2.6) hold for all $n \geq 0$.

Let for all $t > 0, n \geq 0$,

$$\delta_n(t) = M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t).$$

By using (2.5) and (2.6), from (2.1) and (2.4) we have for all $t > 0$ and $n \geq 1$,

$$\begin{aligned} M(gx_n, gx_{n+1}, t) &= M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) \\ &\geq \gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)) \\ &= \gamma(\delta_{n-1}(t)). \end{aligned}$$

Therefore for all $t > 0$ and $n \geq 1$

$$M(gx_n, gx_{n+1}, t) \geq \gamma(\delta_{n-1}(t)). \tag{2.9}$$

Similarly, by using (2.5) and (2.6), from (2.1) and (2.2) we have, for all $t > 0$ and $n \geq 1$,

$$\begin{aligned} M(gy_n, gy_{n+1}, t) &= M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t) \\ &\geq \gamma(M(gy_{n-1}, gy_n, t) * M(gx_{n-1}, gx_n, t)) \\ &= \gamma(\delta_{n-1}(t)). \end{aligned}$$

Therefore for all $t > 0$ and $n \geq 1$

$$M(gy_n, gy_{n+1}, t) \geq \gamma(\delta_{n-1}(t)). \tag{2.10}$$

From (2.9) and (2.10) we obtain for all $t > 0$ and $n \geq 1$,

$$\delta_n(t) \geq \gamma(\delta_{n-1}(t)) * \gamma(\delta_{n-1}(t)) \geq (\gamma(\delta_{n-1}(t)))^2 > \delta_{n-1}(t). \tag{2.11}$$

(by the properties of $*$ and γ).

Thus for each $t > 0$, $\{\delta_n(t); n \geq 0\}$ is an increasing sequence in $[0, 1]$ and hence tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$ for all $t > 0$. If there exists $t_0 > 0$ such that $a(t_0) < 1$, then taking limit as $n \longrightarrow \infty$ for $t = t_0$ in the first part of the above inequality, we get $a(t_0) \geq (\gamma(a(t_0)))^2 > a(t_0)$, which is a contradiction. Hence $a(t) = 1$ for every $t > 0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t) = 1. \tag{2.12}$$

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let, to the contrary, at least one of $\{gx_n\}$ and $\{gy_n\}$ be not a Cauchy sequence. Then there exist $\epsilon, \lambda \in (0, 1)$ such that for each integer k, there are two integers $l(k)$ and $m(k)$ such that $m(k) > l(k) \geq k$ and

$$\text{either } M(gx_{l(k)}, gx_{m(k)}, \epsilon) \leq 1 - \lambda, \text{for all k},$$

$$\text{or} \qquad M(gy_{l(k)}, gy_{m(k)}, \epsilon) \leq 1-\lambda, \text{for all k}.$$

In either case we have, for all $k > 0$,

$$r_k(\epsilon) = M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon) \leq 1-\lambda. \tag{2.13}$$

By choosing $m(k)$ to be the smallest integer exceeding $l(k)$ for which (2.13) holds, we have, for all $k > 0$,

$M(gx_{l(k)}, gx_{m(k)-1}, \epsilon) * M(gy_{l(k)}, gy_{m(k)-1}, \epsilon) > 1-\lambda$.

By continuity of M, we can have some α with $0 < 2\alpha < \epsilon$ such that, for all $k > 0$,

$$M(gx_{l(k)}, gx_{m(k)-1}, \epsilon-2\alpha) * M(gy_{l(k)}, gy_{m(k)-1}, \epsilon-2\alpha) > 1-\lambda. \tag{2.14}$$

From (2.13), (2.14) and by the triangle inequality for all $k > 0$, we have

$$\begin{aligned} 1-\lambda &\geq r_k(\epsilon) \geq M(gx_{l(k)}, gx_{m(k)-1}, \epsilon-\alpha) * M(gx_{m(k)-1}, gx_{m(k)}, \alpha) \\ &\quad *M(gy_{l(k)}, gy_{m(k)-1}, \epsilon-\alpha) * M(gy_{m(k)-1}, gy_{m(k)}, \alpha) \\ &= M(gx_{l(k)}, gx_{m(k)-1}, \epsilon-\alpha) * M(gy_{l(k)}, gy_{m(k)-1}, \epsilon-\alpha) * \delta_{m(k)-1}(\alpha) \\ &> (1-\lambda) * \delta_{m(k)-1}(\alpha). \end{aligned}$$

Taking the limit as $k \to \infty$, we get by (2.12),

$$\lim_{k\to\infty} r_k(\epsilon) = 1-\lambda.$$

Since $M(x, y, t_1) \geq M(x, y, t_2)$ whenever $t_1 \geq t_2$, it follows that $M(x, y, \epsilon) \leq 1-\lambda$ implies $M(x, y, \epsilon_1) \leq 1-\lambda$ for all $x, y \in X$ whenever $\epsilon_1 \leq \epsilon$.

Hence the above derivation is valid if ϵ is replaced by any smaller value. Thus we conclude that

$$\lim_{k\to\infty} r_k(\epsilon_1) = 1-\lambda \text{ for all } \epsilon_1 \leq \epsilon. \tag{2.15}$$

Again, by the triangle inequality for all $k > 0$,

$$\begin{aligned} r_k(\epsilon) &= M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon) \\ &\geq M(gx_{l(k)}, gx_{l(k)+1}, \alpha) * M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon-2\alpha) \\ &\quad *M(gx_{m(k)+1}, gx_{m(k)}, \alpha) * M(gy_{l(k)}, gy_{l(k)+1}, \alpha) \\ &\quad *M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon-2\alpha) * M(gy_{m(k)+1}, gy_{m(k)}, \alpha). \end{aligned}$$

Hence, for all $k > 0$, we have

$$\begin{aligned} r_k(\epsilon) \geq{} & \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon-2\alpha) \\ & * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon-2\alpha). \end{aligned} \tag{2.16}$$

From (2.1) and (2.4), we have, for all $k > 0$,

$$\begin{aligned} M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon-2\alpha) &= M(F(x_{l(k)}, y_{l(k)}), F(x_{m(k)}, y_{m(k)}), \epsilon-2\alpha) \\ &\geq \gamma(M(gx_{l(k)}, gx_{m(k)}, \epsilon-2\alpha) \\ &\quad *M(gy_{l(k)}, gy_{m(k)}, \epsilon-2\alpha)) \\ &= \gamma(r_k(\epsilon-2\alpha)). \end{aligned}$$

Therefore for all $k > 0$,

$$M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon-2\alpha) \geq \gamma(r_k(\epsilon-2\alpha)). \tag{2.17}$$

Also from (2.1) and (2.4) we have, for all $k > 0$,

$$\begin{aligned} M(gy_{m(k)+1}, gy_{l(k)+1}, \epsilon - 2\alpha) &= M(F(y_{m(k)}, x_{m(k)}), F(y_{l(k)}, x_{l(k)}), \epsilon - 2\alpha) \\ &\geq \gamma(M(gx_{l(k)}, gx_{m(k)}, \epsilon - 2\alpha) \\ &\quad *M(gy_{l(k)}, gy_{m(k)}, \epsilon - 2\alpha)) \\ &= \gamma(r_k(\epsilon - 2\alpha)). \end{aligned}$$

Therefore for all $k > 0$,

$$M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon - 2\alpha) \geq \gamma(r_k(\epsilon - 2\alpha)). \tag{2.18}$$

Inserting (2.17) and (2.18) in (2.16) we obtain, for all $k > 0$,

$$\begin{aligned} r_k(\epsilon) &\geq \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * \gamma(r_k(\epsilon - 2\alpha)) * \gamma(r_k(\epsilon - 2\alpha)) \\ &\geq \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * (\gamma(r_k(\epsilon - 2\alpha)))^2 \text{ (since } a * b \geq a.b). \end{aligned}$$

Letting $k \to \infty$ and using (2.12) and (2.15) we get, by continuity of $*$ and the properties of γ,

$$1 - \lambda \geq (\gamma(1 - \lambda))^2 > (1 - \lambda), \tag{2.19}$$

which is a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} gy_n = y. \tag{2.20}$$

Therefore, $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = x$, $\lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = y$.
Since, (g, F) is a compatible pair and using continuity of g, we have
$\lim_{n\to\infty} g(gx_{n+1}) = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n)$,
$\lim_{n\to\infty} g(gy_{n+1}) = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n)$.
By (2.5), (2.6) and (2.20), we have $\{gx_n\}$ is a non-decreasing sequence with $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence with $gy_n \to y$ as $n \to \infty$. Then by (2.2) and (2.3) we have for all $n \geq 0$,

$$gx_n \preceq x \text{ and } gy_n \succeq y.$$

Since, g is monotonic increasing, so

$$g(gx_n) \preceq gx \text{ and } g(gy_n) \succeq gy. \tag{2.21}$$

Now we show that $gx = F(x, y)$ and $gy = F(y, x)$ for all $x, y \in X$.
By using (2.1),(2.4) and (2.21), for all $t > 0$, $n \geq 0$, we have

$$M(F(x, y), g(gx_{n+1}), t) = M(F(x, y), g(F(x_n, y_n)), t).$$

Taking $n \to \infty$ on the both sides of the above inequality, we obtain

$$\begin{aligned} \lim_{n\to\infty} M(F(x, y), g(gx_{n+1}), t) &= \lim_{n\to\infty} M(F(x, y), g(F(x_n, y_n), t), \\ M(F(x, y), gx, t) &= \lim_{n\to\infty} M(F(x, y), (F(gx_n, gy_n)), t), \\ &= \lim_{n\to\infty} (F(gx_n, gy_n)), F(x, y), t), \\ &\geq \lim_{n\to\infty} [\gamma(M(g(gx_n), gx, t) * M(g(gy_n), gy, t))], \\ &= \gamma(M(gx, gx, t) * M(gy, gy, t)), \\ &= \gamma(1 * 1), \\ &= 1, \end{aligned}$$

which implies that $gx = F(x, y)$. (2.22)
Similarly, we can prove $gy = F(y, x)$. (2.23)

Therefore, from (2.22) and (2.23) we conclude that (x, y) is a coupled coincidence point of F and g. Hence proof is completed. □

Example 2.2. Let $X = [0, 1]$. Then $(X, \preceq)$ is a partially ordered set with the natural ordering of the real numbers. Let

$$M(x, y, t) = e^{-\frac{|x-y|}{t}} \quad \text{for all } x, y \in X.$$

Then $(X, M, *)$ be a complete fuzzy metric space, where $a * b = a.b$ for all $a, b \in X$. Let the mapping $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Let the mapping $F : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2-y^2}{3}, & \text{if } x, y \in [0, 1], x \succeq y, \\ 0, & \text{otherwise .} \end{cases}$$

Here F satisfies the mixed g-monotone property. Let $\gamma : [0, 1] \to [0, 1]$ be defined as $\gamma(a) = a^{\frac{1}{3}}$ for each $a \in (0, 1)$.
Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that
$\lim_{n\to\infty} F(x_n, y_n) = a, \quad \lim_{n\to\infty} gx_n = a,$
$\lim_{n\to\infty} F(y_n, x_n) = b$ and $\lim_{n\to\infty} gy_n = b$. Then $a = 0$ and $b = 0$.
Now for all $n \geq 0$,

$$gx_n = x_n^2, \; gy_n = y_n^2,$$

$$F(x_n, y_n) = \begin{cases} \frac{x_n^2-y_n^2}{3}, & \text{if } x_n \succeq y_n, \\ 0, & \text{otherwise .} \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \frac{y_n^2-x_n^2}{3}, & \text{if } y_n \succeq x_n, \\ 0, & \text{otherwise ,} \end{cases}$$

Then it follows that

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(gx_n, gy_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(gy_n, gx_n), t) = 1.$$

Therefore, the mappings F and g are compatible in X. Also, let $x_0 = 0$ and $y_0 = c(> 0)$ are two points in X such that

$gx_0 = g0 = 0 \leq F(0, c) = F(x_0, y_0)$ and $gy_0 = gc = c^2 \geq \frac{c^2}{3} = F(c, 0) = F(y_0, x_0)$.
Now we consider the following cases:

Case-I. For $x \succeq y$ and $u \succeq v$.

$$\begin{aligned} M(F(x, y), F(u, v), t) &= e^{-[\frac{|\frac{(x^2-y^2)}{3} - \frac{(u^2-v^2)}{3}|}{t}]} \\ &= e^{-[\frac{|\frac{(x^2-u^2)}{3} - \frac{(y^2-v^2)}{3}|}{t}]} \\ &\geq e^{-[|\frac{(x^2-u^2)}{3t}|]}.e^{-[|\frac{(y^2-v^2)}{3t}|]} \\ &= [M(gx, gu, t)]^{\frac{1}{3}}.[M(gy, gv, t)]^{\frac{1}{3}} \\ &= \gamma(M(gx, gu, t) * M(gy, gv, t)). \end{aligned}$$

Case-II. For $x \succeq y$ and $u \prec v$.

$$\begin{aligned} M(F(x,y),F(u,v),t) &= e^{-[\frac{|x^2-y^2|}{3t}]} \\ &= e^{-[\frac{|(u^2+x^2-y^2-u^2|}{3t}]} \\ &= e^{-[|\frac{(x^2-u^2)}{3t}|]}.e^{-[|\frac{(u^2-y^2)}{3t}|]} \\ &\geq e^{-[|\frac{(x^2-u^2)}{3t}|]}.e^{-[|\frac{(v^2-y^2)}{3t}|]} \\ &= [M(gx,gu,t)]^{\frac{1}{3}}.[M(gy,gv,t)]^{\frac{1}{3}} \\ &= \gamma(M(gx,gu,t)*M(gy,gv,t)). \end{aligned}$$

Case-III. For $x \prec y$ and $u \succeq v$.

$$\begin{aligned} M(F(x,y),F(u,v),t) &= e^{-[\frac{|u^2-v^2|}{3t}]} \\ &= e^{-[\frac{|(u^2+x^2-v^2-x^2|}{3t}]} \\ &= e^{-[|\frac{(x^2-v^2)}{3t}|]}.e^{-[|\frac{(u^2-x^2)}{3t}|]} \\ &= e^{-[|\frac{(x^2-u^2)}{3t}|]}.e^{-[|\frac{(x^2-v^2)}{3t}|]} \\ &\geq e^{-[|\frac{(x^2-u^2)}{3t}|]}.e^{-[|\frac{(y^2-v^2)}{3t}|]} \\ &= [M(gx,gu,t)]^{\frac{1}{3}}.[M(gy,gv,t)]^{\frac{1}{3}} \\ &= \gamma(M(gx,gu,t)*M(gy,gv,t)). \end{aligned}$$

Case-IV. For $x \prec y$ and $u \prec v$.
This case is obviously satisfied. Here all conditions of the theorem 2.1 are satisfied and $(0,0)$ is the coupled coincidence point of F and g in X.

Remark 2.3. Since commuting mappings are compatible, our present theorem generalizes the result in [21]. In the example 2.2, (g, F) is not a commuting pair. So the result of theorem 2.1 is an actual improvement over the result [21].

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