Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 4, (April 2014), pp. 579–606
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy flip-flop as fuzzy systems

Himadrishekhar Gupta, Swapan Raha

Received 28 March 2013; Accepted 3 July 2013

Abstract. In this paper, an attempt is made to study the problem of constructing a mathematical model with the parametric operations of fuzzy systems which can be used for hardware implementation of a fuzzy flip-flop. The generation of parametric classes of fuzzy systems by means of basic fuzzy operations are considered. The algebraic representation of fuzzy flip-flop circuit is taken into consideration. The algebraic fuzzy flip-flop is one example of the general fuzzy flip-flop concept which has been defined as an extension of the binary J-K flip-flop. Some properties of fuzzy flip-flops are discussed with the help of analytical methods. The fuzzy flip-flop is defined using complement, min and max operations for fuzzy negation, t-norms and s-norms respectively. A rule-base for a fuzzy flip-flop is developed and approximate reasoning methodology is used to generate the next state of the flip-flop when the present state and the present input are known. Results have been extensively discussed with examples.

2010 AMS Classification: 03E72, 08A72

Keywords: Fuzzy flip-flop, Fuzzy automata, Fuzzy system

Corresponding Author: Swapan Raha (swapan.raha@visva-bharati.ac.in)

1. Introduction

The development of basic building blocks for a fuzzy system is intrinsically important, however, the real advantage comes when such techniques are used in application systems. Dynamical systems used in engineering applications have both inputs and outputs. The relationship between the two is a critical feature of the system's behaviour — and thus an important part of its model. A crisp flip-flop is a basic building block of a sequential logic circuit which realises the input - output behaviour of such a system. The function of a flip-flop could be realised through a combination of logic gates and latching action. It is mainly used as a single bit memory element. A series of flip-flops where, the logic operations are performed sequentially is called a sequential circuit. Sequential circuits which are controlled by a repetitive clock signal are called synchronous circuits. Asynchronous circuits are controlled by some random events. A crisp flip-flop has two stable states. It has a number of inputs which can be either 0 or 1. The output of a crisp flip-flop is generated by a set of rules, represented by means of either an external state diagram or a state table.

A fuzzy flip-flop is a basic building block of a fuzzy sequential circuit. A fuzzy flip-flop contains fuzzy inputs and fuzzy outputs. The function of a fuzzy flip-flop could be realised through a combination of fuzzy logic gates and latching action. It can be used as a memory element in a fuzzy logic based function realisation. Unlike a crisp flip-flop a fuzzy flip-flop can have an infinite number of states. A fuzzy sequential machine could be realised through a number of fuzzy flip-flops and circuits realising basic fuzzy operators. Accordingly, a fuzzy flip-flop can have a number of inputs and only one output. The output of a fuzzy flip-flop actually determines the next state of the flip-flop. The next state of a fuzzy flip-flop corresponding to different inputs and present state can be described by a series of fuzzy rules. Nowadays, the high speed fuzzy controller hardware system uses this form of knowledge representation [23]. Some work has been done on the field of intelligent non-industrial robots as an interface for accessing information [20]. The fuzzy flip-flop can also be successfully used in such applications in robotics.

The fuzzy model of flip-flops originates from fuzzy set theory, having the character of being intuitive. The fuzziness of the input and output results from the natural description of the behaviour of the system. It consists of meanings and definitions of terms like 'high', 'low', 'medium', and linguistic modifiers like 'very', 'not very', 'more or less' etc. These are broad concepts that demand subjective meaning. With respect to the definition of the fuzzy sets for each such concepts discussed here, different structures may be imposed on the membership space and assumptions about the membership functions can be made [1]. The fuzzy outputs are generated by means of an inference engine using a fuzzy rule base, the present state and current input. The fuzzy rule base actually represents the human knowledge and the fuzzy inference procedure reproduces the human decision and behaviour [25]. In some decision making problems, conflicting criteria is a major concern. The Pareto sets deal with this problem efficiently [3]. An analytical approach has been studied in this paper. The analytical approach to fuzzy normed linear spaces is studied by N. Thillaigovindan and S. Anita Shanthi [21]. The objective of this paper is to study the behaviour of fuzzy flip-flops under different input and state conditions. Fuzzy flip-flop has been studied from different angles such as Bacterial Memetic Algorithm [2], Non-Associative Fuzzy Flip-Flop with Dual set-reset feature [14].

As the next state function could be different for different implementation under fuzzy logic, many different functions could serve the purpose of realisation. Accordingly, we have examined both the *set* and *reset* type fuzzy J-K flip-flops since these two characteristics are not equivalent in a fuzzy sense. These characteristics are obtained from the formal definition of the crisp J-K flip-flop and are modified with the help of fuzzy set theory. The fuzzy flip-flop has become an important part of the fuzzy sequential system, because the computers capable of performing fuzzy computations is a reality nowadays. Some significant work has already been done in this field.

The paper is organized into seven sections. After a brief Introduction on the research reported in this paper in Section 1, we introduce a few definitions of some basic concepts and their properties as required to make the paper readable in Section 2. The concept of a fuzzy flip-flop is introduced in Section 3. Three different types of fuzzy flip-flops — set, reset and their combination are presented in a subsection of Section 3. Section 4 is devoted to an analytical study of fuzzy flip-flop. Representation of fuzzy flip-flop with a set of fuzzy rules is considered in section 5. With this development, an attempt is made to study Fuzzy flip-flop as a fuzzy system in section 6. The paper is briefly concluded in section 7. A comprehensive list of references is provided at last.

2. Preliminaries

Here is a recapitulation about the crisp flip-flop. It is a single bit memory element. It can be classified in four types according to it's input-output behaviour. The types are T-flip-flop, D flip-flop, S-R flip-flop and J-K flip-flop. Among these types T flip-flop and D flip-flop are single input, whereas S-R and J-K are double input flip-flops. Each flip-flop has a pair of complementary outputs. The S-R flip-flop has a forbidden combination when both the inputs are ON. This difficulty is removed in J-K flip-flop, so the J-K flip-flop demands more attention of all the flip-flops due to it's general and well defined nature.

The characteristic equation for the J-K flip-flop in the minimal disjunctive form is $Q(t+\delta t) = J\overline{Q(t)} + Q(t)\overline{K}$ and in the minimal conjunctive form is $Q(t+\delta t) = (J+Q(t))(\overline{K}+\overline{Q(t)})$, where $Q(t)$ and $Q(t+\delta t)$ are states at time t and $t+\delta t$. J,K are inputs. The state table of the J-K flip-flop is as in Table 2. Another way of

J	K	$Q(t)$	$Q(t+\delta t)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Table 1. State table for the J-K flipflop

representing Table 2 is as given in Table 2.

Postulates of the J-K flip-flop: From the Table 2 it is clear that there exists four postulates of the crisp J-K flip. The postulates, according to the Table 2 can be framed as follows:

Q	F	J	K
0	0	0	Φ
0	1	1	Φ
1	0	Φ	1
1	1	Φ	0

TABLE 2. Steering table for the J-K flip flop

$$\begin{aligned} P1: \ F(0,0,Q) &= Q \\ P2: \ F(0,1,Q) &= 0 \\ P3: \ F(1,0,Q) &= 1 \\ P4: \ F(1,1,Q) &= 1-Q \end{aligned}$$

3. Fuzzy flip-flop

The fuzzy version of the flip-flops are studied by many scientists from different view points and approaches. Among them, Koczy, Hirota and Ozawa established a general algebraic model of a J-K flip-flop [6, 7, 12, 13]. This approach explored the possibilities based on algebraic operations for modelling fuzzy memory and learning systems. The work of Hirota and Pedrycz [8] was on the development of designing a framework for dealing with fuzzy computation using fuzzy J-K flip-flops. They have indicated some links between the fuzzy logic and many valued logics. Different methods of designing the combination part of the sequential circuit was introduced by them. Lovassy, Koczy and Gal proposed an evolutionary approach for optimizing fuzzy flip-flop networks. They also proposed the fuzzy flip-flop neural network architecture that can be used for learning and approximating various simple transcendental functions. Lovassy and Koczy explored for the uniqueness of the definition of the non-associative fuzzy flip-flops. L.Gneiwek and J. Kluska studied the family of fuzzy J-K flip-flops based on bouunded product [4]. Ozawa, Hirota and Omori studied the circuit of Algebraic Fuzzy flip-flop [16], Yamakawa studied a simple fuzzy computer system applying max and min operations [22], B. Kosko explored the Fuzzy entropy and conditioning [11], in an earlier work we studied the Fuzzy mathematical machine as a fuzzy system [17]. The concept of fuzzy finite state machine is also studied by Pamy Sebastian and T.P Johnson [19]. The fuzzy flip-flops can be represented by means of fuzzy functions and the fuzzy functions can also be minimised [10] resulting the study of the fuzzy flip-flops easier. These and some other researches in this field have made a robust base for futher research.

Definition 3.1. A fuzzy flip-flop is an electronic device whose states and inputs are represented by fuzzy sets defined over respective universe of discourses. It has a single output. The output of a fuzzy flip-flop is generated by means of a Fuzzy Rule Base(FRB) and an inference mechanism.

The defuzzification of all the fuzzy states lead to the properties of it's crisp counterparts. Different rules generate different fuzzy flip-flops.

3.1. **Fuzzy J-K flip-flop.** The fuzzy version of a J-K flip-flop is studied here for it's clarity in behaviour. The J-K flip-flop can also be converted to D-type or T-type of flip-flops by suitable arrangement of logic gates. As we have seen, from the logic-table of a J-K flip-flop it's behaviour can be formulated as in the following [5, 15, 18].

$$Q(t+\delta t) = J\overline{Q}(t) + Q(t)\overline{K}. \tag{3.1}$$

$\overline{Q}$ represents the negation of the internal state Q, whereas J, K are inputs. The mathematical model in fuzzy version can be represented as

$$Q(t+\delta t) = \{J \wedge (1 - Q(t))\} \vee \{(1-K) \wedge Q(t)\}. \tag{3.2}$$

We can write 3.2 in a more general way using t-norm T and s-norm T' as in the following:

$$Q(t+\delta t) = (J\ T\ C(Q(t)))\ T'\ (C(K)\ T\ Q(t)). \tag{3.3}$$

One of the most common $t-norm$ and $t-conorm$ frequently used as fuzzy intersection and fuzzy union are min and max operation. The equation 3.3 is modified with these operations as in the following:

$$Q(t+\delta t) = max(min(J, 1-Q), min(1-K, Q) \tag{3.4}$$

where $J, K, Q \in [0,1]$. Although here max and min are used for it's simplicity and clarity but, different other norms could be used instead of them [9, 24]. For the sake of convenience, $Q(t+\delta t)$ will be referred to as F hereinafter.

Let us construct a fuzzified version of the table 2 which will be suitable to study the properties of a fuzzy J-K flip-flop. Since Φ implies 0 or 1, we replace it with the entire interval [0,1]. Instead of 0 we use the linguistic variable 'very small' and instead of 1 we use the linguistic variable 'very large', and we represent these with ϵ and μ respectively. We can now construct the following table using those fuzzy identifiers.

Q	F	J	K
ϵ	ϵ	ϵ	$[0,1]$
ϵ	μ	μ	$[0,1]$
μ	ϵ	$[0,1]$	μ
μ	μ	$[0,1]$	ϵ

Table 3. State table for the fuzzy JK flip flop

We shall be able to construct the rule bases from the Table 3.

3.2. **Generalised fuzzy J-K flip-flop.** In this section, we consider two different types of fuzzy J-K flip-flops — the set type and the reset type behavioural models.

3.2.1. *Reset type fuzzy J-K flip-flop.* **Postulates of the reset type fuzzy J-K flip-flop :**

Let $F^R = \max(\min(J, 1-Q), \min(1-K, Q))$

$$\begin{aligned}
a.\ F^R(0,0,Q) &= max(min(0,1-Q), min(1,Q)\\
&= max(0,Q)\\
&= Q\\
b.\ F^R(0,1,Q) &= max(min(0,1-Q), min(1-1,Q))\\
&= max(min(0,1-Q), min(0,Q))\\
&= max(0,0)\\
&= 0\\
c.\ F^R(1,0,Q) &= max(min(1,1-Q), min(1-0,Q))\\
&= max(min(1,1-Q), min(1,Q))\\
&= max(1-Q,Q)\\
d.\ F^R(1,1,Q) &= max(min(1,1-Q), min(1-1,Q))\\
&= max(1-Q,0)\\
&= 1-Q.
\end{aligned}$$

In case of a fuzzy flip-flop, there may be infinite number of fuzzy inputs and internal states, each combination producing a typical fuzzy output. It is obvious that the most ambiguous inputs will produce most ambiguous output in the fuzzy sense. From this viewpoint we introduce another postulate, which will be helpful to study the properties of a fuzzy flip-flop. This is the fifth postulate, and it is

$e.\ F^R(0.5, 0.5, Q) \;=\; 0.5$ The above postulates are for a reset type fuzzy flip-flop. It is of reset type as we have for all state Q, $F(0,1,Q) = 0$.

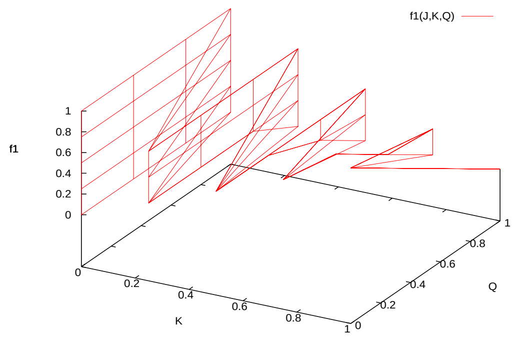

FIGURE 1. Reset type flip-flop

3.2.2. *Set type fuzzy J-K flip-flop.* The equivalent Boolean expression of 3.1 is

$$F = (J+Q)(\overline{K}+\overline{Q}) \tag{3.5}$$

These two equations are different in fuzzy logic. The fuzzy version of 3.5 with min and max compositions will be as follows:

$$F^S = min(max(J,Q), max(1-K, 1-Q)) \tag{3.6}$$

It can be expressed in a more general form as

$$F^S = (JT'Q)T((1-K)T'(1-Q)) \tag{3.7}$$

We can frame postulates for a 'set' type flip-flop from 3.6:

Postulates of the set type fuzzy J-K flip-flop :

$$\begin{aligned} a.\ F^{(S)}(0,0,Q) &= min(max(0,Q), max(1-0, 1-Q) \\ &= min(Q,1) \\ &= Q \\ b.\ F^{(S)}(0,1,Q) &= min(max(0,Q), max(1-1, 1-Q)) \\ &= max(min(0,1-Q), min(0,Q)) \\ &= min(Q, 1-Q) \\ c.\ F^{(S)}(1,0,Q) &= min(max(1,Q), max(1-0, 1-Q)) \\ &= min(1,1) \\ &= 1 \\ d.\ F^{(S)}(1,1,Q) &= min(max(1,Q), max(0, 1-Q)) \\ &= min(1, 1-Q) \\ &= 1-Q \\ e.\ F^{(S)}(0.5,0.5,Q) &= 0.5 \end{aligned}$$

It is called 'set' type because here $F^{(S)}(1,0,Q) = 1$.

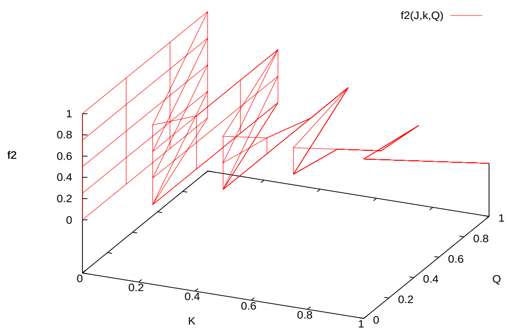

FIGURE 2. Set type flip-flop

3.2.3. *Combined fuzzy J-K flip-flop.* A combination of set and reset type flip-flops gives us a new fuzzy flip-flop. This combined fuzzy flip-flop is defined as in the following:

Definition 3.2. The combined flip-flop $F^{(C)}$ is defined as

$$F^{(C)}(J,K,Q) = \overline{J}\ \overline{K}Q + \overline{J}K\min(Q,1-Q) + JK(1-Q) + J\overline{K}\max(Q,1-Q) \tag{3.8}$$

In fuzzy logic, the combined J-K flip-flop can be wrtitten, using max and min to interpret disjunction and conjunction operations respectively and with usual '$1-$' for an interpretation of complement operation, as

$$F^{(C)}(J,K,Q) = \max(\min(1-J,1-K,Q),\min(1-J,K,\min(Q,1-Q)), \min(J,K,1-Q),\min(J,1-K,\max(Q,1-Q)).$$

In a more general form, the same can be understood as

$$F^{(C)}(J,K,Q) = ((1-J)T(1-K)TQ))T'((1-J)TKT (QT(1-Q)))T'(JTKT(1-Q)T'(JT(1-K)T(QT'(1-Q)))$$

Postulates of the combined fuzzy J-K flip-flop:

a. $F^{(C)}(0,0,Q) = Q$
b. $F^{(C)}(0,1,Q) = min(Q,1-Q)$
c. $F^{(C)}(1,0,Q) = max(Q,1-Q)$
d. $F^{(C)}(1,1,Q) = 1-Q$
e. $F^{(C)}(0.5,0.5,Q) = 0.5$

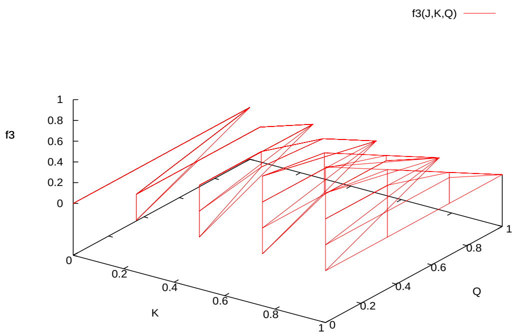

FIGURE 3. Combined type flip-flop

In case of a crisp flip-flop, a D-type flip-flop is represented by $F(D,1-D,Q)$, where $F=0$ if $D=0$ and $F=1$ if $D=1$. But for a fuzzy flip-flop, it's behaviour is different as it is evident from the following observations.

Observation 3.3. $F^{(R)}(D,1-D,Q)$ and $F^{(S)}(D,1-D,Q)$ do not always represent the property of a D-flip-flop.

Proof. We have, from definition

$$\begin{aligned} F^{(R)}(D,1-D,Q) &= max(min(D,1-Q),min(1-(1-D),Q)) \\ &= max(min(D,1-Q),min(D,Q)) \end{aligned}$$

Now, for $D=1$ we have,

$$\begin{aligned} F^{(R)}(1,0,Q) &= max(min(1,1-Q),min(1,Q)) \\ &= max(1-Q,Q) \end{aligned}$$

and for $D=0$ we have,

$$\begin{aligned} F^{(R)}(0,1,Q) &= max(min(0,1-Q),min(0,Q)) \\ &= max(0,0) \\ &= 0. \end{aligned}$$

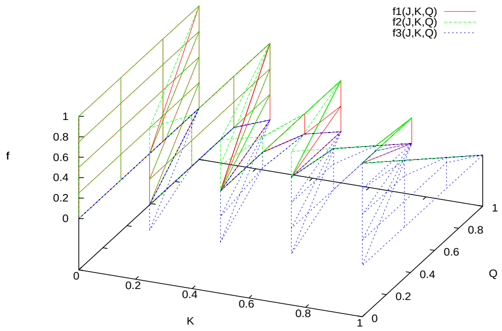

FIGURE 4. Comparison of Set and Reset type flip-flop

Also, $F^{(S)}(D, 1-D, Q) = min(max(D, Q), max(D, 1-Q))$
which for $D = 1$ we have,

$$\begin{aligned} F^{(S)}(1, 0, Q) &= min(max(1, Q), max(1, 1-Q)) \\ &= min(1, 1) \\ &= 1 \end{aligned}$$

and the same for $D = 0$ we have,

$$\begin{aligned} F^{(S)}(0, 1, Q) &= min(max(0, Q), max(0, 1-Q)) \\ &= min(Q, 1-Q) \end{aligned}$$

Thus, it does not represent the property of a D flip-flop for the boundary values of the inputs in the interval [0,1]. Let us now study it for the fuzzy values of the inputs. Let $D = p$ where $p \in (0, 1)$
Then $F^{(R)}(p, 1-p, Q) = max(min(p, 1-Q), min(p, Q))$
Now, for $F^{(R)}(p, 1-p, Q) = p$,

$$\begin{array}{llll} & min(p, 1-Q) & = & p; \\ \text{(a)} & \Rightarrow & 1-Q & \geq p \\ & \Rightarrow & Q & \leq 1-p \end{array}$$

and
(b) $min(p, Q) = p$
$\Rightarrow Q \geq p$
Therefore $Q \in (p, 1-p)$
Thus for being $F^{(R)}$ to be a D-type flip-flop in the fuzzy sense, Q should lie in the interval $(p, 1-p)$ that is, $Q \in (p, 1-p)$.
It is also to be noted that, $F^R(D, 1-D, Q) = Q$, if $0.5 \leq Q \leq D \leq 1$, and $F^R(D, 1-D, Q) = D$ otherwise.
$F^{(S)}(p, 1-p, Q) = min(max(p, Q), max(p, 1-Q))$
For $F^{(S)}(p, 1-p, Q) = p$,
(a) $max(p, Q) = p$
$\Rightarrow Q \leq p$
and
(b) $max(p, 1-Q) = p$

$\Rightarrow p \geq 1 - Q$
$\Rightarrow Q \geq 1 - p$
Therefore, $Q \in (1 - p, p)$
Thus for being $F^{(S)}$ to be a D-type flip-flop in the fuzzy sense, Q should lie in the interval $(1 - p, p)$ that is, $Q \in (1 - p, p)$. □

Observation 3.4. $F^{(R)}(T, T, Q)$ and $F^{(S)}(T, T, Q)$ represent the property of a T flip-flop.

Proof. $F^{(R)}(T, T, Q) = max(min(T, 1 - Q), min(1 - T, Q))$
For $T = 1$,

$$\begin{aligned} F^{(R)}(1, 1, Q) &= max(min(1, 1 - Q), min(0, Q)) \\ &= max(1 - Q, 0) \\ &= 1 - Q \end{aligned}$$

For T=0,

$$\begin{aligned} F^{(R)}(0, 0, Q) &= max(min(0, 1 - Q), min(1, Q)) \\ &= max(0, Q) \\ &= Q \end{aligned}$$

Also,
$F^{(S)}(T, T, Q) = min(max(T, Q), max(1 - T, 1 - Q)$
For T=1,

$$\begin{aligned} F^{(S)}(1, 1, Q) &= min(max(1, Q), max(0, 1 - Q) \\ &= min(1, 1 - Q) \\ &= 1 - Q \end{aligned}$$

For T=0,

$$\begin{aligned} F^{(S)}(0, 0, Q) &= min(max(0, Q), max(1, 1 - Q) \\ &= min(Q, 1) \\ &= Q \end{aligned}$$

Thus, it extends the idea of a T flip-flop for the discrete values of the inputs.
For the fuzzy value $T = q$ where $q \in (0, 1)$, we have, $F^{(R)}(q, q, Q) = max(min(q, 1 - q), min(1 - q, Q))$
$F^{(R)}$ toggles if $F^{(R)}(q, q, Q) = 1 - Q$.
that is if

$$max(min(q, 1 - Q), min(1 - q, Q)) = 1 - Q \tag{3.9}$$

3.9 is satisfied if $min(q, 1 - Q) = 1 - Q$ and $min(1 - q, Q) = Q$.
Now, $min(q, 1 - Q)$ implies that,

$$Q \geq 1 - q \tag{3.10}$$

and, $min(1 - q, Q) = Q$ implies that

$$Q \leq 1 - q \tag{3.11}$$

Combining 3.10 and 3.11 we have,
$Q = 1 - q$.
Also $max(1 - Q, Q) = 1 - Q \Rightarrow 1 - Q \geq Q$
or $Q \leq 0.5$, that is $q \geq 0.5$.
Thus $F^{(R)}$ toggles when $q \geq 0.5$.

Similarly, it can be shown that $F^{(R)} = Q$ if $q \leq 0.5$
Again, $F^{(S)}(J,K,Q) = min(max(J,Q), max(1-K, 1-Q)$
$F^{(S)}$ toggles if $F^{(S)}(q,q,Q) = 1-Q$
that is if,

$$min(max(q,Q), max(1-q, 1-Q)) = 1-Q \tag{3.12}$$

3.12 is satisfied if $max(q,Q) = Q$ and $max(1-q, 1-Q) = 1-Q$.
Now, $max(q,Q) = Q$ implies that,

$$Q \geq q \tag{3.13}$$

and, $max(1-q, 1-Q) = 1-Q$ implies that

$$Q \leq q \tag{3.14}$$

Combining 3.13 and 3.14 we have, $Q = q$
Also $min(Q, 1-Q) = 1-Q \Rightarrow 1-Q \leq Q$
that is $Q \geq 0.5$
that is $q \geq 0.5$
Therefore, $F^{(S)}$ toggles when $q \geq 0.5$. Similarly, it can be shown that $F^{(S)}$ retains it's state when $q \leq 0.5$. □

4. Analytical approach to the fuzzy flip-flop

Let $G^{(R)}(J) = F(J,K,Q)$ and $G^{(S)}(J) = F(J,K,Q)$ where K, Q remains constants with respect to J and let $H^{(R)}(K) = F(J,K,Q)$ and $H^{(S)}(K) = F(J,K,Q)$ where J, Q are constants with respect to K .

Proposition 4.1. *If $J_1 > J_2$ then $G^{(R)}(J_1) > G^{(R)}(J_2)$ and $G^{(S)}(J_1) > G^{(S)}(J_2)$ if Q and K are constants with respect to J and $Q \in (0,1)$.*

Proof. Since $J_1 > J_2$,
therefore ,$max(min(J_1, 1-Q), min(1-K, Q)) > max(min(J_2, 1-Q), min(1-K, Q))$, $1-Q$ being greater than 0
That is $G^{(R)}(J_1) > G^{(R)}(J_2)$
Proof is similar for $G^{(S)}$

□

Proposition 4.2. *If $K_1 > K_2$ then $H^{(R)}(K_1) < H^{(R)}(K_2)$, and $H^{(S)}(K_1) < H^{(S)}(K_2)$ if Q and J are constants with respect to K and $Q \in (0,1)$.*

Both the functions $G^{(R)}$ and $G^{(S)}$ will be referred together as G and $H^{(R)}(K)$, $H^{(S)}(K)$ will be referred together as H .
Since max and min are continuous functions, $G(J)$ and $H(K)$ are continuous functions. Therefore the sequence $\{G(J_i)\}$ is monotone increasing and bounded above and $\{H(K_i)\}$ is monotone decreasing and bounded below. So, we have the following properties,

Proposition 4.3. *Since* $\{G(J_i)\}$ *is monotone increasing and bounded above, it converges to it's exact upper bound.*

Proposition 4.4. *Since* $\{G(K_i)\}$ *is monotone decreasing and bounded below, it converges to it's exact lower bound.*

Proposition 4.5. *The upper bound of* $G^{(R)}(J)$ *is* $max(1-Q, min(1-K, Q))$, *where* $J, K, Q \in [0,1]$.

Proof. $G^{(R)}(J) = max(min(J, 1-Q), min(1-K, Q))$, Q, K are constants w.r.t J.
Therefore,

$$\begin{aligned} G^{(R)}{}_{max}(J) &= max(min(1, 1-Q), min(1-K, Q) \\ &= max(1-Q, min(1-K, Q)) \end{aligned}$$

Proposition 4.6. *The upper bound of* $G^{(S)}(J)$ *is* Q, *where* $J, K, Q \in [0,1]$.

Proposition 4.7. *The upper bound of* $H^{(R)}(K)$ *is* $max(min(J, 1-Q), Q)$, *where* $J, K, Q \in [0,1]$.

Proof. $H^{(R)}(K) = max(min(J, 1-Q), min(1-K, Q))$

Therefore,

$$\begin{aligned} H^{(R)}{}_{max}(K) &= max(min(J, 1-Q), min(1, Q)) \\ &= max(min(J, 1-Q), Q) \end{aligned}$$ □

Proposition 4.8. *The lower bound of* $H^{(S)}(K)$ *is* $max(J, Q)$, *where* $J, K, Q \in [0,1]$.

Theorem 4.9. $Lim_{J \to a}\ G(J) = l$, *where* $a, l \in (0,1)$

Proof. Two variables J_1 and J_2 are chosen in such a way , such that,$\mid G(J_1) - G(J_2) \mid < \epsilon$, where ϵ is a pre-assigned positive number. Then there corresponds a $\delta(>0)$ such that
$0 < \mid J_1 - a \mid < \delta$ and $0 < \mid J_2 - a \mid < \delta$.
Therefore, by Cauchy's general principle of convergence, $Lim_{G \to a}\ G(J) = l$. □

4.1. **Comparative behaviour of $F^{(R)}$,$F^{(S)}$ and $F^{(C)}$.** Here we shall study the comparative behaviour of $F^{(R)}$,$F^{(S)}$ and $F^{(C)}$ with different fuzzy values of the states.
Case I : Let $Q = 0.5$.
Then,

$$\begin{aligned} F^{(R)}(J=0, K=1, Q=0.5) &= max(min(0, 1-0.5), min(0, 0.5)) &&= 0 \\ F^{(R)}(J=1, K=0, Q=0.5) &= max(min(1, 1-0.5), min(0, 0.5)) &&= 0.5 \\ F^{(R)}(J=0, K=0, Q=0.5) &= max(min(0, 1-0.5), min(1, 0.5)) &&= 0.5 \\ F^{(R)}(J=1, K=1, Q=0.5) &= max(min(1, 1-0.5), min(0, 0.5)) &&= 0.5 \end{aligned}$$

$$\text{So, } 0.0 \le F^{(R)} \le 0.5$$

$$\begin{array}{rcl c l}
F^{(S)}(J=0,K=1,Q=0.5) & = & min(max(0,0.5),max(1-1,1-0.5)) & = & 0.5 \\
F^{(S)}(J=1,K=0,Q=0.5) & = & min(max(1,0.5),max(1-0,1-0.5)) & = & 1.0 \\
F^{(S)}(J=0,K=0,Q=0.5) & = & min(max(0,0.5),max(1-0,1-0.5)) & = & 0.5 \\
F^{(S)}(J=1,K=1,Q=0.5) & = & min(max(1,0.5),max(1-1,1-0.5)) & = & 0.5
\end{array}$$

$$\text{So, } 0.5 \le F^{(S)} \le 1.0$$

Also,

$$\begin{array}{rcl}
F^{(C)}(J=0,K=1,Q=0.5) & = & 0.5 \\
F^{(C)}(J=1,K=0,Q=0.5) & = & 0.5 \\
F^{(C)}(J=0,K=0,Q=0.5) & = & 0.5 \\
F^{(C)}(J=1,K=1,Q=0.5) & = & 0.5
\end{array}$$

$$\text{So, } F^{(C)} = 0.5$$

Therefore we have the following inequalities,

$$0.0 \le F^{(R)} \le 0.5 = F^{(C)} \le F^{(S)} \le 1.0$$

Case II : Let $Q \le 0.5$
Then,

$$\begin{array}{rcl c l}
F^{(R)}(J=0,K=1,Q) & = & max(min(0,1-Q),min(1-1,Q)) & = & 0 \\
F^{(R)}(J=1,K=0,Q) & = & max(min(1,1-Q),min(1-0,Q)) & = & 1-Q \\
F^{(R)}(J=0,K=0,Q) & = & max(min(0,1-Q),min(1-0,Q)) & = & Q \\
F^{(R)}(J=1,K=1,Q) & = & max(min(1,1-Q),min(1-1,Q)) & = & 1-Q
\end{array}$$

$$\text{So, } 0.0 \le F^{(R)} \le 1-Q$$

$$\begin{array}{rcl c l}
F^{(S)}(J=0,K=1,Q) & = & min(max(0,Q),max(1-1,1-Q)) & = & Q \\
F^{(S)}(J=1,K=0,Q) & = & min(max(1,Q),max(1-0,1-Q)) & = & 1 \\
F^{(S)}(J=0,K=0,Q) & = & min(max(0,Q),max(1-0,1-Q)) & = & Q \\
F^{(S)}(J=1,K=1,Q) & = & min(max(1,Q),max(1-1,1-Q)) & = & 1-Q
\end{array}$$

$$\text{So, } Q \le F^{(S)} \le 1.0$$

Also,

$$\begin{array}{rcl}
F^{(C)}(J=0,K=1,Q) & = & Q \\
F^{(C)}(J=1,K=0,Q) & = & 1-Q \\
F^{(C)}(J=0,K=0,Q) & = & Q \\
F^{(C)}(J=1,K=1,Q) & = & 1-Q
\end{array}$$

So we have the following inequalities,

$$0.0 \le F^{(R)} \le Q \le F^{(C)} \le 1-Q \le F^{(S)} \le 1.0$$

Case III : Let $Q \ge 0.5$
Proceeding as Case II we can have the following inequalities,

$$0.0 \le F^{(R)} \le 1-Q \le F^{(C)} \le Q \le F^{(S)} \le 1.0$$

Let J and K be fixed in [0,1] . Then $F^{(R)}$, $F^{(S)}$ and $F^{(C)}$ are functions of Q only, and let them be denoted by, $F_1^{(R)}$, $F_1^{(S)}$ and $F_1^{(C)}$ respectively. We immediately have then the following theorem,

Theorem 4.10. *In a certain neighbourhood of Q , if $Lim_{Q\to q}F_1^{(R)} = l = Lim_{Q\to q}F_1^{(S)}$ then $Lim_{Q\to q}F_1^{(C)}$ exists and is equal to l.*

4.2. **Limiting behaviour of fuzzy J-K flip-flop:** In this section, we study the limiting behaviour of the inputs and the states of the fuzzy J-K flip-flops in both max-min compositional form and in algebraic sum and product form. We shall compare their properties of their limiting values also. We have,

$$F^{(R)}(J,K,Q) = max(min(J,1-Q), min(1-K,Q))$$
$$F^{(S)}(J,K,Q) = min(max(J,Q), max(1-K,1-Q))$$

Case I: Let us consider the complementary inputs of the Fuzzy J-K flip-flop.
Let $J=\overline{X}$, $K=X$
Then,

$$F^{(R)}(\overline{X},X,Q) = max(min(1-X,1-Q), min(1-X,Q))$$
$$F^{(S)}(\overline{X},X,Q) = min(max(1-X,Q), max(1-X,1-Q))$$

Let us consider a sequence of inputs and states,$\{X_1, X_2, ...\}$ and $\{Q_1, Q_2, ...\}$. Then,

$$F^{(R)}(\overline{X_i},X_i,Q_i) = max(min(1-X_i,1-Q_i), min(1-X_i,Q_i))$$
$$F^{(S)}(\overline{X_i},X_i,Q_i) = min(max(1-X_i,Q_i), max(1-X_i,1-Q_i))$$

(a) Let $X_i \to 0,\ Q_i \to 0$
Let $X_i = \epsilon,\ Q_i = \delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(1-\epsilon,1-\delta), min(1-\epsilon,\delta)) = 1-\epsilon$
That is, $F^{(R)} \to 1$
$F^{(S)} = min(max(1-\epsilon,\delta), max(1-\epsilon,1-\delta)) = 1-\epsilon$or $1-\delta$ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is $F^{(S)} \to 1$

(b) Let $X_i \to 0,\ Q_i \to 1$
Let $X_i = \epsilon,\ Q_i = 1-\delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(1-\epsilon,\delta), min(1-\epsilon,1-\delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(R)} \to 1$
$F^{(S)} = min(max(1-\epsilon,1-\delta), max(1-\epsilon,\delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(S)} \to 1$

(c) Let $X_i \to 1,\ Q_i \to 0$
Let $X_i = 1-\epsilon,\ Q_i = \delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(1-\epsilon,\delta), min(\epsilon,\delta)) = \epsilon$
That is, $F^{(R)} \to 0$
$F^{(S)} = min(max(\epsilon,\delta), max(\epsilon,1-\delta)) = \epsilon$ or δ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(S)} \to 0$

(d) Let $X_i \to 1,\ Q_i \to 1$
Let $X_i = 1-\epsilon,\ Q_i = 1-\delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(\epsilon,\delta), min(\epsilon,1-\delta)) = \epsilon$ or δ according as $\epsilon \leq \delta$ or $\epsilon \geq \delta$

That is, $F^{(R)} \to 0$
$F^{(S)} = min(max(\epsilon, 1-\delta), max(\epsilon, \delta)) = \epsilon$ or δ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is $F^{(S)} \to 0$
Case II: Let the inputs of the flip-flop be equal.
Let $J = X,\ K = X$. Then

$$F^{(R)}(X, X, Q) = max(min(X, 1-Q), min(1-X, Q))$$
$$F^{(S)}(X, X, Q) = min(max(X, Q), max(1-X, 1-Q))$$

Let us consider a sequence of inputs and states $\{X_1, X_2, ...X_n, ...\}$ and $\{Q_1, Q_2, ...Q_n, ...\}$. Then,
$F^{(R)}(X_i, X_i, Q_i) = max(min(X_i, 1-Q_i), min(1-X_i, Q_i))$
$F^{(S)}(X_i, X_i, Q_i) = min(max(X_i, Q_i), max(1-X_i, 1-Q_i))$
(a) Let $X_i \to 0,\ Q_i \to 0$
Let $X_i = \epsilon,\ Q_i = \delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(\epsilon, 1-\delta), min(1-\epsilon, \delta)) = \epsilon$ or δ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(R)} \to 0$
$F^{(S)} = min(max(\epsilon, \delta), max(1-\epsilon, 1-\delta)) = \epsilon$ or δ according as $\epsilon \leq \delta$ or $\epsilon \geq \delta$
That is, $F^{(S)} \to 0$
(b) Let $X_i \to 0,\ Q_i \to 1$
Let $X_i = \epsilon,\ Q_i = 1-\delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(\epsilon, \delta), min(1-\epsilon, 1-\delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \leq \delta$ or $\epsilon \geq \delta$
That is, $F^{(R)} \to 1$
$F^{(S)} = min(max(\epsilon, 1-\delta), max(1-\epsilon, \delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(S)} \to 1$
(c) Let $X_i \to 1,\ Q_i \to 0$
Let $X_i = \epsilon,\ Q_i = \delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(1-\epsilon, 1-\delta), min(\epsilon, \delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \leq \delta$ or $\epsilon \geq \delta$
That is, $F^{(R)} \to 1$
$F^{(S)} = min(max(1-\epsilon, \delta), max(\epsilon, 1-\delta)) = 1-\epsilon$ or $1-\delta$ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(S)} \to 1$

(d) Let $X_i \to 1,\ Q_i \to 1$
Let $X_i = 1-\epsilon,\ Q_i = 1-\delta$, where $\epsilon,\ \delta > 0$
Then,
$F^{(R)} = max(min(1-\epsilon, \delta), min(\epsilon, 1-\delta)) = \epsilon$ or δ according as $\epsilon \geq \delta$ or $\epsilon \leq \delta$
That is, $F^{(R)} \to 0$
$F^{(S)} = min(max(1-\epsilon, 1-\delta), max(\epsilon, \delta)) = \epsilon$ or δ according as $\epsilon \leq \delta$ or $\epsilon \geq \delta$
That is $F^{(S)} \to 0$.

So far, we have studied the limiting behaviour of Fuzzy J-K flip-flop in max-min form and it is seen that the limiting behaviours match with their functional behaviour. Let us now study the limiting behaviour using algebraic sum and product form.

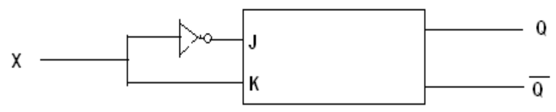

FIGURE 5. A flip-flop with complementary inputs

Case I: Let $J = \overline{X}$ and $K = X$.
Then $F^{(R)}(\overline{X}, X, Q) = max(min(\overline{X}, 1 - Q), min(\overline{X}, Q))$
This equation can be written using algebraic sum and product form as
$F^{(R)}(\overline{X}, X, Q) = (1 - X)(1 - Q) + (1 - X)Q - Q(1 - Q)(1 - X)^2$

$$\Rightarrow F^{(R)}(\overline{X}, X, Q) = (1 - X)\{1 - Q(1 - Q)(1 - X)\} \tag{4.1}$$

When $X = 1$ then $F^{(R)} = 0$ and when $X = 0$ then $F^{(R)} = 1 - Q(1 - Q)$ [Boundary values of the input]
Also, $F^{(S)}(\overline{X}, X, Q) = min(max(\overline{X}, Q), max(\overline{X}, \overline{Q}))$
When $X = 1$ then $F^{(S)} = Q(1 - Q)$ and when $X = 0$ then $F^{(S)} = 1$ [Boundary values of the input]
Using algebraic sum and product form this equation can also be written as

$$F^{(S)}(\overline{X}, X, Q) = (1 - X) + QX^2(1 - Q) \tag{4.2}$$

$F^{(R)}$ and $F^{(R)}$ are both continuous functions defined over the interval [0,1].
Let us consider a sequence of inputs and states , $\{X_1, X_2, X_3, ...\}$ and $\{Q_1, Q_2, Q_3, ...\}$
From 4.1 we obtain,
$F^{(R)}(\overline{X_i}, X_i, Q_i) = (1 - X_i)\{1 - Q_i(1 - Q_i)(1 - X_i)\}$
Here, if $X_i \to 0$,and $Q_i \to 0$ (or $Q_i \to 1$), then $F^{(R)}(X_i, Q_i) \to 1.(1-0)(1-0)) = 1$
If $X_i \to 1$, and $Q_i \to 0$ (or $Q_i \to 1$) then $F^{(R)}(X_i, Q_i) \to 0(1 - 0(1 - 0)) = 0$

Likewise from 4.2 we have,
$F^{(S)}(\overline{X_i}, X_i, Q_i) = (1 - X_i) + Q_i{X_i}^2(1 - Q_i)$
If $X_i \to 0$,and $Q_i \to 0$ (or $Q_i \to 1$), then $F^{(S)}(\overline{X_i}, X_i, Q_i) \to 1 + 0 = 1$
If $X_i \to 1$, and $Q_i \to 0$ (or $Q_i \to 1$) then $F^{(S)}(\overline{X_i}, X_i, Q_i) \to 0 + 0 = 0$

Case II: Let $J = X$ and $K = X$.
Then,

$$F^{(R)}(X, X, Q) = X(1 - Q) + (1 - X)\{Q - XQ(1 - Q)\} \tag{4.3}$$

When $X = 1$ then $F^{(R)} = (1-Q)$ and when $X = 0$ then $F^{(R)} = Q$ [Boundary values of the input]
4.3 gives,
$F^{(R)}(X_i, X_i, Q_i) = X_i(1 - Q_i) + (1 - X_i)\{Q_i - X_iQ_i(1 - Q_i)\}$

If $X_i \to 0$ and $Q_i \to 0$ then $F^{(R)} \to 0$; if $X_i \to 0$ and $Q_i \to 1$ then $F^{(R)} \to 1$.
Also, if $X_i \to 1$ and $Q_i \to 0$, then $F^{(R)} \to 1$; if $X_i \to 1$ and $Q_i \to 1$ then $F^{(R)} \to 0$.
Again we have,

$$F^{(S)}(X, X, Q) = \{X(1 - Q) + Q\}(1 - XQ) \tag{4.4}$$

When $X = 1$ then $F^{(S)} = 1 - Q$, when $X = 0$ then $F^{(S)} = Q$ [Boundary values of the input]
4.4 gives,
$F^{(S)}(X_i, X_i, Q_i) = \{X_i(1 - Q_i) + Q_i\}(1 - X_iQ_i)$
If $X_i \to 0$,$Q_i \to 0$ then $F^{(S)} \to 0$;If $X_i \to 0$,$Q_i \to 1$ then $F^{(S)} \to 1$, If $X_i \to 1$,$Q_i \to 0$ then $F^{(S)} \to 1$,If $X_i \to 1$,$Q_i \to 1$ then $F^{(S)} \to 0$.

Thus, the J-K flip-flop behaves equally in the limiting values in both the forms discussed above. We can choose any one of them as long as the outcomes are concerned.

5. Representing a fuzzy flip-flop with fuzzy rule

In studying the fuzzy flip-flop in a convenient way there are some independent steps the execution of which will lead to the solution of a fuzzy flip-flop related problems. A crisp flip-flop is defined by means of a state-table where the crisp values of the states and inputs are shown. The structure of the table describes the type of the flip-flop. Since the fuzzy flip-flop is the generalisation of the crisp flip-flop, for the defuzzification of it gives it's crisp counterpart, it has been attempted to develope a generalised rule-base for a particular type of flip-flop. The following steps are required for developing the rule-base for a fuzzy flip-flop.

(a) Fuzzification: All the states and inputs of a flip-flop may be considered as having ambiguous characteristics, when their fuzzy versions are studied. Their domain of definition may be considered as consistent fuzzy sets over the respective universe of discourse. For the sake of convenience we consider the finite and ordered domain. The domain considered for fuzzy flip-flop may either be discrete or continuous. For the fuzzification the uniform triangular fuzzy sets are considered for our purpose.

(b) Knowledge Base: This step relates the factor for solving a particular problem. It includes many fuzzy conditional statements for describing a particular situation. A flip-flop has some inputs and states. If the inputs and states are fuzzy sets over their universe of discourse then from the expert's knowledge a set of rules can be constructed. The knowledge base of a fuzzy flip-flop contains the knowledge related to the nature of a fuzzy flip-flop. The knowledge is represented by a fuzzy production rule.
$Rule_i$: If the present input is A_i and the present input is B_i and the present state is S_i then the next state is T_i

(c) Inference: This part consists of the conditional fuzzy statements to represent a particular situation. This step is related to the factors that take place for solving a problem. The fuzzy inference rules may be obtained either from the experienced human operator or from the fuzzified data of a flip-flop. For the determination of proper fuzzy inference rules we need a set of data. The output data is obtained

by the approximate reasoning methodology with some conditional propositions. We state the rules as follows:
Rule: If the present input is A_i and the present input is B_i and the present state is S_i then the next state is S_i.
Fact: The present input is A and the present input is B and the next state is S.
Conclusion: The next state is T.

(d) Defuzzification: The result obtained from the previous steps is in the form of fuzzy statement. To find the deterministic value of the linguistic variables, if necessary, the defuzzification is used. There are many useful methods for defuzzification. We have used the centre of gravity method for our purpose.

5.1. **Generating a Rule-Base from the combined fuzzy flip-flop.** So far, we have considered the $F^{(R)}$ and $F^{(S)}$ types of J-K flip-flops. A natural extension is done for $F^{(C)}$ type flip-flp. From the postulates of combined fuzzy J-K flip-flop we have,

$$\begin{array}{lcl} a.\ F^{(C)}(0,0,Q) & = & Q \\ b.\ F^{(C)}(0,1,Q) & = & min(Q,1-Q) \\ c.\ F^{(C)}(1,0,Q) & = & max(Q,1-Q) \\ d.\ F^{(C)}(1,1,Q) & = & 1-Q \\ e.\ F^{(C)}(0.5,0.5,Q) & = & 0.5 \end{array}$$

We express the above postulates in terms of the linguistic variable. We use here the terms Lo(w) and Hi(gh) as inputs in lieu of 0 and 1 respectively. Of course the intermediate values of the linguistic variables of the states and inputs can also be used wherever it is necessary.
Let us consider at first $Q = Bad$. Then,

$$\begin{array}{lcl} a.\ F^{(C)}(Lo,Lo,Bad) & = & Bad \\ b.\ F^{(C)}(Lo,Hi,Bad) & = & min(Bad,C(Bad)) \\ c.\ F^{(C)}(Hi,Lo,Bad) & = & max(Bad,C(Bad)) \\ d.\ F^{(C)}(Hi,Hi,Bad) & = & C(Bad) \\ e.\ F^{(C)}(0.5,0.5,Q) & = & 0.5 \end{array}$$

Similarly for $Q = Good$ we have,

$$\begin{array}{lcl} a.\ F^{(C)}(Lo,Lo,Good) & = & Good \\ b.\ F^{(C)}(Lo,Hi,Good) & = & min(Good,C(Good)) \\ c.\ F^{(C)}(Hi,Lo,Good) & = & max(Good,C(Good)) \\ d.\ F^{(C)}(Hi,Hi,Good) & = & C(Good) \\ e.\ F^{(C)}(0.5,0.5,Q) & = & 0.5 \end{array}$$

C is a negation function. We can frame the above postulates in a tabular form as in the Table 5.1.

$\odot$	J=Lo K=Lo	J=Lo K=Hi	J=Hi K=Lo	J=Hi K=Hi	J=0.5 K=0.5
Bad	Bad	min(Bad,C(Bad))	max(Bad,C(Bad))	C(Bad)	0.5
Good	Good	min(Good,C(Good))	max(Good,C(Good))	C(Good)	0.5

Table 4. Rule-Base for a fuzzy J-K flip-flop

Let the input $X = \Sigma\mu(\sigma_i)/\sigma_i$, i=0,1,...m-1; and the state be $\Sigma\, \mu(s_i)/s_i$, i=0,1,...n-1. We define here a composition $\odot$ between X and Q as,

$$(X \odot Q)_{Projection\ on\ X} = max_j(min_{i,j}(\mu_X(\sigma_i), \mu_Q(s_j)))$$

and

$$(X \odot Q)_{Projection\ on\ Q} = max_i(min_{i,j}(\mu_X(\sigma_i), \mu_Q(s_j)))$$

5.2. **Representation of a fuzzy state in terms of fuzzy memory elements.** It is known that a flip-flop is used as a memory element. A register is a group of memory elements that work together as a unit. A single fuzzy memory element can be considered as a single state s. A fuzzy buffer register can be considered to store a fuzzy set of states Q, each memory element of this state being fuzzy in nature. We can extend this idea to the collection of all set of states. Here $Q = \Sigma\mu(s^{(t)})/s^{(t)}$. If necessary, we can use a sequence of fuzzy buffer register for a fuzzy system S.

Working Principle: Let $A_1, A_2, ..., A_n$ be 'n' J-K fuzzy flip-flops and it constitutes a 'n' bit fuzzy buffer register. Each flip-flop can store a value between 0 and 1. Let a fuzzy input of the fuzzy system be X. Then the inputs of the J-K flip-flop become function of X, that is, $J = J(X)$ and $K = K(X)$. Let the flip-flops $A_1, A_2, ..., A_n$ initially store the states $s_1, s_2, ...s_n$ respectively. If the input $\sigma_1(\in X)$ is fed to the system then the set of fuzzy states ${Q_i}^{(t)}$ changes to ${Q_i}^{(t+\delta t)}$ according to the postulates of the combined fuzzy flip-flops $F^{(C)}$. Consequently, when the input changes to $\sigma_2(\in X)$ again, the state changes to ${Q_i}^{(t+2\delta t)}$ and so on. After the n^{th} input symbol $\sigma_n(\in X)$ given to the system, the state becomes ${Q_i}^{(t+n\delta t)}$. The fuzzy set thus changes from $Q^{(t)}$ to $Q^{(t+n\delta t)}$.

Let one of the final states of the system be Q_F. If for a pre-assigned positive threshold value ϵ, Sim($Q^{(t+n\delta t)}$,Q_F)$> \epsilon$, then we can say that the fuzzy word $a = \Sigma(\mu(\sigma_i))/\sigma_i$ is accepted by the system. If it is not accepted then another input word a' is applied and the process continues upto a finite number of steps to achieve the desired degree of accuracy.

Example 5.1. Let a four bit register be constituted by four fuzzy J-K flip-flops containing the values 0.1,0.9,0.2, and 0.4 respectively. Let the input of the system be X. Let the four couple of inputs of the J-K flip-flops be designed as $J_1 = X$, $K_1 = X$; $J_2 = \overline{X}$, $K_2 = X$;$J_3 = \overline{X}$, $K_3 = \overline{X}$ and, $J_4 = X$, $K_4 = X$. Let the threshold value $\epsilon = 0.9$. Let Q_F=(0.1,0.2,0.9,0.3).If X=0.9, then the values of the four flip-flops become 0.9,0.1,0.2,0.6 respectively, that is the fuzzy set $Q(t + \delta t)$ becomes $(0.9/s_0, 0.1/s_1, 0.2/s_2, 0.6/s_3)$. Similarly, for X=0.25, $Q(t + 2\delta t) = (0.9/s_0, 0.9/s_1, 0.8/s_2, 0.6/s_3)$, and for $X = 0.8$, $Q(t + 3\delta t) = (0.1/s_0, 0.1/s_1, 0.8/s_2, 0.4/s_3)$. Also, Sim$(Q_F, Q(t + 3\delta t) = 0.91$, which exceeds the threshold value. So the word $(0.9/\sigma_1, 0.25/\sigma_2, 0.8/\sigma_3)$ is accepted by the machine.

5.3. **A sequential circuit with fuzzy inputs and fuzzy states.** We now consider a sequential circuit with input X as follows.

Let $J = X\overline{Q}$, $K = \overline{X}Q$.For our case, let us consider $X = Hi$, $Q = Good$.
Then $J = Hi \odot C(Good)$, $K = C(Hi) \odot Good$
Let the states X and Q be as in the following tabular form,

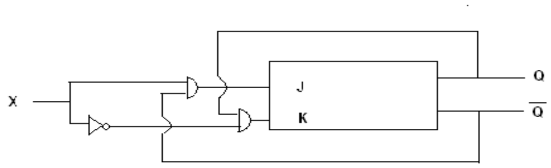

Figure 6.

X	$\mu(X)$	X	$\mu(X)$	X	$\mu(X)$
σ_0		σ_7		σ_{14}	1.0
σ_1		σ_8		σ_{15}	0.8
σ_2		σ_9	0.1	σ_{16}	0.75
σ_3		σ_{10}	0.3	σ_{17}	0.55
σ_4		σ_{11}	0.5	σ_{18}	0.45
σ_5		σ_{12}	0.6	σ_{19}	0.28
σ_6		σ_{13}	0.8	σ_{20}	0.1

Table 5. Input of the machine

Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$
s_0		s_7		s_{14}		s_{21}	0.75
s_1		s_8		s_{15}		s_{22}	0.65
s_2		s_9		s_{16}	0.2	s_{23}	0.45
s_3		s_{10}		s_{17}	0.4	s_{24}	0.35
s_4		s_{11}		s_{18}	0.6	s_{25}	0
s_5		s_{12}		s_{19}	0.8		
s_6		s_{13}		s_{20}	1		

Table 6. State of the machine

We need here a single valued output. So we apply here the centre of gravity defuzzification method. After defuzzification we have, $defuzzy(J) = 0.72$, $defuzzy(K) = 0.41$, $defuzzy(1-Q) = 0.42$, $defuzzy(1-K) = 0.72$, $defuzzy(Q) = 0.81$. Now,

$$\begin{aligned} F^{(R)}(J,K,Q) &= max\{min(J,1-Q), min(1-K,Q)\} \\ &= max\{min(Lo,Lo), min(Hi,Hi)\} \\ &= max(Lo,Hi) \\ &= Hi \end{aligned}$$

Also,

$$\begin{aligned} F^{(S)}(J,K,Q) &= min\{max(J,Q), max(1-K,1-Q)\} \\ &= min\{max(Lo,Hi), max(Hi,Lo)\} \\ &= min(Hi,Hi) \\ &= Hi \end{aligned}$$

So, both the set and reset type of fuzzy J-K flip-flop follows the rule base. With the different combinations of X and Q as Lo and Hi the rule base as obtained from the combined fuzzy flip-flop can be verified.

5.3.1. *Sequential fuzzy flip-flop with cascade connection.* When a set of fuzzy flip-flops work together, they form a fuzzy system. The general structure of the fuzzy flip-flop is as follows:

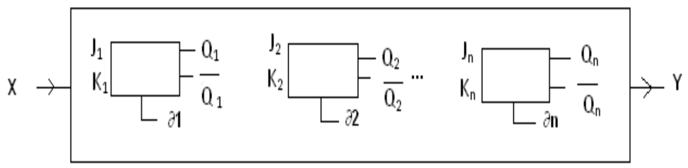

Figure 7.

Here, in figure 7, 'n' fuzzy flip-flops are working together forming a fuzzy system. X and Y are the input and output of the system respectively. Each flip-flop has two inputs J and K. J=J$(Q_1, Q_2, ..., Q_n, X)$, K=K$(Q_1, Q_2, ..., Q_n, X)$, Y=Y$(Q_1, Q_2, ..., Q_n, X)$. We are now interested in a sequential system where the output of each flip-flop is fed as input to the next flip-flop.

A fuzzy automata with output is given by, {S,Σ,M,a,Z,Δ} , where S is the set of states,Σ is the set of input symbols,M is the transition function,a is the initial state, Z is the output function, Δ is the set of output symbols. M: S $\times$ Σ $\times$ S $\rightarrow$ [0,1], Z: S$\times$Σ $\rightarrow$ Δ. Let us consider $J_i = J(Q_i, \sigma_i)$,$K_i = K(Q_i, \sigma_i)$, i=1,2,...n.If the output symbols are used as input symbols to the consecutive flip-flops then $\Delta \subseteq \Sigma$, and consequently Z: S$\times$Σ $\rightarrow$ Σ is a surjective mapping. Our proposed flip-flop has the rule bases for the inputs J and K which are built from the expert's knowledge. From these rule bases the inputs J_i and K_i of the i^{th} flip-flop is determined.Let the states and inputs of a J-K flip-flop be 'Good', 'Bad', and 'Low', 'High', respectively.The rule base for the states and inputs may be as follows:

C is a negation function.

Example 5.2. Let the initial state of the system be 'Bad' and the input of the system be 'Low',.Then, from the Table 7, J_1=$J_1(Bad, Lo) = Lo$ and from the Table 5.3.1, K_1=$K_1(Bad, Lo) = Hi$. Thus the inputs of the first flip-flop of the system is determined.The function Z: S$\times$Σ $\rightarrow$ Σ produces the next input symbol and afterwards J_2 and K_2 are determined and likewise the process goes on upto the last flip-flop of the system. The working principle of the entire fuzzy system is discussed in the next section.

σ	Lo	Hi	C(Lo)	C(Hi)
Good	Lo	Very Hi	Hi	C(Hi)
Bad	Lo	C(Very Hi)	Lo	C(Very Lo)
C(Good)	Lo	Hi	Hi	Very Lo
C(Bad)	Very Hi	C(Very Lo)	Lo	Hi

TABLE 7. Rule Base for J

σ	Lo	Hi	C(Lo)	C(Hi)
Good	C(Very Hi)	More or Less Lo	Hi	Very Hi
Bad	Hi	Very Lo	More or Less Hi	Very Lo
C(Good)	Lo	C(Lo)	Very Hi	C(Very Hi)
C(Bad)	Very Lo	C(Very Lo)	Hi	Lo

TABLE 8. Rule Base for K

6. FUZZY FLIP-FLOP AS FUZZY SYSTEM

A fuzzy flip-flop may be considered as a part of the fuzzy system. The kernel of the fuzzy system consists of a knowledge base, a data base and an inference mechanism. The knowledge base of a fuzzy flip-flop contains the knowledge related to the nature of a fuzzy flip-flop. The knowledge is represented by a fuzzy production rule.

For the fuzzification, the uniform geometric shapes of the fuzzy sets are considered. We have used the triangular shaped fuzzy sets for our purpose. The fuzzy inference rules may be obtained either from the experienced human operators or from the fuzzified empirical data of a flip-flop. For the determination of proper fuzzy inference rules, we need a set of data. The output data is obtained by the approximate reasoning methodology with some conditional fuzzy propositions.

Sometimes, we need defuzzification procedure to convert the modified output values to a single value. We have used modified centre of gravity method for defuzzification. Let us define a fuzzy flip-flop with rules of the form:
R_i: If the present input-1 is A_i and the present input-2 is B_i and the present state is S_i then the next state is T_i.
The present input-1 is A and the present input-2 is B and the present state is S.
Output: The next state is T.

Let us consider the following rule base for a fuzzy $J-K$ flip-flop as in the following:

$\wedge$	J=Low,K=Low	J=Low,K=High	J=High,K=Low	J=High,K=High
Bad	Bad	Bad	Good	C(Bad)
Good	Good	Bad	Good	C(Good)

TABLE 9. Rule Base for a fuzzy flip-flop

6.1. **Algorithm for a fuzzy flip-flop.** A set of if-then rules are used for the determination of the behaviour of a fuzzy flip-flop. In this section, an algorithm is developed to perform the said task of obtaining the behaviour of atypical fuzzy flip-flop.

Step 1. For each rule R_i, similarity value is computed between the input fuzzy sets as follows:
α_i= min $\{ Sim(A_i, A), Sim(B_i, B), Sim(S_i, S) \}$.
Step 2. Translate premise p and compute $R(A_i, B_i, S_i, T_i)$ using some translating rule.
Step 3. Modify $R(A_i, B_i, S_i, T_i)$ with α_i to obtain modified conditional relation $R(A_i, B_i, S_i|A, B, S)$
Step 4. Max projection operation is used on $R(A_i, B_i, S_i|A, B, S)$ to obtain T as follows:
$\mu_T(z) = max_{(u,v,w)}\{\mu_R(A_i, B_i, S_i, T_i|A, B, S)(u, v, w, z)\}$
Step 5. If necessary, defuzzify the fuzzy sets as obtained in Step 4 for a single real value in the output.

Let X_1, X_2, X_3, X_4 be four linguistic variables to denote the *present input*$_1$, *present input*$_2$, *present state* and *next state* respectively defined over the respective universe of discourses U_1, U_1, U_2, and U_2 . A rule base of the state transitions is considered as in Table 9. The exact definitions of the fuzzy sets are to be prescribed. A consequence is derived according to the algorithm of a fuzzy flip-flop.

Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$
s_0		s_7	0.84	s_{14}	0.24	s_{21}		s_{28}	
s_1	0.12	s_8	1.0	s_{15}	0.12	s_{22}		s_{29}	
s_2	0.24	s_9	0.84	s_{16}		s_{23}		s_{30}	
s_3	0.36	s_{10}	0.72	s_{17}		s_{24}			
s_4	0.48	s_{11}	0.60	s_{18}		s_{25}			
s_5	0.60	s_{12}	0.48	s_{19}		s_{26}			
s_6	0.72	s_{13}	0.36	s_{20}		s_{27}			

Table 10. Present state of the machine (Bad)

Observation 6.1. The next state computed is 'Good'. The specificity of this state is less than the present next state.

7. Conclusion

Developing suitable mathematics for the realisation of intelligent systems becomes necessary to handle modern computer based technologies managing different kinds of information and knowledge. This paper discusses one such tool required to help in the design of basic building blocks of relevant circuits for finding solutions to difficult problems in the construction of intelligent systems in which, the available information is supplied by human experts which, at times are found incomplete, imprecise or even uncertain in nature and therefore, inherently ambiguous. It requires a logical

X	$\mu(X)$	X	$\mu(X)$	X	$\mu(X)$
σ_0		σ_7		σ_{14}	1.0
σ_1		σ_8		σ_{15}	0.80
σ_2		σ_9	0.16	σ_{16}	0.64
σ_3		σ_{10}	0.32	σ_{17}	0.48
σ_4		σ_{11}	0.48	σ_{18}	0.32
σ_5		σ_{12}	0.64	σ_{19}	0.16
σ_6		σ_{13}	0.80	σ_{20}	

Table 11. Present Input of the machine(High)

X	$\mu(X)$	X	$\mu(X)$	X	$\mu(X)$
σ_0	0.0	σ_7	0.80	σ_{14}	
σ_1	0.16	σ_8	0.64	σ_{15}	
σ_2	0.32	σ_9	0.48	σ_{16}	
σ_3	0.48	σ_{10}	0.32	σ_{17}	
σ_4	0.64	σ_{11}	0.16	σ_{18}	
σ_5	0.80	σ_{12}		σ_{19}	
σ_6	1.0	σ_{13}		σ_{20}	

Table 12. Present Input of the machine(Low)

Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$
s_0		s_7		s_{14}		s_{21}	0.84	s_{28}	0.24
s_1		s_8		s_{15}	0.12	s_{22}	1.0	s_{29}	0.12
s_2		s_9		s_{16}	0.24	s_{23}	0.84	s_{30}	
s_3		s_{10}		s_{17}	0.36	s_{24}	0.72		
s_4		s_{11}		s_{18}	0.48	s_{25}	0.60		
s_5		s_{12}		s_{19}	0.60	s_{26}	0.48		
s_6		s_{13}		s_{20}	0.72	s_{27}	0.36		

Table 13. Next State of the machine(Good)

framework which, will be able to reason and make decisions in an environment of imprecision, uncertainty, incompleteness of information and partiality of truth.

The concept of a fuzzy flip-flop is defined from widely used two-valued Boolean flip-flops in a systematic way. Some properties of fuzzy flip-flops are studied from the view point of a primitive fuzzy system. An imprecise/incomplete description of the input-output behaviour of a system, as obtained from human experts, containing vague concepts is represented as fuzzy if-then rules — transforming the system into a simple fuzzy rule-based one. Approximate reasoning methodology has been used

Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$
s_0		s_7	0.706	s_{14}	0.057	s_{21}		s_{28}	
s_1	0.014	s_8	1.0	s_{15}	0.014	s_{22}		s_{29}	
s_2	0.057	s_9	0.706	s_{16}		s_{23}		s_{30}	
s_3	0.130	s_{10}	0.518	s_{17}		s_{24}			
s_4	0.230	s_{11}	0.360	s_{18}		s_{25}			
s_5	0.360	s_{12}	0.230	s_{19}		s_{26}			
s_6	0.518	s_{13}	0.130	s_{20}		s_{27}			

TABLE 14. Observed State of the machine (Very Bad)

X	$\mu(X)$	X	$\mu(X)$	X	$\mu(X)$
σ_0		σ_7		σ_{14}	1.0
σ_1		σ_8		σ_{15}	0.640
σ_2		σ_9	0.0256	σ_{16}	0.410
σ_3		σ_{10}	0.102	σ_{17}	0.230
σ_4		σ_{11}	0.230	σ_{18}	0.102
σ_5		σ_{12}	0.410	σ_{19}	0.0256
σ_6		σ_{13}	0.640	σ_{20}	

TABLE 15. Observed Input of the machine(Very High)

X	$\mu(X)$	X	$\mu(X)$	X	$\mu(X)$
σ_0	0.0	σ_7	0.80	σ_{14}	
σ_1	0.16	σ_8	0.64	σ_{15}	
σ_2	0.32	σ_9	0.48	σ_{16}	
σ_3	0.48	σ_{10}	0.32	σ_{17}	
σ_4	0.64	σ_{11}	0.16	σ_{18}	
σ_5	0.80	σ_{12}		σ_{19}	
σ_6	1.0	σ_{13}		σ_{20}	

TABLE 16. Present Input of the machine(Low)

to predict the possible behaviour of the system. The use of fuzzy logic allows us to use different interpretation of the logical operators for flexibility.

Achievement of human-level machine intelligence will have a profound impact on modern society. It is hoped that by upgrading existing methodologies through addition of concepts and techniques drawn from fuzzy set theory open the door to a substantial enhancement of our ability to model reality. Further research on the use of similarity and approximate reasoning is necessary for better understanding of the effect of the same on the cognitive process involved in the modelling and simulation of fuzzy flip-flop. We have suggested relevant issues involved in the design of fuzzy

Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$	Q	$\mu(Q)$
s_0	0.14	s_7	0.14	s_{14}	0.14	s_{21}	0.86	s_{28}	0.35
s_1	0.14	s_8	0.14	s_{15}	0.24	s_{22}	1.0	s_{29}	0.24
s_2	0.14	s_9	0.14	s_{16}	0.35	s_{23}	0.86	s_{30}	0.14
s_3	0.14	s_{10}	0.14	s_{17}	0.45	s_{24}	0.76		
s_4	0.14	s_{11}	0.140	s_{18}	0.55	s_{25}	0.66		
s_5	0.14	s_{12}	0.140	s_{19}	0.66	s_{26}	0.55		
s_6	0.14	s_{13}	0.140	s_{20}	0.76	s_{27}	0.45		

TABLE 17. The next state computed

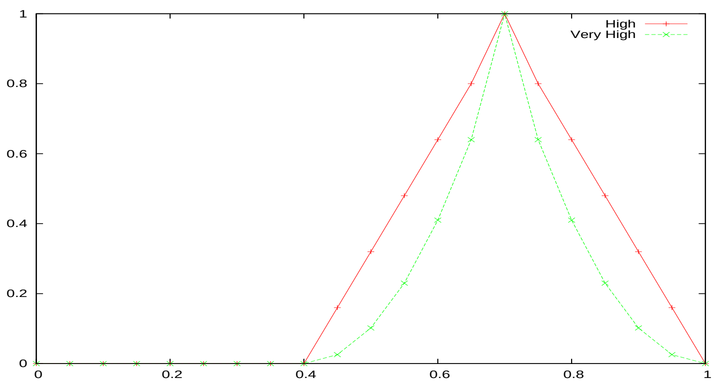

FIGURE 8. Comparison of the inputs applied to the flip-flop

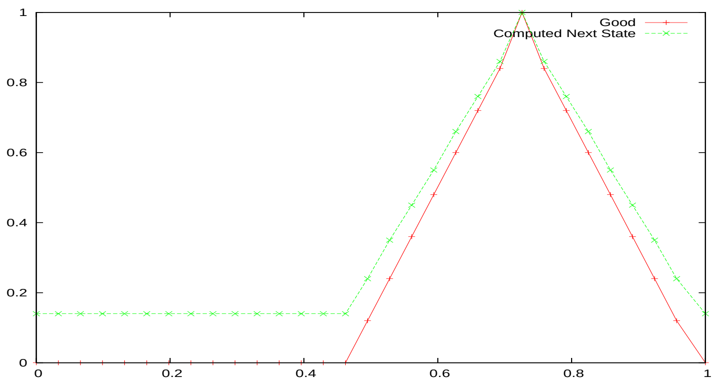

FIGURE 9. Comparison of the given next state with the computed next state

systems — introduced similarity in reasoning, similarity relation in fuzzification and the concept of specificity measure in defuzzification.

Acknowledgements. This research has been financially supported by the UGC's SAP(DRS) Phase-II programme under the Department of Mathematics, Visva-Bharati and the UGC Major Research Project No. F. 36-293/2008(SR).

References

[1] D. Driankov, H. Hellendoorn and M. Reinfrank, An introduction to Fuzzy Control, Narosa Publishing House (2001).

[2] L. Gal, J. Botzheim, L. T. Koczy and A. E. Ruano, Applying Bacterial Memetic Algorithm for Training Feedforward and Fuzzy Flip-Flop, IFSA, EUSFLAT, (2009).

[3] D. Ghosh and D. Chakrabarty, A method to obtain fuzzy Pareto set of fuzzy multi-criteria quadratic programming problems, Ann. Fuzzy Math. Inform. 6(2) (2013) 433–446.

[4] L. Gniewek and J. Kluska, Family of fuzzy j-k flip-flops based on bounded product, IEEE Trans. Systems Man Cybernet. 28(6) (1998) 861–868.

[5] M. M. Gupta and T. Yamakawa, Fuzzy Computing Theory, Hardware and Application, Elsevier Science, (1988).

[6] K. Hirota and K. Ozawa, Fuzzy flip-flops as a basis of fuzzy memory modules, in M.M.Gupta and T.Yamakawa ed. Fuzzy Computing: Theory, Hardware and Application, North-Holland, Amsterdam, (1988) 173-183.

[7] K. Hirota and K. Ozawa, The concept of fuzzy flip-flop, IEEE Trans. Systems Man Cybernet. 19(5) (1989) 980–997.

[8] K. Hirota and W. Pedrycz, Designing sequential systems with Fuzzy J-K flip-flops, Fuzzy Sets and Systems 39 (1991) 202–278.

[9] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall PTR, Upper Saddle River, New Jersey (1995).

[10] A. Kandel, On minimization of fuzzy functions, IEEE Trans. Computers C-22 (1973) 826–832.

[11] B. Kosko, Fuzzy entropy and conditioning, Inform. Sci. 40 (1986) 165–174.

[12] L. T. Koczy, K. Hirota and K. Ozawa, Knowledge representation and accumulation by Fuzzy Flip-Flops, Fuzzy Sets and Systems 39 (1991) 1–13.

[13] L. T. Koczy and R. Lovassy, Fuzzy flip-flops revisited, in O.Castillo et. al. (eds.) Theoretical advances and applications of fuzzy logic and soft computing, ASC 42 (2007) 643–652.

[14] L. T. Koczy and R. Lovassy, Non-associative fuzzy flip-flop with dual set-reset feature, in Proceedings of the 4th Serbian-Hungarian Joint Symposium on Intelligent Systems, Szabadka, (2006) 289–299.

[15] A. P. Malvino, Digital Computer Electronics, Tata Mcgraw-Hill Publishing Company Limited, (1993).

[16] K. Ozawa, K. Hirota, L. T. Koczy and K. Omori, Algebraic fuzzy flip-flop circuits, Fuzzy Sets and Systems 39 (1991) 215–236.

[17] S. Raha and H. Gupta, Fuzzy mathematical machine as a fuzzy system, International Journal of Computational Cognition 6(3) (2008) 13–22.

[18] V.Rajaraman and T.Radhakrishnan, An Introduction to Digital Computer Design, Prentice Hall of India, (1984).

[19] P. Sebastian and T. P. Johnson, Inverse fuzzy automata and inverse fuzzy languages, Ann. Fuzzy Math. Inform. 6(3) (2013) 447-453.

[20] Y. Takama and H. Namba, Investigation on robot user interface for information access, JACIII (Special issue on IWACIII 2009), 14(3) (2010) 316-322.

[21] N. Thillaigovindan and S.A. Shanthi, On intuitionistic fuzzy compact linear operators, Ann. Fuzzy Math. Inform. 3(1) (2012) 133–149.

[22] T. Yamakawa, A simple fuzzy computer system employing min and max operations - A challenge to 6th generation computer, Preprints of Second IFSA Congress, Tokyo (1987) 827-830.

[23] T. Yamakawa, High speed fuzzy controller hardware system, Inform. Sci. 45(2) (1988) 113–128.

[24] L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shimura, Fuzzy sets and their applications to cognitive and decision process, Academic Press, New York (1975).

[25] H. J. Zimmermann, Fuzzy set theory and its applications, Springer, (2001).

Himadrishekhar Gupta (guptahimadrishekhar@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan 731235, West-Bengal, India

Swapan Raha (swapan.raha@visva-bharati.ac.in)
Department of Mathematics, Visva-Bharati, Santiniketan 731235, West-Bengal, India