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# Fuzzy $\sigma$-Baire spaces and functions

Ganesan Thangaraj, Eswaran Poongothai

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Abstract. In this paper some results concerning functions that preserve fuzzy $\sigma$-Baire spaces in the context of images and preimages are obtained.

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Corresponding Author: G. Thangaraj (g.thangaraj@rediffmail.com)

## 1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh in his classical paper [8] in the year 1965. Thereafter the paper of C. L. Chang[3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concept of $\sigma$-Baire spaces in fuzzy setting was introduced and studied by the authors in [5]. The purpose of this paper is to study the effect of fuzzy continuity, fuzzy openness and some decompositions of fuzzy continuity and fuzzy openness on fuzzy $\sigma$-Baire spaces.

## 2. Preliminaries

Now we introduce some basic notions and results that are used in the sequel. In this work, by a fuzzy topological space, we shall mean a non-empty set $X$ together with a fuzzy topology $T$ ( in the sense of Chang) and denote it by $(X, T)$. The interior, closure and the complement of a fuzzy set $\lambda$ will be denoted by $int(\lambda), cl(\lambda)$ and $1 - \lambda$ respectively.

**Definition 2.1.** Let $(X,T)$ be a fuzzy topological space and $\lambda$ be any fuzzy set in $(X,T)$. We define $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1-\mu \in T\}$ and $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$.

For any fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$, it is easy to see that $1 - cl(\lambda) = int(1-\lambda)$ and $1 - int(\lambda) = cl(1-\lambda)$[1].

**Definition 2.2.** Let $(X,T)$ and $(Y,S)$ be any two fuzzy topological spaces. Let $f$ be a function from the fuzzy topological space $(X,T)$ to the fuzzy topological space $(Y,S)$. Let $\lambda$ be a fuzzy set in $(Y,S)$. The inverse image of $\lambda$ under $f$ written as $f^{-1}(\lambda)$ is the fuzzy set in $(X,T)$ defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$. Also the image of $\lambda$ in $(X,T)$ under $f$ written as $f(\lambda)$ is the fuzzy set in $(Y,S)$ defined by

$$f(\lambda)(y) = \begin{cases} sup_{x\in f^{-1}(y)} \quad \lambda(x), & if\ f^{-1}(y)\ is\ non-empty; \\ 0, & otherwise \end{cases} \qquad \text{for each } y \in Y.$$

**Lemma 2.3** ([3])**.** *Let $f : (X,T) \to (Y,S)$ be a mapping. For fuzzy sets $\lambda$ and $\mu$ of $(X,T)$ and $(Y,S)$ respectively, the following statements hold:*

1. $ff^{-1}(\mu) \leq \mu$;
2. $f^{-1}f(\lambda) \geq \lambda$;
3. $f(1-\lambda) \geq 1 - f(\lambda)$;
4. $f^{-1}(1-\mu) = 1 - f^{-1}(\mu)$;
5. *If $f$ is one-to-one, then $f^{-1}f(\lambda) = \lambda$;*
6. *If $f$ is onto, then $ff^{-1}(\mu) = \mu$;*
7. *If $f$ is one-to-one and onto, then $f(1-\lambda) = 1 - f(\lambda)$.*

**Definition 2.4** ([4])**.** A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy dense if there exists no fuzzy closed set $\mu$ in $(X,T)$ such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$.

**Definition 2.5** ([2])**.** A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy $F_\sigma$-set in $(X,T)$ if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $1 - \lambda_i \in T$ for $i \in I$.

**Definition 2.6** ([2])**.** A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy $G_\delta$-set in $(X,T)$ if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$ for $i \in I$.

**Definition 2.7** ([5])**.** A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy $\sigma$-nowhere dense if $\lambda$ is a fuzzy $F_\sigma$-set in $(X,T)$ such that $int(\lambda) = 0$.

**Lemma 2.8** ([1])**.** *For a family $\mathscr{A}$ of $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space $X$, $\vee cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case $\mathscr{A}$ is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee int(\lambda_\alpha) \leq int(\vee\lambda_\alpha)$, where $\alpha \in \mathscr{A}$.*

**Definition 2.9** ([4])**.** A function $f : (X,T) \to (Y,S)$ from a fuzzy topological space $(X,T)$ into another fuzzy topological space $(Y,S)$ is called somewhat fuzzy continuous if $\lambda \in S$ and $f^{-1}(\lambda) \neq 0$ implies that there exists a fuzzy open set $\delta$ in $(X,T)$ such that $\delta \neq 0$ and $\delta \leq f^{-1}(\lambda)$.

**Definition 2.10** ([4])**.** A function $f : (X,T) \to (Y,S)$ from a fuzzy topological space $(X,T)$ into another fuzzy topological space $(Y,S)$ is called somewhat fuzzy open if $\lambda \in T$ and $\lambda \neq 0$ implies that there exists a fuzzy open set $\eta$ in $(Y,S)$ such that $\eta \neq 0$ and $\eta \leq f(\lambda)$.

**Definition 2.11** ([5]). A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy $\sigma$-first category if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$. Any other fuzzy set in $(X,T)$ is said to be of fuzzy $\sigma$-second category.

**Definition 2.12** ([5]). Let $\lambda$ be a fuzzy $\sigma$-first category set in a fuzzy topological space $(X,T)$. Then $1-\lambda$ is called a fuzzy $\sigma$-residual set in $(X,T)$.

## 3. Fuzzy $\sigma$-Baire spaces

**Definition 3.1** ([5]). Let $(X,T)$ be a fuzzy topological space. Then $(X,T)$ is called a fuzzy $\sigma$-Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$.

**Theorem 3.2** ([5]). *Let $(X,T)$ be a fuzzy topological space. Then the following are equivalent:*

(1) *$(X,T)$ is a fuzzy $\sigma$-Baire space.*
(2) *$int(\lambda) = 0$ for every fuzzy $\sigma$-first category set $\lambda$ in $(X,T)$.*
(3) *$cl(\mu) = 1$ for every fuzzy $\sigma$-residual set $\mu$ in $(X,T)$.*

**Definition 3.3** ([6]). A fuzzy topological space $(X,T)$ is called a fuzzy Volterra space if $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where $\lambda_i$'s are fuzzy dense and fuzzy $G_\delta$-sets in $(X,T)$.

**Theorem 3.4** ([5]). *In a fuzzy topological space $(X,T)$, a fuzzy set $\lambda$ is fuzzy $\sigma$-nowhere dense if and only if $1-\lambda$ is a fuzzy dense and fuzzy $G_\delta$-set.*

**Proposition 3.5.** *If a fuzzy topological space $(X,T)$ is a fuzzy Volterra space, then $int(\vee_{i=1}^{N}(\lambda_i)) = 0$, where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$.*

*Proof.* Let $\lambda_i$'s $[i = 1 \text{ } to \text{ } N]$ be fuzzy $\sigma$-nowhere dense sets in $(X,T)$. Then by Theorem 3.4, $(1-\lambda_i)$'s are fuzzy dense and fuzzy $G_\delta$-sets in $(X,T)$. Since $(X,T)$ is a fuzzy Volterra space, $cl(\wedge_{i=1}^{N}(1-\lambda_i)) = 1$. Now $cl(\wedge_{i=1}^{N}(1-\lambda_i)) = 1$ implies that $cl(1-\vee_{i=1}^{N}(\lambda_i)) = 1$. Then we have $1-int(\vee_{i=1}^{N}(1-\lambda_i)) = 1$. This implies that $int(\vee_{i=1}^{N}(\lambda_i)) = 0$. □

**Proposition 3.6.** *If a fuzzy topological space $(X,T)$ is a fuzzy $\sigma$-Baire space, then $(X,T)$ is a fuzzy Volterra space.*

*Proof.* Let $(X,T)$ be a fuzzy $\sigma$-Baire space. Then $int(\vee_{i=1}^{\infty}(\lambda_i) = 0$, where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$. Now $int(\vee_{i=1}^{N}(\lambda_i)) \leq int(\vee_{i=1}^{\infty}(\lambda_i))$ implies that $int(\vee_{i=1}^{N}(\lambda_i)) = 0$. Then $1-int(\vee_{i=1}^{N}(\lambda_i)) = 1-0 = 1$. Then we have $cl(1-\vee_{i=1}^{N}(\lambda_i)) = 1$. This implies that $cl(\wedge_{i=1}^{N}(1-\lambda_i)) = 1$. Since $\lambda_i$'s $[i = 1 \text{ } to \text{ } N]$ are fuzzy $\sigma$-nowhere dense sets in $(X,T)$, by Theorem 3.4, $(1-\lambda_i)$'s are fuzzy dense and fuzzy $G_\delta$-sets in $(X,T)$. Hence $cl(\wedge_{i=1}^{N}(1-\lambda)) = 1$, where $(1-\lambda_i)$'s are fuzzy dense and fuzzy $G_\delta$-sets in $(X,T)$ implies that $(X,T)$ is a fuzzy Volterra space. □

## 4. Fuzzy $\sigma$-Baire spaces and functions

Let $f$ be a function from the fuzzy topological space $(X,T)$ to the fuzzy topological space $(Y,S)$. Under what conditions on $f$ may we assert that if $(X,T)$ is a fuzzy $\sigma$-Baire space, then $(Y,S)$ is a fuzzy $\sigma$-Baire space?.
It may be noticed that the fuzzy continuous image of a fuzzy $\sigma$-Baire space may fail to be a fuzzy $\sigma$-Baire space. For, consider the following example:

**Example 4.1.** Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu, \nu, \alpha, \beta$ and $\eta$ are defined on $X$ as follows:
$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.5$; $\lambda(b) = 0.6$; $\lambda(c) = 0.7$
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.8$; $\mu(b) = 0.4$; $\mu(c) = 0.2$
$\nu : X \to [0, 1]$ defined as $\nu(a) = 0.7$; $\nu(b) = 0.5$; $\nu(c) = 0.8$
$\alpha : X \to [0, 1]$ defined as $\alpha(a) = 0.6$; $\alpha(b) = 0.7$; $\alpha(c) = 0.5$
$\beta : X \to [0, 1]$ defined as $\beta(a) = 0.4$; $\beta(b) = 0.2$; $\beta(c) = 0.8$
$\eta : X \to [0, 1]$ defined as $\eta(a) = 0.5$; $\eta(b) = 0.8$; $\eta(c) = 0.7$

Then $T = \{0, \lambda, \mu, \nu, (\lambda \vee \mu), (\lambda \vee \nu), (\mu \vee \nu), (\lambda \wedge \mu), (\lambda \wedge \nu), (\mu \wedge \nu), \lambda \vee (\mu \wedge \nu), \mu \vee (\lambda \wedge \nu), \nu \wedge (\lambda \vee \mu), (\lambda \vee \mu \vee \nu), 1\}$ and $S = \{0, \alpha, \beta, \eta, (\alpha \vee \beta), (\alpha \vee \eta), (\beta \vee \eta), (\alpha \wedge \beta), (\alpha \wedge \eta), (\beta \wedge \eta), \alpha \vee (\beta \wedge \eta), \beta \vee (\alpha \wedge \eta), \eta \wedge (\alpha \vee \beta), 1\}$ are fuzzy topologies on $X$. Now $1 - [\nu \wedge (\lambda \vee \mu)]$ and $1 - (\mu \wedge \nu)$ are fuzzy $\sigma$-nowhere dense sets in $(X, T)$ and $int(\{1 - [\nu \wedge (\lambda \vee \mu)]\} \vee \{1 - (\mu \wedge \nu)\}) = 0$ and hence $(X, T)$ is a fuzzy $\sigma$-Baire space. Also the fuzzy topological space $(X, S)$ is not a fuzzy $\sigma$-Baire space, since there are no fuzzy $\sigma$-nowhere dense sets $\delta_i$ in $(X, S)$ such that $int(\vee_{i=1}^{\infty}(\delta_i)) = 0$.

Define a function $f : (X, T) \to (X, S)$ by $f(a) = c; f(b) = a; f(c) = b$. Clearly $f$ is a fuzzy continuous function from the fuzzy $\sigma$-Baire space $(X, T)$ to the fuzzy topological space $(X, S)$ which is not a fuzzy $\sigma$-Baire space.

It may also be noticed that the image of a fuzzy $\sigma$-Baire space under a fuzzy open function may fail to be a fuzzy $\sigma$-Baire space. For, consider the following example :

**Example 4.2.** Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu$ and $\delta$ are defined on X as follows :
$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.6$; $\lambda(b) = 0.4$; $\lambda(c) = 0.3$
$\mu : X \to [0, 1]$is defined as $\mu(a) = 0.5$; $\mu(b) = 0.7$; $\mu(c) = 0.2$
$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.7$; $\delta(b) = 0.5$; $\delta(c) = 0.6$

Then $T = \{0, \lambda, \mu, \delta, \lambda \vee \mu, \mu \vee \delta, \lambda \wedge \mu, \mu \wedge \delta, \lambda \vee (\mu \wedge \delta), 1\}$ and $S = \{0, \delta, \lambda \vee \mu, \delta \vee (\lambda \vee \mu), \delta \wedge (\lambda \vee \mu), 1\}$ are fuzzy topologies on $X$. Now $1 - \delta = (1 - \delta) \vee \{1 - [\delta \vee (\lambda \vee \mu)]\}$ and hence $1 - \delta$ is a fuzzy $F_\sigma$-set in $(X, S)$ and $int(1 - \delta) = 0$ and hence $1 - \delta$ is a fuzzy $\sigma$- nowhere dense set in $(X, S)$. Also $1 - \{\delta \wedge (\lambda \vee \mu)\} = [1 - (\lambda \vee \mu)] \vee [1 - \{\delta \wedge (\lambda \vee \mu)\}]$ and hence $1 - \{\delta \wedge (\lambda \vee \mu)\}$ is a fuzzy $F_\sigma$-set in $(X, S)$ and $int[1 - \{\delta \wedge (\lambda \vee \mu)\}] = 0$ and hence $1 - \{\delta \wedge (\lambda \vee \mu)\}$is a fuzzy $\sigma$-nowhere dense set in $(X, S)$. Now $int(\{1 - \delta\} \vee \{1 - [\delta \wedge (\lambda \vee \mu)]\}) = 0$ and hence $(X, S)$ is a fuzzy $\sigma$-Baire space.

Now $1 - \lambda = (1 - \lambda) \vee (1 - [\lambda \vee \mu]) \vee (1 - [\lambda \vee (\mu \wedge \delta)])$ and hence $1 - \lambda$ is a fuzzy $F_\sigma$-set in $(X, T)$ and $int(1 - \lambda) = 0$ and therefore $1 - \lambda$ is a fuzzy $\sigma$-nowhere dense set in $(X, T)$. Also $1 - \mu = (1 - \mu) \vee (1 - [\mu \vee \delta]) \vee (1 - [\lambda \vee \mu])$ and hence $1 - \mu$ is a fuzzy $F_\sigma$-set in $(X, T)$ and $int(1 - \mu) = 0$ and therefore $1 - \mu$ is a fuzzy $\sigma$-nowhere dense set in $(X, T)$. Now $int(\{1 - \lambda\} \vee \{1 - \mu\}) = \mu \wedge \delta \neq 0$ and hence $(X, T)$ is not a fuzzy $\sigma$-Baire space.

Define a function $f : (X, T) \to (X, S)$ by $f(a) = a, f(b) = b$ and $f(c) = c$. Clearly $f$ is an fuzzy open function from the fuzzy $\sigma$-Baire space $(X, S)$ to the fuzzy topological space $(X, T)$ which is not a fuzzy $\sigma$-Baire space.

**Proposition 4.3.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is fuzzy continuous and fuzzy open* *and if $\lambda$ is a fuzzy $\sigma$-nowhere dense set in $(Y, S)$, then $f^{-1}(\lambda)$ is a fuzzy $\sigma$- nowhere dense set in $(X, T)$.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-nowhere dense set in $(Y, S)$. Then $\lambda$ is a fuzzy $F_\sigma$-set in $(Y, S)$ such that $int(\lambda) = 0$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i$'s are fuzzy closed sets in $(Y, S)$. Now $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ implies that $f^{-1}(\lambda) = f^{-1}(\vee_{i=1}^{\infty}(\lambda_i))$. Then $f^{-1}(\lambda) = \vee_{i=1}^{\infty}(f^{-1}(\lambda_i))$. Since the function $f$ is fuzzy continuous and $\lambda_i$'s are fuzzy closed sets in $(Y, S), f^{-1}(\lambda_i)$'s are fuzzy closed sets in $(X, T)$. Hence $f^{-1}(\lambda)$ is a fuzzy $F_\sigma$-set in $(X, T)$.

Now we claim that $int(f^{-1}(\lambda)) = 0$. Suppose that $int(f^{-1}(\lambda)) \neq 0$. Then there exists a fuzzy open set $\mu$ in $(X, T)$ such that $\mu \leq f^{-1}(\lambda)$. Then $f(\mu) \leq ff^{-1}(\lambda) \leq \lambda$. That is, $f(\mu) \leq \lambda$. Since $f$ is fuzzy open, $f(\mu)$ is a fuzzy open set in $(Y, S)$. This implies that $int(\lambda) \neq 0$ which is a contradiction to $int(\lambda) = 0$ in $(Y, S)$. Hence we must have $int(f^{-1}(\lambda)) = 0$. Therefore $f^{-1}(\lambda)$ is a fuzzy $F_\sigma$-set in $(X, T)$ such that $int(f^{-1}(\lambda)) = 0$ and hence $f^{-1}(\lambda)$ is a fuzzy $\sigma$-nowhere dense set in $(X, T)$. □

**Proposition 4.4.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is fuzzy continuous and fuzzy open and if $\lambda$ is a fuzzy $\sigma$-first category set in $(Y, S)$, then $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X, T)$.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-first category set in $(Y, S)$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i$'s are fuzzy $\sigma$ -nowhere dense sets in $(Y, S)$. Now $f^{-1}(\lambda) = f^{-1}(\vee_{i=1}^{\infty}(\lambda_i))$. Then $f^{-1}(\lambda) = \vee_{i=1}^{\infty}(f^{-1}(\lambda_i))$. Since the function $f$ is fuzzy continuous and fuzzy open and $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(Y, S)$, by Proposition 4.3, $f^{-1}(\lambda_i)$'s are fuzzy $\sigma$-nowhere dense sets in $(X, T)$. Hence $f^{-1}(\lambda) = \vee_{i=1}^{\infty}(f^{-1}(\lambda_i))$, where $f^{-1}(\lambda_i)$'s are fuzzy $\sigma$-nowhere dense sets in $(Y, S)$ implies that $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X, T)$. □

**Theorem 4.5** ([7])**.** *Let $f : (X, T) \to (Y, S)$ be a fuzzy open function from a fuzzy topological space $(X, T)$ into a fuzzy topological space $(Y, S)$. Then for every fuzzy set $\beta$ in $(Y, S), f^{-1}(cl(\beta)) \leq clf^{-1}(\beta)$.*

**Proposition 4.6.** *Let $f : (X, T) \to (Y, S)$ be a fuzzy open function from a fuzzy topological space $(X, T)$ into a fuzzy topological space $(Y, S)$. Then for every fuzzy set $\delta$ in $(Y, S), int(f^{-1}(\delta)) \leq f^{-1}(int(\delta))$.*

*Proof.* Let $\delta$ be a fuzzy set in $(Y, S)$. Then for the fuzzy set $1 - \delta$, we have, by Theorem 4.5, $f^{-1}(cl(1 - \delta)) \leq clf^{-1}(1 - \delta)$. Then $f^{-1}(1 - int(\delta)) \leq cl(1 - f^{-1}(\delta))$. This implies that $(1 - f^{-1}(int(\delta)) \leq (1 - int(f^{-1}(\delta)))$. Hence we have $int(f^{-1}(\delta)) \leq f^{-1}(int(\delta))$. □

**Proposition 4.7.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is fuzzy continuous and fuzzy open and if $(Y, S)$ is a fuzzy $\sigma$-Baire space, then $(X, T)$ is a fuzzy $\sigma$-Baire space.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-first category set in $(Y, S)$. Then by Proposition 4.4, $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X, T)$. Since $f$ is a fuzzy open function, by Proposition 4.6, for the fuzzy set $\lambda$ in $(Y, S)$, we have $int(f^{-1}(\lambda)) \leq f^{-1}(int(\lambda)$.......$(A)$. Since $(Y, S)$ is a fuzzy $\sigma$-Baire space, by Theorem 3.2, $int(\lambda) = 0$. Then from $(A)$, we have $int(f^{-1}(\lambda)) \leq f^{-1}(0) = 0$. That is, $int(f^{-1}(\lambda)) = 0$. Hence, by Theorem 3.2, $(X, T)$ is a fuzzy $\sigma$-Baire space. □

**Proposition 4.8.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is fuzzy continuous, one-to-one and fuzzy closed and if $\mu$ is a fuzzy $\sigma$-nowhere dense set in $(X, T)$, then $f(\mu)$ is a fuzzy $\sigma$-nowhere dense set in $(Y, S)$.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-nowhere dense set in $(X, T)$. Then $\mu$ is a fuzzy $F_\sigma$-set in $(X, T)$ such that $int(\mu) = 0$. Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i$'s are fuzzy closed sets in $(X, T)$. Now $f(\mu) = f(\vee_{i=1}^{\infty}(\mu_i)) = \vee_{i=1}^{\infty}(f(\mu_i))$. Since the function $f$ is fuzzy closed and $\mu_i$'s are fuzzy closed sets in $(X, T)$, $f(\mu_i)$'s are fuzzy closed sets in $(Y, S)$. Hence $f(\mu)$ is a fuzzy $F_\sigma$-set in $(Y, S)$.

Now we claim that $int(f(\mu)) = 0$. Suppose that $int(f(\mu)) \neq 0$. Then there exists a fuzzy open set $\eta$ in $(Y, S)$ such that $\eta \leq f(\mu)$. Then $f^{-1}(\eta) \leq f^{-1}f(\mu)$. Since $f$ is one-to-one, $f^{-1}f(\mu) = \mu$. Hence $f^{-1}(\eta) \leq \mu$. Since $f$ is fuzzy continuous and $\eta$ is a fuzzy open set in $(Y, S)$, $f^{-1}(\eta)$ is a fuzzy open set in $(X, T)$. This implies that $int(\mu) \neq 0$ which is a contradiction to $int(\mu) = 0$ in $(X, T)$. Hence we must have $int(f(\mu)) = 0$. Therefore $f(\mu)$ is a fuzzy $F_\sigma$-set in $(Y, S)$ such that $int(f(\mu)) = 0$ and hence $f(\mu)$ is a fuzzy $\sigma$-nowhere dense set in $(Y, S)$. □

**Proposition 4.9.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is fuzzy continuous, one-to-one and fuzzy closed and if $\mu$ is a fuzzy $\sigma$-first category set in $(X, T)$, then $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y, S)$.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-first category set in $(X, T)$. Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i$'s are fuzzy $\sigma$-nowhere dense sets $(X, T)$. Now $f(\mu) = f(\vee_{i=1}^{\infty}(\mu_i)) = \vee_{i=1}^{\infty}(f(\mu_i))$. Since $f$ is a fuzzy continuous, one-to-one and fuzzy closed function and $\mu_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X, T)$, by Proposition 4.8, $f(\mu_i)$'s are fuzzy $\sigma$-nowhere dense sets in $(Y, S)$. Hence $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y, S)$. □

**Proposition 4.10.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is one-to-one, fuzzy continuous and fuzzy closed and if $(X, T)$ is a fuzzy $\sigma$-Baire space, then $(Y, S)$ is a fuzzy $\sigma$-Baire space.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-first category set in $(X, T)$. Since $(X, T)$ is a fuzzy $\sigma$-Baire space, by Theorem 3.2, we have $int(\mu) = 0$. Since $f$ is a fuzzy continuous, one-to-one and fuzzy closed function and $\mu$ is a fuzzy $\sigma$-first category set in $(X, T)$, by Proposition 4.9, $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y, S)$.

Now we claim that $int(f(\mu)) = 0$. Suppose that $int(f(\mu)) \neq 0$. Then there exists a fuzzy open set $\eta$ in $(Y, S)$ such that $\eta \leq f(\mu)$. Then we have $f^{-1}(\eta) \leq f^{-1}f(\mu)$. Since $f$ is one-to-one, $f^{-1}f(\mu) = \mu$. Then $f^{-1}(\eta) \leq \mu$. Since $f$ is fuzzy continuous and $\eta$ is fuzzy open, $f^{-1}(\eta)$ is fuzzy open in $(X, T)$. Hence $int(\mu) \neq 0$ which is a contradiction to $int(\mu) = 0$ in $(Y, S)$. Hence we must have $int(f(\mu)) = 0$. Therefore, for a fuzzy $\sigma$-first category set $f(\mu)$ in $(Y, S)$, we have $int(f(\mu)) = 0$. Hence, by Theorem 3.2, $(Y, S)$ is a fuzzy $\sigma$-Baire space. □

**Theorem 4.11** ([4]). *Suppose $(X,T)$ and $(Y,S)$ are two fuzzy topological spaces. Let $f:(X,T)\to(Y,S)$ be an onto function. Then the following conditions are equivalent:*

(1) *$f$ is somewhat fuzzy open.*
(2) *If $\lambda$ is a fuzzy dense set in $(Y,S)$, then $f^{-1}(\lambda)$ is a fuzzy dense set in $(X,T)$.*

**Proposition 4.12.** *If the function $f:(X,T)\to(Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is somewhat fuzzy open and if $int(\lambda)=0$ in $(Y,S)$, then $int(f^{-1}(\lambda))=0$ in $(X,T)$.*

*Proof.* Let $\lambda$ be a fuzzy set in $(Y,S)$ such that $int(\lambda)=0$. Then $1-int(\lambda)=1$ implies that $cl(1-(\lambda))=1$. That is, $1-(\lambda)$ is a fuzzy dense set in $(Y,S)$. Since $f$ is somewhat fuzzy open, by Theorem 4.11,$f^{-1}(1-(\lambda))$ is a fuzzy dense set in $(X,T)$. That is, $cl(f^{-1}(1-(\lambda)))=1$. This implies that $cl(1-f^{-1}(\lambda))=1$. Then $1-int(f^{-1}(\lambda))=1$. That is, $int(f^{-1}(\lambda))=0$. □

**Proposition 4.13.** *If the function $f:(X,T)\to(Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is fuzzy continuous and if $\lambda$ is a fuzzy $F_\sigma$-set in $(Y,S)$, then $f^{-1}(\lambda)$ is a fuzzy $F_\sigma$-set in $(X,T)$.*

*Proof.* Let $\lambda$ be a fuzzy $F_\sigma$-set in $(Y,S)$. Then $\lambda=\vee_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i$'s are fuzzy closed sets in $(Y,S)$. Now $\lambda=\vee_{i=1}^{\infty}(\lambda_i)$, implies that $f^{-1}(\lambda)=f^{-1}(\vee_{i=1}^{\infty}(\lambda_i))$. Then $f^{-1}(\lambda)=\vee_{i=1}^{\infty}(f^{-1}(\lambda_i))$. Since the function $f$ is fuzzy continuous and $\lambda_i$'s are fuzzy closed sets in $(Y,S), f^{-1}(\lambda_i)$'s are fuzzy closed sets in $(X,T)$. Hence $f^{-1}(\lambda)$ is a fuzzy $F_\sigma$-set in $(X,T)$. □

**Proposition 4.14.** *If the function $f:(X,T)\to(Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is a fuzzy continuous and somewhat fuzzy open function and if $\lambda$ is a fuzzy $\sigma$-nowhere dense set in $(Y,S)$, then $f^{-1}(\lambda)$ is a fuzzy $\sigma$-nowhere dense set in $(X,T)$.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-nowhere dense set in $(Y,S)$. Then $\lambda$ is a fuzzy $F_\sigma$-set in $(Y,S)$ such that $int(\lambda)=0$. Since $f$ is fuzzy continuous, by Proposition 4.13, $f^{-1}(\lambda)$ is a fuzzy $F_\sigma$-set in $(X,T)$. Since $f$ is somewhat fuzzy open, by Proposition 4.12, $int(f^{-1}(\lambda))=0$. Hence $f^{-1}(\lambda)$ is a fuzzy $\sigma$-nowhere dense set in $(X,T)$. □

**Proposition 4.15.** *If the function $f:(X,T)\to(Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is a fuzzy continuous and somewhat fuzzy open function and if $\lambda$ is a fuzzy $\sigma$-first category set in $(Y,S)$, then $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X,T)$.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-first category set in $(Y,S)$. Then $\lambda=\vee_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(Y,S)$. Now $f^{-1}(\lambda)=f^{-1}(\vee_{i=1}^{\infty}(\lambda_i))=\vee_{i=1}^{\infty}(f^{-1}(\lambda_i))$. Since $f$ is a fuzzy continuous, somewhat fuzzy open function and $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets $(Y,S), by Proposition 4.14, f^{-1}(\lambda_i)$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$. Hence $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X,T)$. □

**Proposition 4.16.** *If the function $f:(X,T)\to(Y,S)$ from a fuzzy topological space $(X,T)$ onto another fuzzy topological space $(Y,S)$ is a fuzzy continuous and* *somewhat fuzzy open function and if $(Y, S)$ is a fuzzy $\sigma$-Baire space, then $(X, T)$ is a fuzzy $\sigma$-Baire space.*

*Proof.* Let $\lambda$ be a fuzzy $\sigma$-first category set in $(Y, S)$. Since $(Y, S)$ is a fuzzy $\sigma$-Baire space, by Theorem 3.2, $int(\lambda) = 0$ in $(Y, S)$. Since $f$ is a fuzzy continuous and somewhat fuzzy open function, by Proposition 4.15, $f^{-1}(\lambda)$ is a fuzzy $\sigma$-first category set in $(X, T)$.

Now we claim that $int(f^{-1}(\lambda)) = 0$. Suppose that $int(f^{-1}(\lambda)) \neq 0$. Then there exists a fuzzy open set $\mu$ in $(X, T)$ such that $\mu \leq f^{-1}(\lambda)$. Then $f(\mu) \leq ff^{-1}(\lambda) \leq \lambda$. That is, $f(\mu) \leq \lambda$. Then $int(f(\mu)) \leq int(\lambda)$. Since $f$ is fuzzy somewhat open and $\mu$ is a fuzzy open set in $(X, T), int(f(\mu)) \neq 0$. This implies that $int(\lambda) \neq 0$ which is a contradiction to $int(\lambda) = 0$ in $(Y, S)$. Hence we must have $int(f^{-1}(\lambda)) = 0$. Hence, by Theorem 3.2, $(X, T)$ is a fuzzy $\sigma$-Baire space. □

**Theorem 4.17** ([4]). *Let $(X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a function. Then the following are equivalent:*

(1) *$f$ is somewhat fuzzy continuous.*
(2) *If $\lambda$ is a fuzzy closed set of $(Y, S)$ such that $f^{-1}(\lambda) \neq 1$, then there exists a proper fuzzy closed set $\mu$ of $(X, T)$ such that $\mu > f^{-1}(\lambda)$.*
(3) *If $\lambda$ is a fuzzy dense set in $(X, T)$, then $f(\lambda)$ is a fuzzy dense set in $(Y, S)$.*

**Proposition 4.18.** *If the function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ onto another fuzzy topological space $(Y, S)$ is somewhat fuzzy continuous one-to-one and if $int(\lambda) = 0$ in $(X, T)$, then $int(f(\lambda)) = 0$ in $(Y, S)$.*

*Proof.* Let $\lambda$ be a fuzzy set in $(X, T)$ such that $int(\lambda) = 0$. Then $1 - int(\lambda) = 1$ implies that $cl(1 - (\lambda)) = 1$. That is, $1 - (\lambda)$ is a fuzzy dense set in $(X, T)$. Since $f$ is somewhat fuzzy continuous, by Theorem 4.17, $f(1 - (\lambda))$ is a fuzzy dense set in $(Y, S)$. That is, $cl(f(1 - (\lambda))) = 1$. Since $f$ is one-to-one and onto, $f(1 - \lambda) = 1 - f(\lambda)$. Then $cl(1 - f(\lambda)) = 1$ and hence $1 - int(f(\lambda)) = 1$. That is, $int(f(\lambda)) = 0$ in $(Y, S)$. □

**Proposition 4.19.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is somewhat fuzzy continuous, one-to-one, onto and fuzzy closed and if $\mu$ is a fuzzy $\sigma$-nowhere dense set in $(X, T)$, then $f(\mu)$ is a fuzzy $\sigma$-nowhere dense set in $(Y, S)$.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-nowhere dense set in $(X, T)$. Then $\mu$ is a fuzzy $F_\sigma$-set in $(X, T)$ such that $int(\mu) = 0$. Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i$'s are fuzzy closed sets in $(X, T)$. Now $f(\mu) = f(\vee_{i=1}^{\infty}(\mu_i)) = \vee_{i=1}^{\infty}(f(\mu_i))$. Since the function $f$ is fuzzy closed and $\mu_i$'s are fuzzy closed sets in $(X, T), f(\mu_i)$'s are fuzzy closed sets in $(Y, S)$ .Hence $f(\mu)$ is a fuzzy $F_\sigma$-set in $(Y, S)$. Since the function $f$ is somewhat fuzzy continuous, one-to-one, onto and since $int(\mu) = 0$ in $(X, T)$, by Proposition 4.18 , we have $int(f(\mu)) = 0$ in $(Y, S)$. Therefore $f(\mu)$ is a fuzzy $F_\sigma$-set in $(Y, S)$ such that $int(f(\mu)) = 0$ and hence $f(\mu)$ is a fuzzy $\sigma$-nowhere dense set in $(Y, S)$. □

**Proposition 4.20.** *If a function $f : (X, T) \to (Y, S)$ from a fuzzy topological space $(X, T)$ into another fuzzy topological space $(Y, S)$ is somewhat fuzzy continuous, one-to-one, onto and fuzzy closed and if $\mu$ is a fuzzy $\sigma$-first category set in $(X, T)$, then $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y, S)$.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-first category set in $(X,T)$. Then $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i$'s are fuzzy $\sigma$-nowhere dense sets $(X,T)$. Now $f(\mu) = f(\vee_{i=1}^{\infty}(\mu_i)) = \vee_{i=1}^{\infty}(f(\mu_i))$. Since $f$ is a somewhat fuzzy continuous, one-to-one and fuzzy closed function and $\mu_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X,T)$, by Proposition 4.19, $f(\mu_i)$'s are fuzzy $\sigma$-nowhere dense sets in $(Y,S)$. Hence $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y,S)$. □

**Proposition 4.21.** *If a function $f : (X,T) \to (Y,S)$ from a fuzzy topological space $(X,T)$ into another fuzzy topological space $(Y,S)$ is somewhat fuzzy continuous, one-to-one, onto and fuzzy closed and if $(X,T)$ is a fuzzy $\sigma$-Baire space , then $(Y,S)$ is a fuzzy $\sigma$-Baire space.*

*Proof.* Let $\mu$ be a fuzzy $\sigma$-first category set in $(X,T)$. Since $(X,T)$ is a fuzzy $\sigma$-Baire space, by Theorem 3.2, we have $int(\mu) = 0$. Since $f$ is a somewhat fuzzy continuous, one-to-one and fuzzy closed function and $\mu$ is a fuzzy $\sigma$-first category set in $(X,T)$, by Proposition 4.20, $f(\mu)$ is a fuzzy $\sigma$-first category set in $(Y,S)$.

Now we claim that $int(f(\mu)) = 0$. Suppose that $int(f(\mu)) \neq 0$. Then there exists a fuzzy open set $\eta$ in $(Y,S)$ such that $\eta \leq f(\mu)$. Then we have $f^{-1}(\eta) \leq f^{-1}f(\mu)$. Since $f$ is one-to-one, $f^{-1}f(\mu) = \mu$. Then $f^{-1}(\eta) \leq \mu$. Since $f$ is somewhat fuzzy continuous and $\eta$ is fuzzy open, there exists a fuzzy open set $\delta$ in $(X,T)$ such that $\delta \leq f^{-1}(\eta)$. Then $\delta \leq \mu$.Hence $int(\mu) \neq 0$ which is a contradiction to $int(\mu) = 0$ in $(X,T)$. Hence we must have $int(f(\mu)) = 0$. Therefore, for a fuzzy $\sigma$-first category set $f(\mu)$ in $(Y,S)$, we have $int(f(\mu)) = 0$. Hence, by Theorem 3.2, $(Y,S)$ is a fuzzy $\sigma$-Baire space. □

## 5. Conclusions

In this paper, the interrelation between fuzzy $\sigma$-Baire spaces and fuzzy Volterra spaces are studied. Some results concerning fuzzy closed, fuzzy continuous, somewhat fuzzy continuous and somewhat fuzzy open functions that preserve $\sigma$-Baire spaces in the context of images and pre-images are established.

## References

[1] K. K. Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika (Prague) 31(5) (1995) 459–464.
[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[4] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11(2) (2003) 1–12.
[5] G. Thangaraj and E. Poongothai, On fuzzy $\sigma$-Baire spaces, Inter. J. Fuzzy Math. Sys. 3(4) (2013) 275–283.
[6] G. Thangaraj and S. Soundararajan, On fuzzy volterra spaces, J. Fuzzy Math. 21(4) (2013) 895–904.
[7] T. H. Yalvac, Semi-interior and semi-closure of a fuzzy set, J. Math. Anal. Appl. 132(2) (1988) 356–364.
[8] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

G.Thangaraj (g.thangaraj@rediffmail.com)
Professor and Head, Department of Mathematics, Thiruvalluvar University, Serkkadu, Vellore-632 115, Tamil Nadu, India

E. Poongothai (epoongothai@gmail.com)
Assistant Professor, Department of Mathematics, Shanmuga Industries Arts and Science College, Thiruvannamalai-606 601, Tamil Nadu, India