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Covering dimension of intuitionistic fuzzy topological spaces

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Abstract. The main purpose of this paper is to introduce and study the concept of covering dimension and zero dimensionality in the intuitionistic fuzzy topological spaces. Main results are obtained.

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1. Introduction

After introducing the fuzzy sets by Zadeh [24] in 1965 and fuzzy topology by Chang [11] in 1967, several studies have been conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [4, 5] as a generalization of fuzzy sets. In the last 20 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [12] introduced the concept of intuitionistic fuzzy topological spaces. Recently many fuzzy topological concepts such as, fuzzy compactness [14], fuzzy connectedness [23], separation axioms [7], fuzzy metric spaces [22], fuzzy continuity [16] fuzzy multifunctions [17] have been generalized for intuitionistic fuzzy topological spaces. Some authors have studied the developments of dimension theory in fuzzy topological spaces. Adnadjevic [1, 2] introduced the concept of generalized fuzzy spaces (GF spaces) and defined two dimension functions, *F-ind* and *F-Ind*. Later, Cuchillo and Tarres [15] extended them into fuzzy topological spaces in the case of zero dimensionality. Ajmal and Kohli [3] have studied the concept of covering dimension in fuzzy topological spaces, in [8] T Baiju and Sunil Jacob John studied the covering dimension of fuzzy topological space by using concept of Q- covering [9], and in [10] they studied the covering dimension of normality in L-topological spaces (for further studies, see [20, 21]). However, all these studies have been carried out in the fuzzy topological spaces and in L-topological spaces. In the present paper, we have introduce and investigate the concepts of intuitionistic fuzzy covering dimension of intuitionistic fuzzy topological space, in view of the definition of Chang [11] and study some of their properties.

2. Preliminaries

Definition 2.1 ([4, 5]). Let X be a nonempty fixed set. An intuitionistic fuzzy set A (IFS for short) in X is an object having the form $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$ where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, every fuzzy set $A = \{< \mu_A(x), x >: x \in X\}$ on X is an intuitionistic fuzzy set of the form

$$A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}.$$

Definition 2.2 ([6]). Let X be a non-empty set, and consider the intuitionistic fuzzy sets $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \gamma_B(x) > x \in X\}$ and let $\{A_\lambda : \lambda \in A\}$ be an arbitrary family of intuitionistic fuzzy sets in X. Then

(a) $A \subseteq B$ if $[\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)], \forall x \in X$
(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$
(c) $A^c = \{< x, \gamma_A(x), \mu_A(x) >: x \in X\}$
(d) $\cap A_\lambda = \{< x, \wedge\mu_{A_\lambda}(x), \vee\gamma_{A_\lambda}(x) >: x \in X\}$, in particular
$A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) >: x \in X\}$
(e) $\cup A_\lambda = \{< x, \vee\mu_{A_\lambda}(x), \wedge\gamma_{A_\lambda}(x) >: x \in X\}$, in particular
$A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) >: x \in X\}$
(f) $0_{\tilde{X}} = \{< x, 0, 1 >: x \in X\}$ and $1_{\tilde{X}} = \{< x, 1, 0 >: x \in X\}$

Definition 2.3 ([13]). Let $\alpha, \beta \in [0, 1]$, $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP for short) of nonempty set X is an IFS of X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } x = y \\ (0, 1) & \text{if } x \neq y \end{cases}$$

In this case, x is called the support of $x_{(\alpha,\beta)}$ and α, β are called the value and non-value of $x_{(\alpha,\beta)}$ respectively.

Clearly an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy point as follows:

$$x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{1-\beta})$$

In IFP $x_{(\alpha,\beta)}$ is said to belong to an IFS $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$ denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Definition 2.4 ([6]). Let X and Y be two nonempty sets and $f : X \to Y$ be a function. Then

(a) If $B = \{< y, \mu_B(y), \gamma_B(y) >: y \in Y\}$ is an intuitionistic fuzzy set in Y, then the preimage of B under f denoted by $f^{-1}(B)$ is the IFS in X defined by

$$f^{-1}(B) = \{< x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x) : x \in X\}$$

(b) If $A = \{< x, \lambda_A(x), \nu_A(x) >: x \in X\}$ is an intuitionistic fuzzy set in X, then the image of A under f denoted by $f(A)$ is the intuitionistic fuzzy set in Y defined by

$$f(A) = \{< y, f(\lambda_A)(y), 1 - f(1 - \nu_A)(y) >: y \in Y\}$$

Where,

$$f(\lambda_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\lambda_A(x)\} & \text{if } f^{-1}(x) \neq \varnothing, \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$1 - f(1 - \nu_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\lambda_A(x)\} & \text{if } f^{-1}(x) \neq \varnothing, \\ 1 \quad , & \text{otherwise} \end{cases}$$

Definition 2.5 ([12]). An intuitionistic fuzzy topology (IFT for short) on a non-empty set X is a family of intuitionistic fuzzy sets in X satisfy the following axioms:

(T1) $0_{\widetilde{X}}, 1_{\widetilde{X}} \in \tau$

(T2) If $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

(T3) If $A_\lambda \in \tau$ for each λ in Λ, then $\bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \tau$.

In this case the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS for short) and each intuitionistic fuzzy set in τ is known as an intuitionistic fuzzy open sets (IFOSs for short) of X, and the complement an intuitionistic fuzzy closed set (IFCSs for short).

Example 2.6. Let $X = [0, 1]$ and let $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}, B = \{< x, \mu_B(x), \gamma_B(x) > x \in X\}$ be IFSS on X defined by

$$\mu_A(x) = \begin{cases} 0 & , \quad 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases},$$

$$\gamma_A(x) = \begin{cases} 1 & , \quad 0 \leq x \leq \frac{1}{2} \\ 2(1 - x) & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_B(x) = \begin{cases} 1 & , \quad 0 \leq x \leq \frac{1}{4} \\ 2 - 4x & , \quad \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases},$$

$$\gamma_B(x) = \begin{cases} 0 & , \quad 0 \leq x \leq \frac{1}{4} \\ 4x - 1 & , \quad \frac{1}{4} \leq x \leq \frac{1}{2} \\ 1 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then $\tau = \{0_{\widetilde{X}}, 1_{\widetilde{X}}, B, A \cup B\}$ is an IFTS on X.

Definition 2.7. An intuitionstic fuzzy set in X is said to be clopen if it is intuitionstic fuzzy closed and open.

Definition 2.8 ([12]). Let (X, τ) be an IFTS and $A =< x, \mu_A(x), \gamma_A(x) >$ be an IFS in X. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of A are defined by

$$cl(A) = \cap\{F : F \text{ is an IFCs in } X \text{ and } A \subseteq F\},$$
$$int(A) = \cup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}$$

Definition 2.9 ([14]). Let (X, τ) and (Y, φ) be intuitionistic fuzzy topological spaces then a map $f : X \to Y$ is said to be

(1) Continuous if $f^{-1}(B)$ is an intuitionistic fuzzy open set in X, for each intuitionistic fuzzy open set B in Y, or equivalently, $f^{-1}(B)$ is an intuitionistic fuzzy closed set in X, for each intuitionistic fuzzy closed set B in Y,

(2) Open if $f(A)$ is an intuitionistic fuzzy open set in Y, for each intuitionistic fuzzy open set A in X,

(3) Closed if $f(A)$ is an intuitionistic fuzzy closed set in Y for each intuitionistic fuzzy closed set A in X,

(4) A homeomorphism if f is bijective, continuous, and open.

Definition 2.10 ([13]). Two intuitionstic fuzzy sets A and B of X said to be quasi-coincident AqB if and only if there exists an element x in X such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$. In this, case A and B are said to be quasi-coinciden at x.

Otherwise A is not quasi-coincident with B, denote $A\overline{q}B)$ and. $A\overline{q}B$ if and only if $A \subseteq B^c$.

Remark 2.11. If AqB, we have that $A \cap B \neq 0_{\widetilde{X}}$. ($\mu_A q \mu_B$ implies that $\mu_A \wedge \mu_B \neq 0_{\widetilde{X}}$, then $A \cap B \neq 0_{\widetilde{X}}$).

Definition 2.12. Let X be a nonempty set. A family $\mathbf{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of IFS in X is said to be quisi-coincident family if there exists, $x \in X$ such that $\mu_{U_\alpha}(x) > \gamma_{U_\beta}(x)$, for all $\alpha, \beta \in \Lambda$.

A family $\{U_\lambda\}_{\lambda \in \Lambda}$ is not quisi-coincident that is for every $x \in X$, there exist $\alpha, \beta \in \Lambda$. Such that

$$\mu_{U_\alpha}(x) \leq \gamma_{U_\beta}(x)$$

Definition 2.13 ([12]). A family $\mathbf{U} = \{U_\lambda : \lambda \in \Lambda\}$ where $U_\lambda = \langle x, \mu_{U_\lambda}, \gamma_{U_\lambda} \rangle, (\lambda \in \Lambda)$ of IFOS in IFTS $\mathbf{X} = (X, \tau)$ is said to be an intuitionistic fuzzy open cover (or intuitionistic covering for short) of IFS A if and only if $A \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$.

A sub cover of an intuitionistic fuzzy open cover $\mathbf{U}$ of A is a subfamily of $\mathbf{U}$, which is still an intuitionistic fuzzy open cover of A.

Definition 2.14. A family $\mathbf{U} = \{U_\lambda : \lambda \in \Lambda\}$ of IFOSs in IFTS $\mathbf{X} = (X, \tau)$ is said to be an intuitionistic fuzzy open cover of $\mathbf{X}$ if and only if $\bigcup_{\lambda \in \Lambda} U_\lambda = 1_{\widetilde{X}}$, and a collection $\mathbf{V} = \{V_\alpha : \alpha \in \Delta\}$ where $\mathbf{V}_\alpha = \langle x, \mu_{V_\alpha}, \gamma_{V_\alpha} \rangle, (\alpha \in \Delta)$ is said to be intuitionistic fuzzy refinement of $\mathbf{U}$ if $\bigcup_{\alpha \in \Delta} V_\alpha = 1_{\widetilde{X}}$, and each V_α is contained in some members U_λ of $\mathbf{U}$.

Definition 2.15. Let Y be IFS in IFTS $\mathbf{X} = (X, \tau)$ then the set of restriction $\tau_Y = \{U|Y : U \in \tau\}$ is an intuitionistic fuzzy topology on Y, and $\mathbf{Y} = (Y, \tau_Y\}$ is called an intuitionistic fuzzy subspace of IFTS. $\mathbf{X} = (X, \tau)$.

3. Intuitionsitic fuzzy covering dimension

Definition 3.1. Let X be a nonempty set. A family $\mathbf{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of IFSs in X is said to be of order $n(n > -1)$ written $ord_{\mathrm{If}}\mathbf{U} = n$, if n is the greatest integer such that there exists a quisi-coinsident subfamily of $\mathbf{U}$ having $n + 1$ elements.

Remark 3.2. From the above definition if $ord_{\mathrm{If}}\mathbf{U} = n$ then for each $n+2$ distinct indexes $\lambda_1, \lambda_2, \ldots, \lambda_{n+2} \in \Lambda$ we have $U_{\lambda_1} \cap U_{\lambda_2} \cap \ldots \cap U_{\lambda_{n+2}} = \varnothing$, then it is non quisi-coinsident, in particular if $ord_{\mathrm{If}}\mathbf{U} = -1$, then $\mathbf{U}$ consists of the empty IFS and $ord_{\mathrm{If}}\mathbf{U} = 0$, then $\mathbf{U}$ consist of pairwise disjoint IFSs which are not all empty.(For ordinary case see [18])

Definition 3.3. The covering dimension of a IFTS $X = (X, \tau)$ denoted $dim_{\mathrm{If}}(X)$ is the least integer n such that every finite intuitionistic fuzzy open cover of $1_X^{\sim}$ has a finite open refinement of order not exceeding n or $+\infty$ if there exists no such integer.

Thus it follows that $dim_{\mathrm{If}}(\mathbf{X}) = -1$ if and only if $X = \varnothing$ and $dim_{\mathrm{If}}(\mathbf{X}) \leq n$ if every finite intuitionistic fuzzy open cover of $1_X^{\sim}$ has a finite open refinement of order $\leq n$. We have $dim_{\mathrm{If}}(\mathbf{X}) = n$ if it is true that $dim_{\mathrm{If}}(\mathbf{X}) \leq n$, but it is false that $dim_{\mathrm{If}}(\mathbf{X}) \leq n-1$.

Finally $dim_{\mathrm{If}}(\mathbf{X}) = +\infty$ if for every positive integer n it is false that $dim_{\mathrm{If}}(\mathbf{X}) \leq n$.

Remark 3.4. The notion of covering dimension of a IFTS X is an *intuitionistic fuzzy* topological invariant. Moreover, the covering dimension of a topological space is n if and only if the covering dimension of its characteristic IFTS is n.

The following theorem is an important result in this paper.

Theorem 3.5. *The following statements are equivalent for IFTS* $\mathbf{X} = (X, \tau)$

(1) $dim_{\mathrm{If}}(\mathbf{X}) \leq n$.

(2) *For every finite intuitionistic fuzzy open cover* $\{U_1, \ldots, U_k\}$ *of* $1_X^{\sim}$ *there exists a finite intuitionistic fuzzy open cover* $\{\eta_1, \ldots, \eta_k\}$ *of* $1_X^{\sim}$ *of order less than or equal to* n *and* $\eta_i \leq U_i$ *for* $i = 1, 2, \ldots, k$.

(3) *If* $\{U_1, \ldots, U_{n+2}\}$ *is an intuitionistic fuzzy open cover of* $1_X^{\sim}$ *then there exists a non- quasi-coinsident intuitionistic fuzzy open cover* $\{\eta_1, \ldots, \eta_{n+2}\}$ *of* $1_X^{\sim}$ *such that* $\eta_i \leq U_i$, *for* $i = 1, 2, \ldots, n+2$

Proof. (1) $\Rightarrow$ (2) Suppose that $dim_{\mathrm{If}}(\mathbf{X}) \leq n$ and let $\mathbf{U} = \{U_1, \ldots, U_k\}$ be an intuitionistic fuzzy open cover of $1_X^{\sim}$. Let $\mathbf{V}$ be a finite intuitionistic fuzzy open refinement of $\mathbf{U}$ such that $ord_{\mathrm{If}}(\mathbf{V}) \leq n$, if $V \in \mathbf{V}$, then $V \subseteq U_i$, for some i, let each $V \in \mathbf{V}$ be associated with one IFSs U_i containing it. Let η_i be the union of all those members of V thus associated with U_i. Then each η_i is an IFOS and $\eta_i \subseteq U_i$.

Let $\mathbf{N} = \{\eta_i, \ldots, \eta_k\}$, we want to show that $ord_{\mathrm{If}}\mathbf{N} \leq n$, that is, every quasi-coinsident subfamily of $\mathbf{N}$ contains at most $n+1$ members.

Suppose if possible, there exists a quisi-coinsident subfamily N_i of $\mathbf{N}$ containing $(n+2)$ members. Then there exists $x \in X$ such that

$$\mu_{\eta_\alpha}(x) + \mu_{\eta_\beta}(x) > 1, \text{ i.e. } \mu_{\eta_\alpha}(x) > \gamma_{\eta_\beta}(x)$$

for every pair $\eta_\alpha, \eta_\beta \in N_i$.

Now, since

$$\eta_\sigma = \bigcup \{V_{i_\sigma} \in V : V_{i_\sigma} \leq U_i, \text{ as associated in the construction of } \eta_\sigma\},$$

$(\sigma = \alpha, \beta)$, and since $\mathbf{V}$ is a finite cover of $1_X^{\sim}$ and

$$\mu_{\eta_\beta}(x) = max\{\mu_{V_{1_\beta}}(x), \ldots, \mu_{V_{s_\beta}}(x)\}$$

Choose V_{k_α} and V_{t_β} such that

$$\mu_{\eta_\alpha}(x) = \mu_{V_{k_\alpha}}(x) \text{ and } \mu_{\eta_\beta}(x) = \mu_{V_{t_\beta}}(x).$$

Clearly V_{k_α} and V_{t_β} quasi-coincident at x.

In this way we obtain corresponding to every quasi-coincident pair η_α, η_β at x, a pair V_{i_α} and V_{j_β} of V's which are distinct in themselves as well as distinct from others and quasi-coincident at x.

The collection of all these members of $\mathbf{V}$ chosen above constitute a quasi-coincident subfamily of $\mathbf{V}$ having $n+2$ members.

This is contradiction to the fact that $ord_{\mathrm{If}}(\mathbf{V}) \leq n$. Thus $ord_{\mathrm{If}}(\mathbf{V}) \leq n$.

The statement (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are trivial.

To complete the proof, we will show that (3) $\Rightarrow$ (2). To this end. Let $\mathbf{X}$ be a IFTS. satisfying (3), and let $\{U_1, \ldots, U_k\}$ be an intuitionistic fuzzy open cover of $1_{\tilde{X}}$. Assume that $k > n+1$.

Let $\delta_i = U_i$ if $1 \leq i \leq n+1$.

And let $\delta_{n+2} = \bigcup_{i=n+2}^{k} U_i$. Clearly, $\{\delta_1, \ldots, \delta_{n+2}\}$ is an intuitionistic fuzzy open cover of $1_{\tilde{X}}$.

By hypothesis, there exists an open cover $\mathbf{N} = \{\eta_1, \ldots, \eta_{n+2}\}$ of $1_{\tilde{X}}$ such that $\eta_i \leq \delta_i$ for each i, and $\mathbf{N}$ is non-quasi-coincident family.

Define intuitionistic fuzzy open sets $V_i = U_i$ if $1 \leq i \leq n+1$, and $V_i = U_i \cap \eta_{n+2}$ if $i > n+1$

Then $\mathbf{V} = \{V_1, \ldots, V_k\}$ is intuitionistic fuzzy open cover of $1_{\tilde{X}}$ such that $V_i \subseteq U_i$, for each i and the subfamily $\{V_1, \ldots, V_{n+2}\}$ of v is non-quasi-coincident.

If there is a subset $\mathbf{B}$ of $\{1, \ldots, K\}$ having $n+2$ elements such that the family $\{V_j : j \in B\}$ is quasi-coincident, then let the members of v be renumbered to give a family $P = \{p_1, \ldots, p_K\}$ such that the subfamily $\{p_1, \ldots, p_{n+2}\}$ is quasi-coincident.

By applying above construction to P, we obtain an intuitionistic fuzzy open cover $V' = \{V'_1, \ldots, V'_k\}$ of $1_{\tilde{X}}$ such that $V'_i \subseteq P_i$ and the subfamily $\{V'_1, \ldots, V'_{n+2}\}$ is non-quasi-coincident. Clearly, if C is a subset of $\{1, \ldots, K\}$ with $n+2$ elements such that the family $\{p_i : i \in C\}$ is non-quasi-coincident, then so is the family $\{V'_i : i \in C\}$. Thus by a finite number of repetitions of this process we obtain an intuitionistic fuzzy open cover $\mathbf{W} = \{W_1, \ldots, W_K\}$ of $1_{\tilde{X}}$ such that $W_i \leq U_i$ for each i and $ord_{\mathrm{If}}(\mathbf{W}) \leq n$. □

The following proposition follows directly from Theorem 3.5.

Proposition 3.6. *Let $Y = (Y, \tau_Y)$ be a intuitionistic closed fuzzy subspace of IFTS $X = (X, \tau)$. Then $dim_{If}(Y) \leq dim_{If}(X)$.*

Proof. Since Y is an intuitionistic closed fuzzy subspace of X, μ_Y is a intuitionistic closed fuzzy set in X. We must show that if $dim_{\mathrm{If}}(X) \leq n$, then $dim_{\mathrm{If}}(Y) \leq n$. Clearly if $dim_{\mathrm{If}}(X) = -1$ then $X = \varnothing$ and hence $Y = \varnothing$, then $dim_{\mathrm{If}}(\mathbf{Y}) = -1$, and if $dim_{\mathrm{If}}(\mathbf{X}) = \infty$, then the theorem is obvious. Suppose that $dim_{\mathrm{If}}(\mathbf{X}) \leq n < \infty$, and let $U^Y = \{U_1^Y, \ldots, U_n^Y\}$ be an intuitionistic fuzzy open cover of $1_{\tilde{Y}}$. Then $\mathbf{U} = \{U_I, \ldots, U_K, Y^c\}$ is an intuitionistic fuzzy open cover of $1_{\tilde{X}}$, and so $\mathbf{U}$ has an intuitionistic fuzzy open refinement V such that $ord_{\mathrm{If}}(\mathbf{V}) \leq n$, let $\mathbf{V}^Y = \{V|Y : v \in V\}$, we claim that $ord_{\mathrm{If}} V^Y \leq n$. Let $\{V_{i_1}^Y, \ldots, V_{i_{n+2}}^Y\}$ be a subfamily of V^Y, since $ord_{\mathrm{If}} v \leq n$, and since $\{V_i, \ldots, V_{i_{n+2}}\}$ is a subfamily of v having $n+2$ members which is non-quasi-coincident. That is for each $x \in X$ and in particular for each $x \in Y$

there exists subscripts i_q and i_r such that

$$\mu_{V_{i_q}}(x) + \mu_{V_{i_r}}(x) \leq 1 \text{ .i.e. } \mu_{V_{i_q}}(x) \leq \gamma_{V_{i_r}}(x).$$

This in turn implies that every subfamily of v^Y having $n+2$ members is non-quasi-coincident and hence $ord_{\mathrm{If}} V^Y \leq n$. Thus $dim_{\mathrm{If}}(\mathbf{Y}) \leq dim_{\mathrm{If}}(\mathbf{X})$. □

Remark 3.7. The condition that Y is closed is necessary for this theorem to hold, as well as in classical topological space (see Pears [18]).

We prove the following theorem.

Theorem 3.8. *If $X = (X, \tau)$ and $Y = (Y, \sigma)$ are two IFTS, and f is an ituistionistic fuzzy homeomorphism between them, then $dim_{If}(X) = dim_{If}(Y)$.*

Proof. If $f : X \to Y$ is an IF-home. And U is a finite
intuitionistic fuzzy open cover of $1_{\tilde{Y}}$, then $f^{-1}(U)$ is a finite
intuitionistic fuzzy open cover of $1_{\tilde{X}}$, also if A is a finite
intuitionistic fuzzy open cover of $1_{\tilde{X}}$ then $f(A)$ is a finite
intuitionistic fuzzy open cover of $1_{\tilde{Y}}$. (See. 2.4)
The case $dim_{\mathrm{If}}(\mathbf{X}) = \infty$ follows directly

Now, clearly if $dim_{\mathrm{If}}(\mathbf{X}) = -1$ and $\mathbf{Y}$ is IF homeomorphic to $\mathbf{X}$ then $Y = \varnothing$ and $dim_{\mathrm{If}}(\mathbf{Y}) = -1$, since as the only IFTS.X homeomorphic to the empty IFTS. is itself, and as the only IFTS, with covering dimension -1 is the empty IFTS.

Let $dim_{\mathrm{If}}(\mathbf{X}) = n$ and let $\mathbf{Y}$ be IFTS. homeomorphic to $\mathbf{X}$ then clearly $dim_{\mathrm{If}}(\mathbf{Y}) \geq n$. To show that $dim_{\mathrm{If}}(\mathbf{Y}) \leq n$.

Let A be a finite intuitionistic fuzzy open cover of $1_{\tilde{X}}$, since f is an ituistionistic fuzzy homeomorphism then $f^{-1}(\mathbf{U}) = A$ for a finite intuitionistic fuzzy open cover U of $1_{\tilde{Y}}$.

As $dim_{\mathrm{If}}(\mathbf{X}) = n$, then there is an intuitionistic fuzzy open refinement B of A such that $ord_{\mathrm{If}} B \leq n$, as f is an intuitionistic fuzzy homeomorphism then $B = f^{-1}(W)$, $\mathbf{W}$ is an intuitionistic fuzzy open refinement of $\mathbf{U}$, and the order of B is homeomorphic to the order of $\mathbf{W}$ then $ord_{\mathrm{If}} W \leq n$ and hence $dim_{\mathrm{If}}(\mathbf{Y}) \leq n = dim_{\mathrm{If}}(\mathbf{X})$, as $dim_{\mathrm{If}}(\mathbf{Y}) \geq n = dim_{\mathrm{If}}(\mathbf{X})$, it follows that $dim_{\mathrm{If}}(\mathbf{X}) = dim_{\mathrm{If}}(\mathbf{Y})$. □

4. Zero dimensional in ituitionistic fuzzy topological spaces

Many authors studied zero- dimensionality in fuzzy topological spaces such as L.Pujate and A.B.Šostak[19],they used precisely the definition for a zero dimensional Chang's space, that is, a Chang's space is zero dimensional if there is a base for the space consisting of clopen fuzzy sets, we study this concepts for covering dimension in IFTS. According to covering dimension of IFTS, we introduced some results about zero-dimensional in Ifts as follows.

Proposition 4.1. *Let $X = (X, \tau)$ be a IFTS, then $dim_{If}(X) = 0$ if and only if every finite intuitionistic fuzzy open cover of $1_{\tilde{X}}$ has a refinement consisting of disjoint crisp clopen intuistionistic fuzzy sets.*

Proof. (⇐) By Remark (3.2) if $\mathbf{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ is a disjoint crisp intuitionistic clopen cover of $1_{\tilde{X}}$ then $order_{\mathrm{f}} \mathbf{U} = 0$ and hence $dim_{\mathrm{If}}(\mathbf{X}) = 0$.

($\Rightarrow$) let $\mathbf{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a finite *intuitionistic* fuzzy open cover of $1_{\tilde{X}}$, since $dim_{\rm If}(\mathbf{X}) = 0$ there exists a finite *intuitionistic* fuzzy open refinement $\mathbf{V} = \{V_1, \ldots, V_k\}$ of $\mathbf{U}$ such that $ord_{\rm f}V = 0$, it follows that every pair of elements of $\mathbf{V}$ are non-quasi-coincident. Now, we show that each member of $\mathbf{V}$ is crisp *intuitionistic* clopen fuzzy set, let $V_i \in \mathbf{V}$ then V_i is non-quasi-coincident with the union of remaining members of $\mathbf{V}$ which is an *intuitionistic* fuzzy open set, since V is also *intuitionistic* fuzzy cover of $1_{\tilde{X}}$ i.e. $V_i \bigcup(\bigcup_{i\neq j} V_j) = 1_{\tilde{X}}$, and since for each i, V_i is non-quasi-coincident with $\bigcup_{i\neq j}V_j$, we have $V_i \bigcap(\bigcup_{i\neq j}V_j) = 0$, for each i.

Hence $V_i + \bigcup_{i\neq j}V_j = 1_{\tilde{X}}$, by definition each $V_i = 1_{\tilde{X}} - \bigcup_{i\neq j}V_j$ is crisp and clopen intuitionistic fuzzy set in $\mathbf{X}$ and by Remark (3.2) the members of $\mathbf{V}$ are pairwise disjoint. □

For a singleton space, we prove the following proposition.

Proposition 4.2. *If $X = \{x\}$ is singleton space and $\mathbf{X} = (X, \tau)$ is an IFTS Then $dim_{\rm If}(\mathbf{X}) = 0$.*

Proof. Let $\mathbf{U} = \{U\}$ be a singleton family of *intuitionistic* fuzzy open sets which is cover of $1_{\tilde{X}}$, there is an intuitionistic fuzzy open refinement of U which is U, then $U = 1_{\tilde{X}}$ but the $ord_{\rm If}\{U\} = 0$ it follows that $dim_{\rm If}(\mathbf{X}) = 0$. □

Now, we give and prove the following theorem.

Theorem 4.3. *A closed subspace $\mathbf{Y} = (Y, \tau_Y)$ of zero dimensional IFTS $\mathbf{X} = (X, \tau)$ is also zero-dimensional.*

Proof. Let $\mathbf{U}^Y = \{U_1^Y, \ldots, U_n^Y\}$ be an intuitionistic fuzzy open cover of $1_{\tilde{Y}}$, then $\mathbf{U} = \{U_1, \ldots, U_n, Y^C\}$ is an intuitionistic fuzzy open cover of $1_{\tilde{X}}$, by Proposition (4.1) $\mathbf{U}$ has an intuitionistic fuzzy open refinement $\mathbf{V}$ consisting of disjoint crisp clopen intuistionistic fuzzy sets such that $ord_{\rm f}\mathbf{V} = 0$, let $\mathbf{V}^Y = \{V|Y, V \in \mathbf{V}\}$ we claim that $ord_{\rm If}V^Y = 0$, since $\mathbf{V}$ consisting of disjoint crisp clopen intuistionistic fuzzy sets, this implies that V^Y consisting of disjoint crisp clopen intuistionistic fuzzy set, since $ord_{\rm If}\mathbf{V} = 0$ and $dim_{\rm If}(\mathbf{X}) = 0$, then by Proposition (4.2) we get $ord_{\rm If}V^Y = 0$ and $dim_{\rm If}(\mathbf{Y}) = 0$. □

In the final of this section we prove the following proposition.

Proposition 4.4. *If $dim_{\rm If}(\mathbf{X}) = 0$, then for every pair A_1, A_2 of disjoint intuitionistic closed fuzzy sets, there exist crisp clopen intuitionistic fuzzy sets B_1 and B_2 such that $A_1 \subseteq B_1, A_2 \subseteq B_2$ and $B_1 + B_2 = 1_{\tilde{X}}$.*

Proof. Let A_1, A_2 be a pair of disjoint intuitionistic closed fuzzy sets in $\mathbf{X}$, then $\{A_1^c, A_2^c\}$ is an intuitionistic fuzzy open cover of $1_{\tilde{X}}$, since $dim_{\rm If}(\mathbf{X}) = 0$ by Proposition (4.1) the cover $\{A_1^c, A_2^c\}$ has an intuistionistic fuzzy open refinement $\{B_1, B_2\}$ consisting of crisp clopen intuitionistic fuzzy sets, such that $B_1 \cap B_2 = 0_{\tilde{X}}$, and $B_1 + B_2 = 1_{\tilde{X}}$, thus $B_1 \subseteq A_2^c$, $B_2 \subseteq A_1^c$, this implies that $A_2 \subseteq B_1^c = B_2$ and $A_1 \subseteq B_2^c = B_1$ i.e. $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. □

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