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Common fixed point theorems using property $(E.A.)$ and its variants involving quadratic terms

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Abstract. The purpose of this paper is to utilize the property $(E.A.)$ and its variants to prove existence of coincidence and common fixed points for occasionally weakly compatible and weakly compatible maps in fuzzy metric space. Our results generalize, extend and improve several relevant common fixed point theorems from the literature. We also furnish an illustrative example in support of our result.

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1. Introduction

The notion of a fuzzy set was introduced by Zadeh [34] in 1965 which is being extensively used by economist, biologists, engineers, computer scientists, and many others who use mathematical methods in their subject. For a good bibliography on fundamental and development of fuzzy mathematics, refer to Shostak [28]. In 1975, Kramosil et al. [15] introduced the concept of a fuzzy metric space by generalizing the concept of probabilistic metric to fuzzy situation. It appears that Kramosil et al.[15] laid down the foundation of a very soothing machinery to develop fixed point theorems for contractive and nonexpansive type maps in fuzzy metric spaces. Later on, Grabiec [7] defined the completeness of the fuzzy metric space (now known as G-complete fuzzy metric space) and formulated Banach contraction principle in the sense of Kramosil et al.[15]. George et al. [6] modified the definition of the Cauchy sequence introduced by Grabiec [7] because even R is not complete with Grabiec's[7] completeness definition. They also slightly modified the concept of fuzzy metric space introduced by Kramosil et al. and defined a Hausdorff first countable topology for fuzzy metric space. Since then, the notion of a complete fuzzy metric space presented by George et al. [6] (now known as an M-complete fuzzy metric space) has emerged as another characterization of completeness. Note that every G-complete fuzzy metric space is M-complete; the construction of fixed point theorem in M-complete fuzzy metric space seems to be more valuable.

Aamri et al.[1] introduced the notion of property $(E.A.)$ which contains the class of compatible as well as noncompatible maps and this is the motivation to use the property $(E.A)$ instead of compatibility or non-compatibility in common fixed point theorems. Liu et al. [19] further improved it by common property $(E.A)$. The utility of study of noncompatible maps can be understood from the fact that while studying the common fixed point theorem for compatible maps we often require completeness of the space or the continuity of the maps involved besides some contractive conditions but the study of noncompatible maps can be extended to the class of nonexpansive or Lipschitz type maps even without assuming continuity of the maps involved or completeness of the space. In literature, many results have been proved for contraction maps satisfying property $(E.A.)$ and its variants in different settings such as metric space [10, 12], probabilistic metric space [5, 11, 22], fuzzy metric space [4, 17, 18, 21, 23, 25, 26, 27, 30] and intuitionistic fuzzy metric space [9, 16]. In this paper, we prove the existence of coincidence and common fixed points for two pairs of occasionally weakly compatible and weakly compatible self maps satisfying property $(E.A)$ and its variants using an inequality involving quadratic terms. Our results generalize , extend and improve multitude of common fixed point results existing in the literature [2, 3, 32] and guarantee the existence of coincidence and common fixed point for noncompatible maps even when all the maps may be discontinuous. We also furnish an illustrative example in support of our results.

2. Preliminaries

Definition 2.1 ([29])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if * satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Example 2.2. $a * b = min\{a, b\}$ and $a * b = a.b$ are t-norms, for all $a, b \in [0,1]$.

Definition 2.3 ([6])**.** The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(5) $M(x, y, .) : [0, 1) \to [0, 1]$ is continuous,

for all $x, y, z \in X$ and $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.
Now, we give some interesting examples of fuzzy metric spaces:

Example 2.4. Let (X, d) be a metric space. Define $a * b = a + b$, for all $a, b \in [0, 1], x, y \in X$ and $t > 0$. Define $M(x, y, t) = \frac{t}{t+d(x,y)}$. Then $(X, M, *)$ is a fuzzy metric space.

Moreover, fuzzy metric M is induced by a metric d and is called the Standard fuzzy metric.

Example 2.5 ([8])**.** Let (X, d) be a bounded metric space with $d(x, y) < k$ for all $x, y \in X$. Let $g : R^+ \to (k, \infty)$ be an increasing continuous function M as then $(X, M, *)$ is a fuzzy metric space on X where * is a Lukasievicz t - norm i.e. $a * b = \max\ \{a + b - 1, 0\}$ for all $a, b \in [0, 1]$.

Example 2.6 ([8])**.** Define a function M as $M(x, y, t) = e^{(\frac{-d(x,y)}{g(t)})}$ then $(X, M, *)$ is a fuzzy metric space on X where * is the product t-norm and $g : R^+ \to [0, \infty)$ is an increasing continuous function.

In all that follows X is a fuzzy metric space with the following property:
(6) $lim_{t\to\infty} M(x, y, t) = 1$
for all $x, y \in X$ and $t > 0$.

Definition 2.7 ([7])**.** A sequence $\{x_n\}$ in an fuzzy metric space X is
(a) cauchy sequence if $lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, for each $t > 0$ and $n, p \in N$,
(b) convergent to a point $x \in X$ if $lim_{n\to\infty} M(x_n, x, t) = 1$, for each $t > 0$,
(c) complete if every cauchy sequence in X is convergent to some point in X.

Definition 2.8 ([24])**.** Self maps S and T of a fuzzy metric space X are compatible if and only if $lim_{n\to\infty} M(STx_n, TSx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.9 ([33])**.** A point x in fuzzy metric space X is a coincidence point of maps S and T if $Sx = Tx = w$(say), where $w \in X$. In this case, w is a point of coincidence of S and T.

Definition 2.10 ([13])**.** Self maps S and T of fuzzy metric space X are weakly compatible if S and T commute at coincidence points, that is, $STx = TSx$ whenever $Sx = Tx$.

Definition 2.11 ([33])**.** Self maps S and T of fuzzy metric space X are occasionally weakly compatible (owc) iff there is a point $x \in X$ which is a coincidence point of S and T at which S and T commutes, that is, there exists at least one point $x \in X$ such that $Sx = Tx$ implies $STx = TSx$.

Notice that weak compatibility implies occasionally weak compatibility but reverse implication is not true.

Example 2.12. Let $X = [0, \infty)$ be a fuzzy metric with the usual fuzzy metric as Example 2.4. Define selfmaps S, T of X as $Sx = 3x$ and $Tx = x^2$ for all $x \in X$. Then $Sx = Tx$ for $x = 0$, 3 but $ST0 = TS0$ and $ST3 \neq TS3$. Thus S and T are occasionally weakly compatible maps but not weakly compatible.

Motivated from [1], one can have the following:

Definition 2.13. Self maps S and T of a fuzzy metric space X satisfy the property $(E.A)$ if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Sx_n = lim_{n\to\infty}Tx_n = z$ for some $z \in X$.

Example 2.14. Let $X = [0,\infty)$ be fuzzy metric space and * the continuous t-norm defined by $a * b = ab$ for all $a, b \in [0,1]$. For $x, y \in X$, define $M(x,y,t) = \frac{t}{t+|x-y|}$ if $t > 0$ and $M(x,y,0) = 0$. Define self maps S and T as $Sx = \frac{2x}{5}$ and $Tx = \frac{x}{5}$ for all $x \in X$. Then for sequence $\{x_n\} = \{\frac{1}{n}\}$, maps S and T satisfy property $(E.A)$.

Also on the lines of Liu et al.[19], one can have the following:

Definition 2.15. Two pairs of self maps (A, S) and (B, T) of a fuzzy metric space X satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = z$ for some $z \in X$.

Example 2.16. Let $X = [-1,1]$ be a fuzzy metric with the usual fuzzy metric as defined in Example 2.14. Define self maps A, B, S and T on X as $Ax = \frac{x}{3}, Bx = \frac{-x}{3}, Sx = x, Tx = -x$ for all $x \in X$. Then, with sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = 0$. So pairs (A, S) and (B, T) satisfy the common property $(E.A.)$.

Definition 2.17 ([31])**.** A pair of self maps (S, T) on a fuzzy metric space X satisfy the common limit in the range property (CLR) if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Sx_n = lim_{n\to\infty}Tx_n = Tz$ for some $z \in X$.

Inspired by Sintunavarat et al.[31], S. Manro et. al. [20] introduced the following:

Definition 2.18. Two pairs of self maps (A, S) and (B, T) on a fuzzy metric space X share the common limit in the range of S property if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = Sz$ for some $z \in X$.

Example 2.19 ([20])**.** Let X = [-1, 1] be a fuzzy metric space and for all $x, y \in X$, $M(x,y,t) = e^{(\frac{-|x-y|}{t})}$ if $t > 0$, $M(x,y,0) = 0$. Define self maps A, B, S and T on X by $Ax = \frac{x}{3}, Bx = \frac{-x}{3}, Sx = x, Tx = -x$ for all $x \in X$. Then with sequences $\{x_n\} = \{1/n\}$ and $\{y_n\} = \{-1/n\}$ in X, one can easily verify that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = 0 = S0$. This shows that the pairs (A, S) and (B, T) share the common limit in the range of S property.

It is interesting to note here that common limit in the range property imply common property $(E.A.)$.

3. Main result

Let $C(A, S)$ denote the set of coincidence point of A and S.

Proposition 3.1. *Let A, B, S and T be four self maps of a fuzzy metric space X such that*

(3.1)

$$\begin{aligned}[M(Ax,By,t)]^2 \geq\; & c_1 \min\{[M(Sx,Ax,t)]^2, M(Ty,By,t)^2, [M(Sx,Ty,t]^2\} \\ & + c_2 \min\{[M(Sx,Ax,t)], [M(Sx,By,t)], [M(Ty,By,t)], \\ & \quad [M(Ty,Ax,t)]\} \\ & + c_3\{[M(Sx,By,t)].[M(Ty,Ax,t)]\}\end{aligned}$$

for all $x, y \in X$ and $t > 0$, where $c_1, c_2, c_3 \geq 0$. Suppose that
(i) $c_1 + c_2 + c_3 = 1$;
(ii) $BX \subseteq SX$ *(or* $AX \subseteq TX$*);*
(iii) *pair (B,T) (or (A,T)) satisfies property (E.A.);*
(iv) $T(X)$ *(or $S(X)$) is a closed subspace of X.*
Then $C(B,T) \neq \Phi$ and $C(A,S) \neq \Phi$.

Proof. Let the pair (B,T) satisfies property $(E.A.)$, there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since $BX \subseteq SX$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence, $lim_{n\to\infty} Sy_n = z$. First, we claim that $lim_{n\to\infty} Ay_n = z$.
For proving this, take $x = y_n$ and $y = x_n$ in 3.1, we get

$$\begin{aligned}[M(Ay_n,Bx_n,t)]^2 \geq\; & c_1 \min\{[M(Sy_n,Ay_n,t)]^2, M(Tx_n,Bx_n,t)^2, [M(Sy_n,Tx_n,t]^2\} \\ & + c_2 \min\{[M(Sy_n,Ay_n,t)], [M(Sy_n,Bx_n,t)], [M(Tx_n,Bx_n,t)], \\ & \quad [M(Tx_n,Ay_n,t)]\} \\ & + c_3\{[M(Sy_n,Bx_n,t)].[M(Tx_n,Ay_n,t)]\}.\end{aligned}$$

On making $n \to \infty$, we get

$$\begin{aligned}[M(lim_{n\to\infty}Ay_n,z,t)]^2 \geq\; & c_1 \min\{[M(z,lim_{n\to\infty}Ay_n,t)]^2, M(z,z,t)^2, [M(z,z,t]^2\} \\ & + c_2 \min\{[M(z,lim_{n\to\infty}Ay_n,t)], [M(z,z,t)], [M(z,z,t)], \\ & \quad [M(z,lim_{n\to\infty}Ay_n,t)]\} \\ & + c_3\{[M(z,z,t)].[M(z,lim_{n\to\infty}Ay_n,t)]\} \\ \geq\; & c_1[M(z,lim_{n\to\infty}Ay_n,t)]^2 + c_2[M(z,lim_{n\to\infty}Ay_n,t)] \\ & + c_3[M(z,lim_{n\to\infty}Ay_n,t)]\end{aligned}$$

$$[M(lim_{n\to\infty}Ay_n,z,t)]^2(1-c_1) \geq (c_2+c_3)[M(z,lim_{n\to\infty}Ay_n,t)]$$

$$[M(lim_{n\to\infty}Ay_n,z,t)] \geq \frac{(c_2+c_3)}{(1-c_1)} = 1.$$

This gives, $lim_{n\to\infty}Ay_n = z$.
Since $T(X)$ is a closed subspace of X, we have $z = Tv$ for some $v \in X$.
Next, we claim that $Bv = z$.

By (3.1), we have

$$\begin{aligned}[M(Ay_n, Bv, t)]^2 \geq\ & c_1 \min\{[M(Sy_n, Ay_n, t)]^2, M(Tv, Bv, t)^2, [M(Sy_n, Tv, t]^2\} \\ & + c_2 \min\{[M(Sy_n, Ay_n, t)], [M(Sy_n, Bv, t)], [M(Tv, Bv, t)], \\ & \quad [M(Tv, Ay_n, t)]\} \\ & + c_3\{[M(Sy_n, Bv, t)].[M(Tv, Ay_n, t)]\}.\end{aligned}$$

On making $n \to \infty$, we get

$$\begin{aligned}[M(z, Bv, t)]^2 \geq\ & c_1 \min\{[M(z, z, t)]^2, M(z, Bv, t)^2, [M(z, z, t]^2\} \\ & + c_2 \min\{[M(z, z, t)], [M(z, Bv, t)], [M(z, Bv, t)], [M(z, z, t)]\} \\ & + c_3\{[M(z, Bv, t)].[M(z, z, t)]\} \\ & \geq c_1[M(z, Bv, t)]^2 + c_2[M(z, Bv, t)] + c_3[M(z, Bv, t)]\end{aligned}$$

$$[M(z, Bv, t)]^2(1 - c_1) \geq (c_2 + c_3)[M(z, Bv, t)]$$

$$[M(z, Bv, t)] \geq \frac{(c_2 + c_3)}{(1 - c_1)} = 1.$$

This gives, $Bv = z = Tv$. Hence $C(B, T) \neq \Phi$.
Since $BX \subseteq SX$, there exists a point $u \in X$ such that $z = Su$.
Lastly, we claim that $Au = z$.
By (3.1), we have

$$\begin{aligned}[M(Au, z, t)]^2 =\ & [M(Au, Bv, t)]^2 \\ & \geq c_1 \min\{[M(Su, Au, t)]^2, M(Tv, Bv, t)^2, [M(Su, Tv, t]^2\} \\ & + c_2 \min\{[M(Su, Au, t)], [M(Su, Bv, t)], [M(Tv, Bv, t)], [M(Tv, Au, t)]\} \\ & + c_3\{[M(Su, Bv, t)].[M(Tv, Au, t)]\}\end{aligned}$$

$$\begin{aligned}[M(Au, z, t)]^2 \geq\ & c_1 \min\{[M(z, Au, t)]^2, M(z, z, t)^2, [M(z, z, t]^2\} \\ & + c_2 \min\{[M(z, Au, t)], [M(z, z, t)], [M(z, z, t)], [M(z, Au, t)]\} \\ & + c_3\{[M(z, z, t)].[M(z, Au, t)]\} \\ & \geq c_1[M(z, Au, t)]^2 + c_2[M(z, Au, t)] + c_3[M(z, Au, t)]\end{aligned}$$

$$[M(Au, z, t)]^2(1 - c_1) \geq (c_2 + c_3)[M(z, Au, t)]$$

$[M(Au, z, t)] \geq 1$.
This gives $Au = z$. Hence, $Su = Au = z$.
Therefore $C(A, S) \neq \Phi$.
Similarly if the pair (A, S) satisfies property $(E.A.)$, $AX \subseteq TX$ and $S(X)$ is a closed subspace of X then also $C(B, T) \neq \Phi$ and $C(A, S) \neq \Phi$. □

Now we prove our main result.

Theorem 3.2. *In addition to the hypotheses of Proposition 3.1, if both the pairs (A, S) and (B, T) are owc on X then the self maps A, B, S and T have a unique common fixed point in X.*

Proof. By Proposition 3.1, we have $C(B,T) \neq \Phi$ and $C(A,S) \neq \Phi$, there exists point $u \in C(A,S)$ such that $Au = Su = z$ and also there exists point $v \in C(B,T)$ such that $Bv = Tv = z$ for some $z \in X$. Since the pair (A,S) is *owc*, $ASu = SAu$. This gives, $Az = Sz = z'$ (say).
Also, since the pair (B,T) is *owc*, there exists $v \in C(B,T)$ such that $BTv = TBv$. This gives $Bz = Tz = w$ (say).
Now we show that $z' = w$.
By (3.1), we have

$$\begin{aligned}
[M(z',w,t)]^2 &= [M(Az,Bz,t)]^2\\
&\geq c_1 \min\{[M(Sz,Az,t)]^2, M(Tz,Bz,t)^2, [M(Sz,Tz,t]^2\}\\
&+ c_2 \min\{[M(Sz,Az,t)], [M(Sz,Bz,t)], [M(Tz,Bz,t)], [M(Tz,Az,t)]\}\\
&+ c_3\{[M(Sz,Bz,t)].[M(Tz,Az,t)]\}\\
&\geq c_1 \min\{[M(z',z',t)]^2, M(w,w,t)^2, [M(z',w,t]^2\}\\
&+ c_2 \min\{[M(z',z',t)], [M(z',w,t)], [M(w,w,t)], [M(w,z',t)]\}\\
&+ c_3\{[M(z',w,t)].[M(w,z',t)]\}\\
&\geq c_1[M(z',w,t]^2 + c_2[M(z',w,t] + c_3[M(z',w,t]
\end{aligned}$$

$$[M(z',w,t)]^2(1-c_1) \geq (c_2+c_3)[M(z',w,t]$$

$$[M(z',w,t)] \geq \frac{(c_2+c_3)}{(1-c_1)} = 1.$$

This gives $z' = w$. Finally, we show that $z = w$.
By (3.1), we have

$$\begin{aligned}
[M(z,w,t)]^2 &= [M(Az,Bz,t)]^2\\
&\geq c_1 \min\{[M(Sz,Az,t)]^2, M(Tz,Bz,t)^2, [M(Sz,Tz,t]^2\}\\
&+ c_2 \min\{[M(Sz,Az,t)], [M(Sz,Bz,t)], [M(Tz,Bz,t)], [M(Tz,Az,t)]\}\\
&+ c_3\{[M(Sz,Bz,t)].[M(Tz,Az,t)]\}\\
&\geq c_1 \min\{[M(z,z,t)]^2, M(w,w,t)^2, [M(z,w,t]^2\}\\
&+ c_2 \min\{[M(z,z,t)], [M(z,w,t)], [M(w,w,t)], [M(w,z,t)]\}\\
&+ c_3\{[M(z,w,t)].[M(w,z,t)]\}\\
&\geq c_1[M(z,w,t]^2 + c_2[M(z,w,t] + c_3[M(z,w,t]
\end{aligned}$$

$$[M(z,w,t)]^2(1-c_1) \geq (c_2+c_3)[M(z,w,t]$$

$$[M(z,w,t)] \geq \frac{(c_2+c_3)}{(1-c_1)} = 1.$$

Hence, $z = w$. Therefore, $Az = Sz = z' = w = z = Bz = Tz$.
Therefore, z is common fixed point of A, B, S and T.
The uniqueness of z follows from inequality (3.1). □

Now we attempt to drop containment of range from the above theorem using common property $(E.A.)$.

Proposition 3.3. *Let A, B, S and T be four self maps of a fuzzy metric space X satisfying the inequality* (3.1) *of Proposition* 3.1 *such that*

(i) *the pairs (A, S) and (B, T) satisfy a common property (E.A.);*

(ii) *SX and TX are closed subspaces of X.*

Then $C(B, T) \neq \Phi$ and $C(A, S) \neq \Phi$.

Proof. As the pairs (A, S) and (B, T) satisfy a common property $(E.A.)$, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ay_n = lim_{n\to\infty} Sy_n = lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since SX and TX are closed subspaces of X. Then, $z = Su = Tv$ for some $u, v \in X$.
We first claim that $Bv = z$.
For proving this, by (3.1), we have

$$\begin{aligned}[M(Ay_n, Bv, t)]^2 \geq\ & c_1 \min\{[M(Sy_n, Ay_n, t)]^2, M(Tv, Bv, t)^2, [M(Sy_n, Tv, t]^2\} \\ & + c_2 \min\{[M(Sy_n, Ay_n, t)], [M(Sy_n, Bv, t)], [M(Tv, Bv, t)], \\ & \quad [M(Tv, Ay_n, t)]\} \\ & + c_3\{[M(Sy_n, Bv, t)].[M(Tv, Ay_n, t)]\}.\end{aligned}$$

On making $n \to \infty$, we get

$$\begin{aligned}[M(z, Bv, t)]^2 \geq\ & c_1 \min\{[M(z, z, t)]^2, M(z, Bv, t)^2, [M(z, z, t]^2\} \\ & + c_2 \min\{[M(z, z, t)], [M(z, Bv, t)], [M(z, Bv, t)], [M(z, z, t)]\} \\ & + c_3\{[M(z, Bv, t)].[M(z, z, t)]\} \\ \geq\ & c_1[M(z, Bv, t)]^2 + c_2[M(z, Bv, t)] + c_3[M(z, Bv, t)]\end{aligned}$$

$$[M(z, Bv, t)]^2(1 - c_1) \geq (c_2 + c_3)[M(z, Bv, t)]$$

$$[M(z, Bv, t)] \geq \frac{(c_2 + c_3)}{(1 - c_1)} = 1,$$

which gives, $z = Bv = Tv$. Hence, $C(B, T) \neq \Phi$.
Next, we claim that $Au = z$.
By (3.1), we have

$$\begin{aligned}[M(Au, z, t)]^2 =\ & [M(Au, Bv, t)]^2 \\ \geq\ & c_1 \min\{[M(Su, Au, t)]^2, M(Tv, Bv, t)^2, [M(Su, Tv, t]^2\} \\ & + c_2 \min\{[M(Su, Au, t)], [M(Su, Bv, t)], [M(Tv, Bv, t)], [M(Tv, Au, t)]\} \\ & + c_3\{[M(Su, Bv, t)].[M(Tv, Au, t)]\} \\ \geq\ & c_1 \min\{[M(z, Au, t)]^2, M(z, z, t)^2, [M(z, z, t]^2\} \\ & + c_2 \min\{[M(z, Au, t)], [M(z, z, t)], [M(z, z, t)], [M(z, Au, t)]\} \\ & + c_3\{[M(z, z, t)].[M(z, Au, t)]\} \\ \geq\ & c_1[M(z, Au, t)]^2 + c_2[M(z, Au, t)] + c_3[M(z, Au, t)]\end{aligned}$$

$$[M(Au, z, t)]^2(1 - c_1) \geq (c_2 + c_3)[M(z, Au, t)]$$

$[M(Au, z, t)] \geq 1$.
This gives, $Au = z = Su$. Hence $C(A, S) \neq \Phi$. □

Theorem 3.4. *In addition to the hypotheses of Proposition 3.3, if both the pairs (A, S) and (B, T) are owc on X then the self maps A, B, S and T have a unique common fixed point in X.*

Proof. By Proposition 3.3, we get $C(B, T) \neq \Phi$ and $C(A, S) \neq \Phi$. The proof follows on the same lines as of Theorem 3.2. □

Remark 3.5. Observe that the notion of common property $(E.A.)$ relaxes containment requirement of range of one map into the range of other which is utilized to construct the sequence of joint iterates. As a consequence, a multitude of recent common fixed point theorems for two pair of self maps existing in the literature are sharpened and enriched.

We now attempt to relax closedness of subspace and drop the containment requirement of range from Theorem 3.2 using common limit in the range property.

Proposition 3.6. *Let A, B, S and T be four self maps of a fuzzy metric space X satisfying the inequality* (3.1) *of Proposition 3.1 such that*
(i) *the pairs (A, S) and (B, T) share common limit in the range of S property,*
(ii) *TX is closed subspace of X.*
Then $C(B, T) \neq \Phi$ and $C(A, S) \neq \Phi$.

Proof. As the pairs (A, S) and (B, T) share a common limit in the range of S property, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ay_n = lim_{n\to\infty} Sy_n = lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = Su = z$ for some $u, z \in X$.
Since TX is closed subspace of X. Then, $z = Su = Tv$ for some $v \in X$.
Rest of proof is same as of Proposition 3.3. □

Theorem 3.7. *In addition to the hypotheses of Proposition 3.6, if both the pairs (A, S) and (B, T) are owc on X then the self maps A, B, S and T have a unique common fixed point in X.*

Proof. The proof follows on the same lines as that of Theorem 3.2. □

On taking $A = B$ and $S = T$ in Theorem 3.2 we get the following interesting result.

Corollary 3.8. *Let A and S be self maps of a fuzzy metric space X such that*
(3.2)

$$\begin{aligned}
[M(Ax, Ay, t)]^2 \geq\; & c_1 \min\{[M(Sx, Ax, t)]^2, M(Sy, Ay, t)^2, [M(Sx, Sy, t]^2\} \\
& + c_2 \min\{[M(Sx, Ax, t)], [M(Sx, Ay, t)], [M(Sy, Ay, t)], \\
& \quad [M(Sy, Ax, t)]\} \\
& + c_3\{[M(Sx, Ay, t)].[M(Sy, Ax, t)]\}
\end{aligned}$$

for all $x, y \in X$ and $t > 0$, where $c_1, c_2, c_3 \geq 0$. Suppose that
(i) $c_1 + c_2 + c_3 = 1$;

(ii) *pair (A, S) satisfies property $(E.A.)$;*
(iii) *SX is a closed subspace of X.*
Then $C(A, S) \neq \Phi$.
Further, if the pair (A, S) is owc on X then the maps A and S have a unique common fixed point in X.

Remark 3.9. Observe that containment requirement of range of maps is not needed for the existence of common fixed point for a pair of self maps satisfying property $(E.A.)$.

Remark 3.10. Theorem 3.2, 3.4 and 3.7 remains valid if condition (3.1) is replaced by
(3.3)

$$\begin{aligned}[M(Ax, By, t)]^2 \geq c_1 \min\{&[M(Sx, Ax, t)]^2, M(Ty, By, t)^2, [M(Sx, Ty, t]^2\} \\ &+ c_2 \min\{[M(Sx, Ax, t)].[M(Sx, By, t)], \\ &\quad [M(Ty, By, t)].[M(Ty, Ax, t)]\} \\ &+ c_3\{[M(Sx, By, t)].[M(Ty, Ax, t)]\}.\end{aligned}$$

So Theorem 3.2,3.4 and 3.7 fuzzify and improve the result of Babu et al.[2].

Remark 3.11. Notice that for two pairs of self maps the notion of occasionally weak compatibility reduces to weak compatibility due to unique coincidence point of underlying self maps which is ensured by the used contractive condition. Hence all these results remain valid even if occasionally weak compatibility is replaced by weak compatibility.

Remark 3.12. Since occasionally weak compatibility and weak compatibility coincide in the presence of contraction condition, weak compatibility still remain the minimal commutativity condition for the existence of common fixed point for contractive maps.

Finally, we conclude this paper by furnishing example to demonstrate Theorem 3.2 besides exhibiting its superiority over earlier relevant results.

Example 3.13. Let $X = [\frac{1}{3}, 1)$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Define self maps A, B, S and T on X by
$Ax = \frac{1}{3}$ if $\frac{1}{3} \leq x < \frac{2}{3}$, $Ax = \frac{2}{3}$ if $\frac{2}{3} \leq x < 1$,
$Bx = \frac{3}{4}$ if $\frac{1}{3} \leq x < \frac{2}{3}$, $Bx = \frac{2}{3}$ if $\frac{2}{3} \leq x < 1$,
$Sx = \frac{1}{2}$ if $\frac{1}{3} \leq x < \frac{2}{3}$, $Sx = \frac{1}{3} + \frac{x}{2}$ if $\frac{2}{3} \leq x < 1$
and
$Tx = \frac{1}{2}$ if $\frac{1}{3} \leq x < \frac{2}{3}$, $Tx = 1 - \frac{x}{2}$ if $\frac{2}{3} \leq x < 1$.
Clearly,$BX \subseteq SX$, TX is a closed subspace of X. The self maps A, B, S and T satisfy both the inequalities (3.1) and (3.1)' with $c_1 = \frac{1}{3}, c_2 = \frac{1}{2}, c_3 = \frac{1}{6}$ where the sequence for which pair (B, T) satisfies property $(E.A.)$ is $\{x_n\} = \{\frac{2}{3} + \frac{1}{n+3}\}$. Clearly, the pairs (A, S) and (B, T) are *owc*. Hence, the self maps A, B, S and T satisfy all the conditions of Theorem 3.2 and $x = \frac{2}{3}$ is the unique coincidence and common fixed point of A, B, S and T.
Further, all self maps A, B, S and T are discontinuous at common fixed point $x = \frac{2}{3}$. Moreover, neither $AX \subseteq TX$ nor $TX \subseteq AX$.

Thus by imposing the 'property $(E.A.)$' in the results of Tas et al.[32], Babu et al. [2], Kameswari [14], Babu et al [3], we are able to relax one of the containment of $BX \subseteq SX$ and $AX \subseteq TX$ together with two restrictions $c_1+2c_2 < 1$ and $c_1+c_3 < 1$ by a single restriction $c_1+c_2+c_3 = 1$. Also relaxed continuity hypothesis of A, B, S and T in Tas et al. [32]. Here it is worth mentioning that none of the results can be used in the context of this example.

Remark 3.14. It is interesting to note that this example cannot be covered by all those common fixed point theorems which require containment of both the pairs, continuity requirement of self maps along with completeness (or closedness) of underlying space. Moreover Theorem 3.7 neither requires any condition on the containment of ranges for two pairs of self maps nor any restriction on $c's$. Also only one subspace is closed.

Remark 3.15. The above coincidence and common fixed point theorems extend, generalize and improve several results on metric, menger, uniform and fuzzy metric space without any requirement of completeness (or closedness) of the underlying space, containment of ranges amongst involved maps and continuity in respect of any one of the involved map. For example, Tas et al.[32], Babu et al. [2], Kameswari [14], Babu et al [3].

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