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On fuzzy Euler-Lagrange equations

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Abstract. Farhadinia [11] based on concept of Buckley and Feuring's derivative proposed the fuzzy Euler-Lagrange conditions for fuzzy constrained and unconstrained variational problems. In 2012, Fard and Zadeh [10] by using α-differentiability concept obtained an extended fuzzy Euler-Lagrange condition. The main purpose of this study is to improve the fuzzy Euler-Lagrange conditions under the generalized Hukuhara differentiability.

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1. Introduction

The ideas of the generalized Hukuhara difference (gH-difference for short) and generalized Hukuhara differentiability (gH-differentiability for short) come from a generalization of the Hukuhara difference and Hukuhara differentiable.

The Hukuhara derivative of a fuzzy-number-valued function was first introduced in [13] and it has its starting point in the Hukuhara derivative of multivalued functions. The approaches based on the Hukuhara derivative have a well-known drawback: a differentiable function has increasing length of its support interval [6]. This shortcoming was overcome by concepts of strongly generalized differentiability and it's generalization (weakly generalized Hukuhara differentiability) which are introduced and studied in [1]. In these cases, a differentiable function may have decreasing length of its support. Recently this line of research has been extended by introducing gH-derivative and the generalized derivative [2] of a fuzzy-valued function. The gH-differentiability concept is slightly more general than the notion of strongly Hukuhara differentiability and equivalent with the concept of weakly generalized

Hukuhara differentiability [2]. It should be mentioned that gH-differentiability exists under much less restrictive conditions, however it does not always exist. Indeed, the gH-derivative is defined for a larger class of fuzzy-number-valued functions than the other derivatives (except concept of generalized derivative).

The fuzzy Euler-Lagrange conditions for fuzzy constrained and unconstrained variational problems was first introduced by Farhadinia in [11] based on Buckley and Feuring's derivative [3]. Using α-differentiability concept [15], in [10], Fard et al. has been presented an extended fuzzy Euler-Lagrange condition. In the present work, following approaches given in [2] and [10], a generalization of the fuzzy Euler-Lagrange conditions for fuzzy constrained and unconstrained variational problems is investigated.

The rest of the paper is organized as follows: in Section 2, the basic notations of fuzzy concepts are briefly presented. In Section 3, the fuzzy Euler-Lagrange condition for the fuzzy unconstrained variational problems is described. In Section 4, we establish the modified fuzzy Euler-Lagrange condition for the fuzzy constrained variational problems, called isoperimetric problems. In Section 5, we show that this method is applicable to a large class of problems. Finally, Section 6 presents concluding remarks.

2. Preliminaries

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy numbers, i.e., normal, convex, upper semi-continuous and compactly supported fuzzy subsets of the real numbers. Obviously, $R \subset \mathbb{R}_{\mathcal{F}}$. Here $R \subset \mathbb{R}_{\mathcal{F}}$ is understood as $R = \{\chi_x; x \text{ is usual real number}\}$. The fuzzy zero define as $\tilde{0} = \chi_0$. The α-level set of $u \in \mathbb{R}_{\mathcal{F}}$, denoted by $u[\alpha]$ where for $0 < \alpha \leq 1$, $u[\alpha] = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $u[0] = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Then it is well known that for any $\alpha \in [0,1], u[\alpha] = [u^l(\alpha), u^r(\alpha)]$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $(u+v)[\alpha] = u[\alpha] + v[\alpha], (\lambda.u)[\alpha] = \lambda.u[\alpha], \forall \alpha \in [0,1]$, where $u[\alpha] + v[\alpha]$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda.u[\alpha]$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Proposition 2.1 ([2])**.** *A fuzzy number u is completely determined by any pair $u = (u^l, u^r)$ of functions $u^l, u^r : [0,1] \to \mathbb{R}$, defining the end-points of the α-levels, satisfying the three conditions:*

(i) $u^l : \alpha \to u^l(\alpha) \in \mathbb{R}$ *is a bounded monotonic nondecreasing left-continuous function* $\forall \alpha \in]0,1]$ *and right-continuous for* $\alpha = 0$;

(ii) $u^r : \alpha \to u^r(\alpha) \in \mathbb{R}$ *is a bounded monotonic nonincreasing left-continuous function* $\forall \alpha \in]0,1]$ *and right-continuous for* $\alpha = 0$;

(iii) $u^l(1) \leq u^r(1)$ *for* $\alpha = 1$, *which implies* $u^l(\alpha) \leq u^r(\alpha) \forall \alpha \in [0,1]$.

Definition 2.2 ([11])**.** (Partial ordering) Let $u, v \in \mathbb{R}_{\mathcal{F}}$. We write $u \preceq v$, if $u^l(\alpha) \leq v^l(\alpha)$ and $u^r(\alpha) \leq v^r(\alpha)$ for all $\alpha \in [0,1]$. Moreover, $u \approx v$, if $u \preceq v$ and $u \succeq v$. In the other words, $u \approx v$, if $u[\alpha] = v[\alpha]$ for all $\alpha \in [0,1]$.

The Hausdorff distance is defined by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}, D(u,v) = \sup_{\alpha \in [0,1]} \max\{|u^l(\alpha) - v^l(\alpha)|, |u^r(\alpha) - v^r(\alpha)|\}$, where $u[\alpha] = [u^l(\alpha), u^r(\alpha)], v[\alpha] = [v^l(\alpha), v^r(\alpha)]$. The following properties are well-known:

$D(u+w, v+w) = D(u, v), \forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,
$D(k.u, k.v) = |k|D(u, v), \forall k \in \mathbb{R}, u, v \in \mathbb{R}_{\mathcal{F}}$,
$D(u+v, w+e) \leq D(u, w) + D(v, e), \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$,
and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space [1]. We define $\|.\|_{\mathcal{F}} = D(., 0)$.

Definition 2.3 ([7])**.** Let $f : S \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function. We say that f is continuous at $c \in S$ if for every $\epsilon_1 > 0$, there exists a $\delta = \delta(c, \epsilon_1) > 0$ such that

$$D(f(x), f(c)) < \epsilon_1$$

for all $x \in S$ with $\|x - c\| < \delta$.

Definition 2.4 ([17])**.** The gH-difference of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$, is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} & (i) u = v + w, \\ \text{or} & (ii) v = u + (-1)w. \end{cases} \tag{2.1}$$

If $w = u \ominus_{gH} v$ exists as a fuzzy number, its level sets $[w^l(\alpha), w^r(\alpha)]$ are obtained by $w^l(\alpha) = \min\{u^l(\alpha) - v^l(\alpha), u^r(\alpha) - v^r(\alpha)\}$ and $w^r(\alpha) = \max\{u^l(\alpha) - v^l(\alpha), u^r(\alpha) - v^r(\alpha)\}$ for all $\alpha \in [0, 1]$. Based on the gH-difference, Bede et al. [2] obtained the following definition:

Definition 2.5 ([2, 16])**.** Let $x_0 \in]a, b[$ and h be such that $x_0 + h \in]a, b[$, then the gH-derivative of a function $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h}[f(x_0 + h) \ominus_{gH} f(x_0)]. \tag{2.2}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (2.2) exists, we say that f is gH-differentiable at x_0.

Definition 2.6. Let $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in]a, b[$. We say that f is n-order gH-differentiable at x_0, if there exists an element $f^{(n)}(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that for all h sufficiently small, $\exists f_{gH}^{(n-1)}(x_0 + h) \ominus_{gH} f_{gH}^{(n-1)}(x_0)$ and

$$f_{gH}^{(n)}(x_0) = \lim_{h \to 0} \frac{1}{h}[f_{gH}^{(n-1)}(x_0 + h) \ominus_{gH} f_{gH}^{(n-1)}(x_0)].$$

Definition 2.7. Let f be a fuzzy function defined on an open subset X of $\mathbb{R}^n$ and let $x_0 = (x_0^1, \cdots, x_0^n) \in X$ be fixed. We say that f has i^{th} partial gH-derivative $f'_{i;gH}(x_0)$ at x_0 if the fuzzy function $g(x_i) = f(x_0^1, \cdots, x_0^{i-1}, x_0^i, x_0^{i+1}, \cdots, x_0^n)$ is gH-differentiable at x_0^i with gH-derivative $f'_{i;gH}(x_0)$.

Definition 2.8. A fuzzy function f is gH-differentiable at x_0 if one of the partial gH-derivatives $f'_{1;gH}, \cdots, f'_{n;gH}$ exists at x_0 and the remaining $n-1$ partial gH-derivatives exist on some neighborhoods of x_0 and are continuous at x_0.

Definition 2.9. A fuzzy function f is said to be continuously gH-differentiable at x_0 if all of the partial gH-derivatives $f'_{i;gH}, i = 1, \cdots, n$, exist on some neighborhoods of x_0 and are continuous at x_0.
We say that f is continuously gH-differentiable on X if it is continuously gH-differentiable at every $x_0 \in X$.

The next theorem gives the expression of the gH-derivative in terms of the derivatives of the endpoints of the level sets.

Theorem 2.10 ([2]). *Let $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ be such that $f(x)[\alpha] = [f^l(x, \alpha), f^r(x, \alpha)]$. Suppose that functions $f^l(x, \alpha)$ and $f^r(x, \alpha)$ are real-valued functions, differentiability w.r.t. x, uniformly w.r.t. $\alpha \in [0, 1]$. Then the function $f(x)$ is gH-differentiable at a fixed $x \in]a, b[$ if and only if one of the following two cases holds:*

(a) *$(f^l)'(x, \alpha)$ is increasing, $(f^r)'(x, \alpha)$ is decreasing as functions of α, and $(f^l)'(x, 1) \le (f^r)'(x, 1)$,*

or

(b) *$(f^l)'(x, \alpha)$ is decreasing, $(f^r)'(x, \alpha)$ is increasing as functions of α, and $(f^r)'(x, 1) \le (f^l)'(x, 1)$.*

Also, $\forall \alpha \in [0, 1]$ we have

$$f'_{gH}(x)[\alpha] = [\min\{(f^l)'(x, \alpha), (f^r)'(x, \alpha)\}, \max\{(f^l)'(x, \alpha), (f^r)'(x, \alpha)\}].$$

According to Theorem 2.10, for the definition of gH-differentiability when $f^l(x, \alpha)$ and $f^r(x, \alpha)$ are both differentiable, we distinguish two cases, corresponding to (i) and (ii) of (2.1).

Definition 2.11 ([2]). Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in]a, b[$ with $f^l(x, \alpha)$ and $f^r(x, \alpha)$ both differentiable at x_0.
− f is (i)-gH-differentiable at x_0 if

$$\text{(i)} f'_{gH}(x_0)[\alpha] = [(f^l)'(x_0, \alpha), (f^r)'(x_0, \alpha)], \quad \forall \alpha \in [0, 1],$$

− f is (ii)-gH-differentiable at x_0 if

$$\text{(ii)} f'_{gH}(x_0)[\alpha] = [(f^r)'(x_0, \alpha), (f^l)'(x_0, \alpha)], \quad \forall \alpha \in [0, 1].$$

It is possible that $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at x_0 and not (i)-gH-differentiable nor (ii)-gH-differentiable.

Definition 2.12 ([5]). A switching point $x_0 \in]a, b[$ is such that gH-differentiability changes from type (i) to type (ii) or from type (ii) to type (i).

Definition 2.13 ([8]). A mapping $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be strongly measurable if the level set mapping $f(x)[\alpha]$ are measurable for all $\alpha \in [0, 1]$. Here measurable means Borel measurable.

A fuzzy-valued mapping $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is called integrably bounded if there exists an integrable function $h : [a, b] \to \mathbb{R}_{\mathcal{F}}$, such that

$$||f(t)||_{\mathcal{F}} \le h(t), \forall t \in [a, b].$$

A strongly measurable and integrably bounded fuzzy-valued function is called integrable. If $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is integrable such that $f(x)[\alpha] = [f^l(x, \alpha), f^r(x, \alpha)]$ for all $\alpha \in [0, 1]$, then $\int_a^b f(x)dx$ is obtained by integrating the α-level curve, that is,

$$\int_a^b f(x)dx[\alpha] = \left[\int_a^b f^l(x, \alpha)dx, \int_a^b f^r(x, \alpha)dx\right].$$

3. Fuzzy variational problem

In this section, we try to obtain the fuzzy Euler-Lagrange conditions in the sense of gH-differentiability for finding a fuzzy curve $x = x(t)$, which minimizes the following cost fuzzy function subject to $x(t) \approx x_0, x(t_f) \approx x_f$:

$$\text{(FVP) Minimize} J(x) := \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt,$$

where the fuzzy curve $x = x(t)$ is a fuzzy function of $t \in [t_0, t_f] \subseteq \mathbb{R}$ and belongs to the class of fuzzy functions with continuous first gH-derivatives w.r.t. $t \in [t_0, t_f]$ and g assigns a fuzzy number to the fuzzy point $(x(t), \dot{x}(t), t) \in \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}$. We assume that the integrand g has continuous first and second partial gH-derivatives w.r.t. all of its arguments. The endpoints (or end-conditions) of the fuzzy curve are characterized by x_0 and x_f corresponding to the fixed points t_0 and t_f, respectively.

Definition 3.1 ([11])**.** (*Admissible curve*) Fuzzy curve $x = x(t)$ is admissible, if it satisfies the end-conditions and is also twice continuously gH-differentiable with respect to $t \in [0, 1]$. The set of all admissible admissible curves is denoted by X_{ad}.

Definition 3.2 ([11])**.** (*Fuzzy weak neighborhood*) A fuzzy weak neighborhood $N_\varepsilon(\hat{x})$ of a fuzzy curve $\hat{x} = \hat{x}(t)$ is the set of all admissible curves $x_s \in X_{ad}$ satisfying for all $t \in [t_0, t_f]$

$$D(x(t), \hat{x}(t)) \leq \varepsilon, \quad \forall t \in [t_0, t_f], \tag{3.1}$$

$$D(\dot{x}(t), \dot{\hat{x}}(t)) \leq \varepsilon, \quad \forall t \in [t_0, t_f], \tag{3.2}$$

where $\varepsilon > 0$ is a small real number.

In the case that the requirement imposed by (3.2) is waived, $N_\varepsilon(\hat{x})$ is a fuzzy *strong neighborhood* and if $x \in N_\varepsilon(\hat{x})$, x and $\hat{x}$ are close together.

The goal of FVP is to find the fuzzy curve $x_s \in X_{ad}$ in a fuzzy weak neighborhood, if any exists, such that minimize J. The fuzzy curve $x^* = x^*(t)$ is a minimizing curve for FVP if for all admissible fuzzy curves $x_s \in X_{ad}$ in the fuzzy weak neighborhood $N_\varepsilon(x^*)$

$$\Delta J := J(x) \ominus_{gH} J(x^*) \succeq \tilde{0},$$

or equivalently

$$J(x) \succeq J(x^*). \tag{3.3}$$

The fuzzy curve x^* with the mentioned properties, is also called a *relative (or local) minimizer.*

Now we may deform the $\alpha-$cuts of fuzzy curve $x^*(t), x^{*l}(t, \alpha)$ and $x^{*r}(t, \alpha)$, by using $\alpha-$cuts of an arbitrary twice continuously gH-differentiable fuzzy function $\eta(t)$ as follows

$$x^l(t, \alpha) := x^{*l}(t, \alpha) + \epsilon\eta^l(t, \alpha), \tag{3.4}$$

$$x^r(t, \alpha) := x^{*r}(t, \alpha) + \epsilon\eta^r(t, \alpha), \tag{3.5}$$

such that $x(t)[\alpha] = [x^l(t, \alpha), x^r(t, \alpha)]$ is admissible for any real number ϵ, i.e., $x \in X_{ad}$ and further $\eta(t_0) \approx \eta(t_f) \approx 0$. The class of such deformations is known as *fuzzy* *weak variations* of the fuzzy curve x^* because given $\varepsilon > 0$, choosing $|\epsilon|$ sufficiently small, we are able to make $x(t)$ lie in the fuzzy weak neighborhood $N_\varepsilon(x^*)$.

Although the necessary and sufficient condition (3.3) for x^* being the solution of FVP does not give any hint as to how x^* might be found, but as will be seen, it leads to an equation which gives useful information about the treatment of minimizing curves.

3.1. **Fuzzy Euler-Lagrange condition.** It follows from Definition 2.2 that the inequality (3.3) holds if and only if

$$J^l(x,\alpha) \geq J^l(x^*,\alpha), \quad \text{and} \quad J^r(x,\alpha) \geq J^r(x^*,\alpha), \tag{3.6}$$

for all $\alpha \in [0,1]$ and all admissible $x_s' \in X_{ad}$ close to x^*.

From (3.6), inequality (3.3) holds if and only if the *left-increment* and *right-increment* are non-negative (in the sense of α-cuts), that is,

$$\Delta J^l(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) := (\Delta_l J^l, \Delta_r J^l) \geq (0,0), \tag{3.7}$$

$$\Delta J^r(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) := (\Delta_l J^r, \Delta_r J^r) \geq (0,0), \tag{3.8}$$

for all $\alpha \in [0,1]$ and all admissible $x_s \in X_{ad}$ close to x^*. We see that (3.7) and (3.8) hold if and only if

$$\Delta_l J^l = J^l(x^l(t,\alpha), x^{*r}(t,\alpha), \alpha) - J^l(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) \geq 0, \tag{3.9}$$

$$\Delta_r J^l = J^l(x^{*l}(t,\alpha), x^r(t,\alpha), \alpha) - J^l(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) \geq 0, \tag{3.10}$$

$$\Delta_l J^r = J^r(x^l(t,\alpha), x^{*r}(t,\alpha), \alpha) - J^r(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) \geq 0, \tag{3.11}$$

$$\Delta_r J^r = J^r(x^{*l}(t,\alpha), x^r(t,\alpha), \alpha) - J^r(x^{*l}(t,\alpha), x^{*r}(t,\alpha), \alpha) \geq 0. \tag{3.12}$$

We consider only equation (3.9). Using (3.4) and (3.9), one can easily verify that

$$\begin{aligned} \Delta_l J^l(x^{*l}, x^{*r}, \alpha) = \int_{t_0}^{t_f} \{ & g^l(x^{*l} + \epsilon\eta^l, \dot{x}^{*l} + \epsilon\dot{\eta}^l, x^{*r}, \dot{x}^{*r}, t, \alpha) \\ & - g^l(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\} dt \geq 0, \end{aligned} \tag{3.13}$$

where $x^{*l} + \epsilon\eta^l$ (or $x^{*r} + \epsilon\eta^r$) stands for $x^{*l}(t,\alpha) + \epsilon\eta^l(t,\alpha)$ (or $x^{*r}(t,\alpha) + \epsilon\eta^r(t,\alpha)$).
Corresponding to Definition 2.11 the following two cases just can occur.
Case (i): g is (i)-gH differentiable ((ii)-gH differentiable) w.r.t. x and $\dot{x}$.
Expanding the integrant $g^l(x^{*l} + \epsilon\eta^l, \dot{x}^{*l} + \epsilon\dot{\eta}^l, x^{*r}, \dot{x}^{*r}, t, \alpha)$ of (3.13) in a Taylor series about the point $(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r})$ gives

$$\Delta_l J^l(x^{*l}, x^{*r}, \alpha) = \epsilon J_1^l(x^{*l}, x^{*r}, \alpha) + \epsilon^2 J_2^l(x^{*l}, x^{*r}, \alpha) + O(\epsilon^3), \tag{3.14}$$

where

$$J_1^l(x^l, x^r, \alpha) = \int_{t_0}^{t_f} (\eta^l \frac{\partial g^l}{\partial x^l} + \dot{\eta}^l \frac{\partial g^l}{\partial \dot{x}^l}) dt,$$

$$J_2^l(x^l, x^r, \alpha) = \frac{1}{2}\int_{t_0}^{t_f} (\eta^{l^2}\frac{\partial^2 g^l}{\partial x^{l^2}} + 2\eta\dot{\eta}^l \frac{\partial^2 g^l}{\partial x^l \partial \dot{x}^l} + \dot{\eta}^{l^2}\frac{\partial^2 g^l}{\partial \dot{x}^{l^2}})dt.$$

The integral $J_1^l(x^l, x^r, \alpha)$ is called the *first variation* of J^l, since it is expressed in terms containing the first-order change in J^l w.r.t. the deformations $x^{*l} + \epsilon\eta^l$ and $x^{*r} + \epsilon\eta^r$. Similarly, the integral $J_2^l(x^l, x^r, \alpha)$ is called the *second variation* of J^l. The notation $O(\epsilon^3)$ denotes terms in the expansion of order 3 and greater in ϵ.

By (3.13), the right-hand side of equation (3.14) is non-negative. On the other hand, ϵ is a arbitrary and may be positive or negative. Hence, dividing the right-hand side of (3.14) by ϵ, the two following inequalities can be taken into consideration:

$$J_1^l(x^{*l}, x^{*r}, \alpha) + \epsilon J_2^l(x^{*l}, x^{*r}, \alpha) + O(\epsilon^2) \geq 0, \qquad if\ \epsilon > 0, \tag{3.15}$$

$$J_1^l(x^{*l}, x^{*r}, \alpha) + \epsilon J_2^l(x^{*l}, x^{*r}, \alpha) + O(\epsilon^2) \leq 0, \qquad if\ \epsilon < 0. \tag{3.16}$$

Now, the two inequalities (3.15) and (3.16) can be reduced to $J_1^l(x^{*l}, x^{*r}, \alpha) \geq 0$ and $J_1^l(x^{*l}, x^{*r}, \alpha) \leq 0$, respectively, as ϵ approaches zero. This means that

$$\begin{aligned} J_1^l(x^{*l}, x^{*r}, \alpha) &= \int_{t_0}^{t_f} (\eta^l \frac{\partial g^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \dot{\eta}^l \frac{\partial g^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha))dt \\ &= 0, \end{aligned} \tag{3.17}$$

for all admissible $\eta^l(t, \alpha)$. Since $\eta^l(t_0, \alpha) = \eta^l(t_f, \alpha) = 0$, solving integral involves integration by part, the equation (3.17) becomes

$$\int_{t_0}^{t_f} \eta^l \Big(\frac{\partial g^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) - \frac{d}{dt}\big(\frac{\partial g^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\big)\Big)dt = 0. \tag{3.18}$$

for all admissible $\eta^l(t, \alpha)$.

Lemma 3.3 ([4])**.** *Assume that $h = h(t)$ is a continuous real-valued function. If it holds*

$$\int_{t_0}^{t_f} h(t)g(t)dt = 0,$$

for every continuous real-valued function $g = g(t)$ in the interval $[t_0, t_f]$, then $h = h(t)$ must be zero everywhere in $[t_0, t_f]$.

By applying Lemma 3.3 to (3.18), we have

$$\frac{\partial g^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) - \frac{d}{dt}\Big(\frac{\partial g^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big) = 0. \tag{3.19}$$

Following the scheme of obtaining (3.19) and adapting it to the cases (3.10)-(3.12), one can easily show that

$$\frac{\partial g^l}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) - \frac{d}{dt}\Big(\frac{\partial g^l}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big) = 0, \tag{3.20}$$

$$\frac{\partial g^r}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) - \frac{d}{dt}\Big(\frac{\partial g^r}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big) = 0, \tag{3.21}$$

$$\frac{\partial g^r}{\partial x^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)-\frac{d}{dt}\Big(\frac{\partial g^r}{\partial \dot{x}^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big)=0. \tag{3.22}$$

Case (ii): g is (i)-gH differentiable ((ii)-gH differentiable) w.r.t. x and (ii)-gH differentiable ((i)-gH differentiable) w.r.t. $\dot{x}$.
Similar the procedure of obtaining the fuzzy Euler-Lagrange conditions for case (i), one can show that the conditions for this case are

$$\frac{\partial g^l}{\partial x^l}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)-\frac{d}{dt}\Big(\frac{\partial g^r}{\partial \dot{x}^l}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big)=0, \tag{3.23}$$

$$\frac{\partial g^l}{\partial x^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)-\frac{d}{dt}\Big(\frac{\partial g^r}{\partial \dot{x}^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big)=0, \tag{3.24}$$

$$\frac{\partial g^r}{\partial x^l}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)-\frac{d}{dt}\Big(\frac{\partial g^r}{\partial \dot{x}^l}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big)=0, \tag{3.25}$$

$$\frac{\partial g^r}{\partial x^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)-\frac{d}{dt}\Big(\frac{\partial g^l}{\partial \dot{x}^r}(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big)=0. \tag{3.26}$$

4. Fuzzy isoperimetric problem

The problem involving minimization of a fuzzy functional while giving a fuzzy integral constraints is called the fuzzy isoperimetric problem and it is stated as follows:

$$\begin{aligned}\text{(FIP) Minimize } J(x) &:= \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt\\ \text{subject to } \ I(x) &:= \int_{t_0}^{t_f} h(x(t),\dot{x}(t),t)dt \approx c\\ x(t) &\approx x_0, x(t_f)\approx x_f,\end{aligned}$$

where $c\in\mathbb{R}_{\mathcal{F}}$ is a given fuzzy number.
For FIP, consider the following deformations of the $\alpha-$cuts of fuzzy curve x^* by taking into consideration $\alpha-$cuts of an arbitrary twice continuously gH-differentiable fuzzy function $\delta(t)$ as

$$x^l(t,\alpha):=x^{*l}(t,\alpha)+\epsilon\delta^l(t,\alpha), \tag{4.1}$$

$$x^r(t,\alpha):=x^{*r}(t,\alpha)+\epsilon\delta^r(t,\alpha), \tag{4.2}$$

where ϵ is a small real number and $\delta(t):=\sigma\eta(t)+\beta\zeta(t)$. In the last equation σ,β are real constants and the arbitrary independent fuzzy functions $\eta(t)$ and $\zeta(t)$ vanish in the fuzzy sense at the endpoints.

Since I is equal to the fuzzy number c, therefore its increment is identically zero, particularly, the first variation must be zero. According to Definition 2.11 eight cases can be occur.
Case (i): g and h are both (i)-gH differentiable ((ii)-gH differentiable) w.r.t. x and

$\dot{x}$.
In this case, for all $\alpha \in [0,1]$, we have

$$\int_{t_0}^{t_f} \Big\{\delta^l \frac{\partial h^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \dot{\delta}^l \frac{\partial h^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big\} dt = 0, \tag{4.3}$$

$$\int_{t_0}^{t_f} \Big\{\delta^r \frac{\partial h^l}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \dot{\delta}^r \frac{\partial h^l}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big\} dt = 0, \tag{4.4}$$

$$\int_{t_0}^{t_f} \Big\{\delta^l \frac{\partial h^r}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \dot{\delta}^l \frac{\partial h^r}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big\} dt = 0, \tag{4.5}$$

$$\int_{t_0}^{t_f} \Big\{\delta^r \frac{\partial h^r}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \dot{\delta}^r \frac{\partial h^r}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big\} dt = 0. \tag{4.6}$$

We consider only equation (4.3). By integrating by parts the terms involving $\dot{\delta}^l$ and letting $\delta^l(t,\alpha) := \sigma\eta^l(t,\alpha) + \beta\zeta^l(t,\alpha)$, we may find that
(4.7)

$$\int_{t_0}^{t_f} (\sigma\eta^l + \beta\zeta^l) \Bigg\{ \frac{\partial h^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) - \frac{d}{dt}\Bigg(\frac{\partial h^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Bigg)\Bigg\} dt = 0.$$

Let us define the fuzzy operator $\mathcal{L}^l(*)$ as

$$\mathcal{L}^l(*) := \frac{\partial *}{\partial x^l} - \frac{d}{dt}\Big(\frac{\partial *}{\partial \dot{x}^l}\Big). \tag{4.8}$$

Hence, we see that (4.7) can be written as

$$\int_{t_0}^{t_f} (\sigma\eta^l + \beta\zeta^l)\mathcal{L}^l(h^l) dt = 0. \tag{4.9}$$

Observe that x^* is not the minimizer of I therefore, $\mathcal{L}^l(h^l(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)) \neq 0$. Furthermore, for any η^l, ζ^l, the constants σ, β are related together by (4.9). The assumption that x^* is the minimizer of J grantees the increment of J must be non-negative in the fuzzy sense with respect to the deformation given in (4.1). Consequently, the first variation is zero and after integrating by parts, it holds that

$$\int_{t_0}^{t_f} (\sigma\eta^l + \beta\zeta^l)\mathcal{L}^l(g^l) dt = 0, \tag{4.10}$$

where σ, β are those satisfy (4.9). Solving by elimination σ and β between (4.9) and (4.10), for every independent and twice continuously differentiable functions η^l and ζ^l, one can show that

$$\frac{\int_{t_0}^{t_f} \eta^l \mathcal{L}^l(g^l) dt}{\int_{t_0}^{t_f} \eta^l \mathcal{L}^l(h^l) dt} = \frac{\int_{t_0}^{t_f} \zeta^l \mathcal{L}^l(g^l) dt}{\int_{t_0}^{t_f} \zeta^l \mathcal{L}^l(h^l) dt}. \tag{4.11}$$

Introducing the constant $-\lambda_1^l = -\lambda_1^l(\alpha)$ which is equal to both sides of the equality in (4.11) gives that

$$\int_{t_0}^{t_f} \eta^l \mathcal{L}^l\Big(g^l(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha) + \lambda_1^l(\alpha) h^l(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, t, \alpha)\Big) dt = 0, \tag{4.12}$$

for any admissible $\eta^l = \eta^l(t,\alpha)$.

Applying Lemma 3.3, we derive from (4.12) that

$$\mathcal{L}^l\Big(g^l(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha) + \lambda^l(\alpha)h^l(x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)\Big) = 0. \tag{4.13}$$

Taking into account the structure of $\mathcal{L}^l$ in (4.8), we get from (4.13) that

$$\frac{\partial}{\partial x^l}\Big(g^l(z) + \lambda_1^l h^l(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z) + \lambda_1^l h^l(z)\Big)\right) = 0, \tag{4.14}$$

where $z^l = (x^{*l},\dot{x}^{*l},x^{*r},\dot{x}^{*r},t,\alpha)$.

Now following the scheme of obtaining (4.14) and adapting it to the case under consideration involving (4.4)-(4.6), we may show that

$$\frac{\partial}{\partial x^r}\Big(g^l(z) + \lambda_2^r h^l(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z) + \lambda_2^r h^l(z)\Big)\right) = 0, \tag{4.15}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z) + \lambda_2^l h^r(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z) + \lambda_2^l h^r(z)\Big)\right) = 0, \tag{4.16}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z) + \lambda_1^r h^r(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z) + \lambda_1^r h^r(z)\Big)\right) = 0. \tag{4.17}$$

Following the scheme of obtaining the Euler-Lagrange conditions for case (i), one can show that these conditions for other cases are as follows:
Case (ii): h, g w.r.t. x and g w.r.t. $\dot{x}$ are both (i)-gH differentiable ((ii)-gH differentiable) and h is (ii)-gH differentiable ((i)-gH differentiable) w.r.t $\dot{x}$.

$$\frac{\partial}{\partial x^l}\Big(g^l(z) + \lambda_1^l h^l(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z) + \lambda_1^l h^r(z)\Big)\right) = 0, \tag{4.18}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z) + \lambda_2^r h^l(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z) + \lambda_2^r h^r(z)\Big)\right) = 0, \tag{4.19}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z) + \lambda_2^l h^r(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z) + \lambda_2^l h^l(z)\Big)\right) = 0, \tag{4.20}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z) + \lambda_1^r h^r(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z) + \lambda_1^r h^l(z)\Big)\right) = 0. \tag{4.21}$$

Case (iii): h w.r.t. x and $\dot{x}$ is (ii)-gH differentiable ((i)-gH differentiable) and g is (i)-gH differentiable ((ii)-gH differentiable) w.r.t x and $\dot{x}$.

$$\frac{\partial}{\partial x^l}\Big(g^l(z) + \lambda_1^l h^r(z)\Big) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z) + \lambda_1^l h^r(z)\Big)\right) = 0, \tag{4.22}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_2^r h^r(z)\Big)\right)=0, \tag{4.23}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_2^l h^l(z)\Big)\right)=0, \tag{4.24}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_1^r h^l(z)\Big)\right)=0. \tag{4.25}$$

Case (iv): h, g w.r.t. $\dot{x}$ and g w.r.t. x are both (i)-gH differentiable ((ii)-gH differentiable) and h is (ii)-gH differentiable ((i)-gH differentiable) w.r.t x.

$$\frac{\partial}{\partial x^l}\Big(g^l(z)+\lambda_1^l h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z)+\lambda_1^l h^l(z)\Big)\right)=0, \tag{4.26}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_2^r h^l(z)\Big)\right)=0, \tag{4.27}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_2^l h^r(z)\Big)\right)=0, \tag{4.28}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_1^r h^r(z)\Big)\right)=0. \tag{4.29}$$

Case (v): h, g w.r.t. x and h w.r.t. $\dot{x}$ are both (i)-gH differentiable ((ii)-gH differentiable) and g is (ii)-gH differentiable ((i)-gH differentiable) w.r.t $\dot{x}$.

$$\frac{\partial}{\partial x^l}\Big(g^l(z)+\lambda_1^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_1^l h^l(z)\Big)\right)=0, \tag{4.30}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_2^r h^l(z)\Big)\right)=0, \tag{4.31}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z)+\lambda_2^l h^r(z)\Big)\right)=0, \tag{4.32}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_1^r h^r(z)\Big)\right)=0. \tag{4.33}$$

Case (vi): h, g w.r.t. x are both (i)-gH differentiable ((ii)-gH differentiable) and h, g are both (ii)-gH differentiable ((i)-gH differentiable) w.r.t $\dot{x}$.

$$\frac{\partial}{\partial x^l}\Big(g^l(z)+\lambda_1^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_1^l h^r(z)\Big)\right)=0, \tag{4.34}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_2^r h^r(z)\Big)\right)=0, \tag{4.35}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z)+\lambda_2^l h^l(z)\Big)\right)=0, \tag{4.36}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_1^r h^l(z)\Big)\right)=0. \tag{4.37}$$

Case (vii): h w.r.t. $\dot{x}$ and g w.r.t. x are both (i)-gH differentiable ((ii)-gH differentiable) and h w.r.t. x and g w.r.t. $\dot{x}$ are both (ii)-gH differentiable ((i)-gH differentiable).

$$\frac{\partial}{\partial x^l}\Big(g^l(z)+\lambda_1^l h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_1^l h^l(z)\Big)\right)=0, \tag{4.38}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_2^r h^l(z)\Big)\right)=0, \tag{4.39}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z)+\lambda_2^l h^r(z)\Big)\right)=0, \tag{4.40}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_1^r h^r(z)\Big)\right)=0. \tag{4.41}$$

Case (viii): g w.r.t. x is (i)-gH differentiable ((ii)-gH differentiable) and h, g w.r.t. $\dot{x}$ and h w.r.t. x are both (ii)-gH differentiable ((i)-gH differentiable).

$$\frac{\partial}{\partial x^l}\Big(g^l(z)+\lambda_1^l h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^r(z)+\lambda_1^l h^r(z)\Big)\right)=0, \tag{4.42}$$

$$\frac{\partial}{\partial x^r}\Big(g^l(z)+\lambda_2^r h^r(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^r(z)+\lambda_2^r h^r(z)\Big)\right)=0, \tag{4.43}$$

$$\frac{\partial}{\partial x^l}\Big(g^r(z)+\lambda_2^l h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^l}\Big(g^l(z)+\lambda_2^l h^l(z)\Big)\right)=0, \tag{4.44}$$

$$\frac{\partial}{\partial x^r}\Big(g^r(z)+\lambda_1^r h^l(z)\Big)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^r}\Big(g^l(z)+\lambda_1^r h^l(z)\Big)\right)=0. \tag{4.45}$$

5. Numerical example

In this section, we give an example which illustrates the applicability of results of the paper. Note that there is no solution for this example using the results obtained by Farhadinia [11].

Example 5.1. Find the minimum of

$$J(x) := \int_0^1 -\dot{x}^2(t)dt,$$

subject to

$$I(x) := \int_0^1 x(t)dt \approx c =< 0,1,3 >,$$

$$x(0) \approx 2 =< 0,2,4 >, x(1) \approx 4 =< 2,4,6 > .$$

Solution. In order to find the optimal solution of the above problem, it suffices to find the optimal solution of

$$\mathcal{J}(x) := \int_0^1 (-\dot{x}^2(t) + \lambda x(t))dt,$$

given that

$$x(0) \approx 2 =< 0,2,4 >, x(1) \approx 4 =< 2,4,6 > .$$

We first derive α-level set of $\mathcal{J}$ as follows:

$$\mathcal{J}(x)[\alpha] = \int_0^1 \Big[-\dot{x}^{r^2}(t,\alpha) + \lambda^l(\alpha)x^l(t,\alpha), -\dot{x}^{l^2}(t,\alpha) + \lambda^r(t)x^r(t,\alpha)\Big]dt. \tag{5.1}$$

We have $g := -\dot{x}^2(t)$ and $h := x(t)$. Suppose that case (i) is fulfilled in this FIP, i.e. g and h are (i)-gH differentiable ((ii)-gH differentiable) w.r.t. x and $\dot{x}$. Using (4.16) for equation (5.1), we have $\ddot{x}^l(t,\alpha) = 0$. Hence, by virtue of the classical differential equation theory, we may solve it analytically for fixed $\alpha \in [0,1]$ to get

$$x^l(t,\alpha) = k_1 t + k_2.$$

Here, the constants of integration i.e. k_1, k_2, might be given by the endpoint conditions, so,

$$x^l(t,\alpha) = 2t + 2\alpha.$$

On the other hand, in view of $< 0,1,3 > [\alpha] = [\alpha, 3-2\alpha]$, the above left-hand endpoint of the α-level set of extremal must satisfy the fuzzy constraint $I(x)$. That is

$$\int_0^1 (2t + 2\alpha)dt = \alpha,$$

which is contradiction with $\alpha \in [0,1]$. Then this problem has no solution with Euler-Lagrange conditions obtained in [11].

In case (ii), by using (4.20), we have $\ddot{x}^l(t,\alpha) = 0$. Similar previous case, we cannot find the solution of problem.

Now, suppose that h w.r.t. x and $\dot{x}$ be (ii)-gH differentiable ((i)-gH differentiable) and g be (i)-gH differentiable ((ii)-gH differentiable) w.r.t x and $\dot{x}$ (according to case (iii)). In this case, the fuzzy Euler-Lagrange conditions (4.22)-(4.25) say

$$\lambda_2^r(\alpha) - \frac{d}{dt}(-2\dot{x}^r(t,\alpha)) = 0,$$

$$\lambda_2^l(\alpha) - \frac{d}{dt}(-2\dot{x}^l(t,\alpha)) = 0.$$

From the classical differential equation theory and the endpoint conditions, we have

$$x^r(t,\alpha) = -\frac{\lambda_2^r(\alpha)}{4}t^2 + (2 + \frac{\lambda_2^r(\alpha)}{4})t + 4 - 2\alpha, \tag{5.2}$$

$$x^l(t,\alpha) = -\frac{\lambda_2^l(\alpha)}{4}t^2 + (2 + \frac{\lambda_2^l(\alpha)}{4})t + 2\alpha. \tag{5.3}$$

Now by virtue of $< 0, 1, 3 > [\alpha] = [\alpha, 3 - 2\alpha]$ and the fact that the above left-hand and right-hand endpoints of α-level set of extremal must satisfy the fuzzy constraint $I(x)$, $\lambda_2^l(\alpha), \lambda_2^r(\alpha)$ are determined by considering

$$3 - 2\alpha = \int_0^1 \Big(-\frac{\lambda_2^r(\alpha)}{4}t^2 + (2 + \frac{\lambda_2^r(\alpha)}{4})t + 4 - 2\alpha\Big)dt,$$

$$\alpha = \int_0^1 \Big(-\frac{\lambda_2^l(\alpha)}{4}t^2 + (2 + \frac{\lambda_2^l(\alpha)}{4})t + 2\alpha\Big)dt,$$

which result in $\lambda_2^r(\alpha) = -48$ and $\lambda_2^l(\alpha) = -24(\alpha+1)$. According to this results (5.2)-(5.3) turn to

$$x^r(t,\alpha) = 12t^2 - 10t + 4 - 2\alpha,$$

$$x^l(t,\alpha) = 6(\alpha+1)t^2 - (4 + 6\alpha)t + 2\alpha.$$

One can easily show that

$$\frac{\partial x^r(t,\alpha)}{\partial\alpha} = -2 \leq 0, \quad \forall t \in [0,1],$$

$$\frac{\partial x^l(t,\alpha)}{\partial\alpha} = 6t^2 - 6t + 2 \geq 0, \quad \forall t \in [0,1],$$

that is, $x^r(t,\alpha)$ and $x^l(t,\alpha)$ are continuous nonincreasing and nondecreasing functions of α respectively (conditions (ii) and (i) of Proposition 2.1).
Moreover, for all $0 \leq t \leq 1, x^l(t,1) = 12t^2 - 10t + 2$, and $x^r(t,1) = 12t^2 - 10t + 2$. Hence, it holds $x^l(t,1) \leq x^r(t,1)$ (condition (iii) of Proposition 2.1).

Consequently, $x^* \in X_{ad}$ parameterized by

$$x^*(t)[\alpha] = [x^{*l}(t,\alpha), x^{*r}(t,\alpha)] = [6(\alpha+1)t^2 - (4+6\alpha)t + 2\alpha, 12t^2 - 10t + 4 - 2\alpha],$$

that defines α-level set of a fuzzy number which minimizes J in the fuzzy sense.

Concluding remarks

In this paper, we presented the fuzzy Euler-Lagrange condition for the fuzzy unconstrained and constraint variational problems using the concept of gH-differentiability. As it is shown in the paper, Case (i) of fuzzy variational and isoperimetric problems is coincident with the fuzzy Euler-Lagrange conditions in [11]. Thus, the results (3.19)-(3.26) and (4.14)-(4.45) are the generalization and extension of the results in [11] and they are also more applicable to a large class of problems.

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