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Fuzzy subsystems of fuzzy automata based on lattice-ordered monoid

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Abstract. The purpose of the present work is to introduce the concepts of L-fuzzy subsystems and strong L-fuzzy subsystems of fuzzy automata having the membership values in lattice-ordered monoid. Unlike to the usual fuzzy automata, we show that such concepts for fuzzy automata having the membership values in lattice-ordered monoid depend on the fact that whether associated monoid is with or without zero divisors.

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1. Introduction

The study of fuzzy automata was initiated by Santos [26] and Wee [32] in 1960's after the introduction of fuzzy set theory by Zadeh [35]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced and studied by Malik, Mordeson and Sen [17] (cf. [19], for more details). Somewhat different notions were introduced subsequently in [5, 6, 7, 8, 9, 22]. In these studies, the membership values in the closed interval $[0, 1]$ were considered. During the recent years, the researchers began to work on fuzzy automata with membership values in complete residuated lattice, lattice-ordered monoid and some other kind of lattices (cf., [4, 10, 11, 12, 13, 14, 15, 20, 21, 24, 27, 33, 34]). In application point of view, fuzzy automata have been shown to be useful in numerous engineering applications such as pattern recognition, clinical monitoring, and also used to model fuzzy discrete event systems (cf., [16, 19, 23, 25]).

In view of the fact that the algebraic properties play a key role in the development of fundamentals of computer science (cf. [3]), the concepts of separatedness, connectedness and retrievability of a fuzzy automaton were introduced and studied by Mordeson and Malik [17]. In [1, 28, 29], it is shown that certain topological and fuzzy topological concepts can be used in fuzzy automata theory to throw light on the structure of such fuzzy automata, particularly, to obtain certain results pertaining to their connectivity and separation properties. Similar studies for fuzzy automata with membership values in complete residuated lattice were proposed in [20]. But, interestingly, for fuzzy automata with membership values in lattice-ordered monoid, it is shown that the results discussed in [17, 28] depend on the associated monoid structure (cf. [30]).

Chiefly inspired from [30], in this paper, we study the concepts of L-fuzzy subsystem and strong L-fuzzy subsystem of fuzzy automata with membership values in lattice-ordered monoid and thereafter introduce the L-fuzzy topological concepts. We show that the results for L-fuzzy subsystem and strong L-fuzzy subsystem introduced in [1] and [18] may not hold well in the case of fuzzy automata with membership values in lattice-ordered monoid.

2. Preliminaries

In this section, we recall some basic concepts related to lattice-ordered monoid, fuzzy automaton and fuzzy point, which we shall need in the subsequent section.

We begin with the following from [2].

Definition 2.1. An algebra $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is called a *lattice-ordered monoid* if

(1) $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(2) $(L, \bullet, e)$ is a monoid with identity $e \in L$ such that for all $a, b, c \in L$

(i) $a \bullet 0 = 0 \bullet a = 0$,

(ii) $a \leq b \Rightarrow \forall x \in L, a \bullet x \leq b \bullet x$ and $x \bullet a \leq x \bullet b$,

(iii) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Definition 2.2. A monoid $(L, \bullet, e)$ is called *monoid without zero divisors* if for all $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b \neq 0$.

Definition 2.3. A monoid $(L, \bullet, e)$ is called *monoid with zero divisors* if there exist $a, b \in L, a \neq 0, b \neq 0$ such that $a \bullet b = 0$.

The concepts of L-fuzzy sets, L-fuzzy topologies and L-fuzzy automata, we study in this paper, have the membership values in lattice-ordered monoid. For example, an *L-fuzzy subset* of a nonempty set X is a function from X to L. Throughout, L^X denotes the family of all L-fuzzy subsets of X and $\underline{\alpha}$ denotes the α-valued constant L-fuzzy subset of X. For an L-fuzzy subset A of X and $t \in L$, the *level set* of A is $A_t = \{x \in X : A(x) \geq t\}$.

Now, we recall the following concepts related to a fuzzy automaton based on lattice-ordered monoid.

Definition 2.4 ([30]). An *L-fuzzy automaton* is a triple $M = (Q, X, \delta)$, where Q is a nonempty set (of *states* of M), X is a monoid (the *input monoid* of M), whose identity shall be denoted as e_X, and $\delta : Q \times X \times Q \to L$ is a map, such that

$\forall q, p \in Q, \forall x, y \in X$,

$$\delta(q, e_X, p) = \begin{cases} e & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\text{and} \quad \delta(q, xy, p) = \vee\{\delta(q, x, r) \bullet \delta(r, y, p) : r \in Q\}.$$

Definition 2.5 ([30])**.** Let (Q, X, δ) be an L-fuzzy automaton and $A \subseteq Q$. The *source* and the *successor* of A are respectively the sets

$$\sigma_Q(A) = \{q \in Q : \delta(q, x, p) > 0, \text{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s_Q(A) = \{p \in Q : \delta(q, x, p) > 0, \text{ for some } (x, q) \in X \times A\}.$$

We shall frequently write $\sigma_Q(A)$ and $s_Q(A)$ as just $\sigma(A)$ and $s(A)$, and $\sigma(\{q\})$ and $s(\{q\})$ as just $\sigma(q)$ and $s(q)$.

Remark 2.6 ([30])**.** Let (Q, X, δ) be an L-fuzzy automaton and $p, q, r \in Q$. Then $p \in \sigma(q), q \in \sigma(r)$ does not imply $p \in \sigma(r)$ in general.

Proposition 2.7 ([30])**.** *Let $(L, \bullet, e)$ be a monoid without zero divisors and (Q, X, δ) be an L-fuzzy automaton. Then for all $p, q, r \in Q$, $p \in \sigma(q), q \in \sigma(r) \Rightarrow p \in \sigma(r)$.*

We close this section by recalling the following from [31].

Definition 2.8. Let X be a nonempty set. An L-fuzzy point x_t is an L-fuzzy set defined by

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

$y \in X, t \in L$.

3. L-fuzzy subsystems of L-fuzzy automata

In this section, we introduce the concepts of L-fuzzy subsystems and strong L-fuzzy subsystems of an L-fuzzy automaton as a generalization of similar concepts for fuzzy automata introduced in [1] and [18]. We show that several results for such concepts introduced in [1] and [18] may not hold in this generalized setup.

We begin with the following.

Definition 3.1. $A \in L^Q$ is called an *L-fuzzy subsystem* of an L-fuzzy automaton (Q, X, δ) if

$$A(p) \geq (A(q) \bullet \delta(q, x, p)), \forall p, q \in Q, x \in X.$$

Proposition 3.2. *Let (Q, X, δ) be an L-fuzzy automaton and $A \in L^Q$. Define an L-fuzzy subset $c(A)$ of Q as:*

$$c(A)(p) = \vee\{A(q) \bullet \delta(q, x, p) : (x, q) \in X \times A\}.$$

Also, let $A, A_i (i \in J) \in L^Q$. Then

(i) $c(\underline{\alpha}) = \underline{\alpha}, \forall \alpha \in L$,
(ii) $A \leq c(A)$,
(iii) $c(\vee\{A_i : i \in J\}) = \vee\{c(A_i) : i \in J\}$.

Proof. The proofs are straightforward. □

Remark 3.3. From the above proposition, it is obvious that $c(A) \leq c(c(A))$, but in general $c(A) \neq c(c(A))$ as the following counter-example shows.

Counter-example 3.4. *For the lattice-ordered monoid L, consider the monoid $(L, \bullet, e)$, where $L = [0,1], a \bullet b = max(0, a+b-1), \forall a, b \in L$ and $e = 1$. Consider an L-fuzzy automaton (Q, X, δ), where $Q = X = \{0, 1, 2, ...\}$ and $\delta : Q \times X \times Q \to L$ is given by*

$$\delta(q, 0, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$\forall p, q \in Q$, and $\delta(q, x_0, p) = 0.6, \delta(q, x_0, r) = 0.4, \delta(r, x_0, p) = 0.35$, $\delta(p, x_0, q) = 1, \delta(r, x_0, q) = 0.3, \delta(p, x_0, r) = 0.4$, $\delta(p, x_0, p) = 0.6, \delta(q, x_0, q) = 0.5, \delta(r, x_0, r) = 0.7$ for fixed $p, q, r \in Q$ and for fixed $x_0 \in X (x_0 \neq 0)$. For other $p, q \in Q$ and $x \in X$, let $\delta(p, x, q) = 0$. Also, let $A \in L^Q$ such that $A(p) = 0.8, A(q) = 0.5, A(r) = 0.4$ for fixed $p, q, r \in Q$ and for other $p \in Q, A(p) = 0$. Then $c(A)(q) = 0.8$, while $c(c(A))(q) = 0.5$, showing that $c(c(A)) \neq c(A)$.

Proposition 3.5. *Let $(L, \bullet, e)$ be a monoid without zero divisors and (Q, X, δ) be an L-fuzzy automaton. Then for all $A \in L^Q, c(c(A)) = c(A)$.*

Proof. Let $A \in L^Q$. Then $c(c(A))(q) = \vee\{c(A)(p) \bullet \delta(p, x, q) : (x, p) \in X \times A\} = \vee\{\vee\{A(r) \bullet \delta(r, y, p) : (y, r) \in X \times A\} \bullet \delta(p, x, q) : (x, p) \in X \times A\} = \vee\{A(r) \bullet \delta(r, y, p) \bullet \delta(p, x, q) : x, y \in X, p, r \in A\}$. As $(L, \bullet, e)$ being a monoid without zero divisors, $\delta(r, y, p) > 0$ and $\delta(p, x, q) > 0$ implying that $\delta(r, y, p) \bullet \delta(p, x, q) = \delta(r, yx, q) > 0$. Thus $c(c(A))(q) = \vee\{\delta(r, z, q) \bullet A(r) : (z, r) \in X \times A\} = c(A)(q)$. Hence $c(c(A)) = c(A)$. □

Remark 3.6. Let (Q, X, δ) be an L-fuzzy automaton, where the monoid $(L, \bullet, e)$ is without zero divisors. Then from Propositions 3.2 and 3.5, it is clear that c is a Kuratowski saturated L-fuzzy closure operator on Q. Thus c induces an L-fuzzy topology on Q, say, τ_c. (A Kuratowski L-fuzzy closure operator c on X is being called here saturated if the(usual) requirement $c(A \vee B) = c(A) \vee c(B)$ is replaced by $c(\vee\{A_i : i \in J\}) = \vee\{c(A_i) : i \in J\}, \forall A, B, A_i \in L^X, i \in J)$.

Proposition 3.7. *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton, where the monoid $(L, \bullet, e)$ is without zero divisors. Then $A \in L^Q$ is an L-fuzzy subsystem of M if and only if A is τ_c-closed.*

Proof. Let A be an L-fuzzy subsystem of an L-fuzzy automaton M. Then $A(p) \geq A(q) \bullet \delta(q, x, p), \forall p, q \in Q, \forall x \in X$, whereby $A(p) \geq \vee\{A(q) \bullet \delta(q, x, p) : (x, q) \in X \times A\}$, or that $A(p) \geq c(A)(p)$. Also, $A(p) \leq c(A)(p), \forall p \in Q$. Thus $c(A) = A$. Hence A is an L-fuzzy subsystem of M. Converse follows similarly. □

Proposition 3.8. *Let (Q, X, δ) be an L-fuzzy automaton. Then for all $A \in L^Q$ and $\forall \alpha \in L, c(\underline{\alpha} \bullet A) = \underline{\alpha} \bullet c(A)$.*

Proof. Let $A \in L^Q$ and $\alpha \in L$. Then $c(\underline{\alpha} \bullet A)(q) = \vee\{(\underline{\alpha} \bullet A)(p) \bullet \delta(p, x, q) : (x, p) \in X \times A\} = \vee\{\underline{\alpha}(p) \bullet A(p) \bullet \delta(p, x, q) : (x, p) \in X \times A\} = \underline{\alpha} \bullet \vee\{A(p) \bullet \delta(p, x, q) : (x, p) \in X \times A\} = \underline{\alpha} \bullet c(A)(q)$. Thus $c(\underline{\alpha} \bullet A) = \underline{\alpha} \bullet c(A)$. □

Before stating the next, recall the following concept of an L-fuzzy subautomaton from [31].

Definition 3.9. Let $M = (Q, X, \delta)$ be an L-fuzzy automaton. $R \subseteq Q$ is called an *L-fuzzy subautomaton* of M if $s(R) \subseteq R$ and $\lambda = \delta|_{R \times X \times R}$.

Remark 3.10. Let A be an L-fuzzy subsystem of an L-fuzzy automaton $M = (Q, X, \delta)$. Then $N = (Supp(A), X, \lambda)$ may not be an L-fuzzy subautomaton of M as the following counter-example shows, where $\lambda = \delta|_{Supp(A) \times X \times Supp(A)}$.

Counter-example 3.11. *For the lattice-ordered monoid L, consider the monoid $(L, \bullet, e)$, where $L = [0, 1], a \bullet b = max(0, a + b - 1), \forall a, b \in L$ and $e = 1$. Consider an L-fuzzy automaton (Q, X, δ), where $Q = X = \{0, 1, 2, ...\}$ and $\delta : Q \times X \times Q \to L$ is given by*

$$\delta(q, 0, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p, \end{cases}$$

$\forall p, q \in Q$, and $\delta(q, x_0, p) = 1/2, \delta(p, x_0, q) = 1/2$ for fixed $x_0 \in X (x_0 \neq 0)$ and for fixed $p, q \in Q$. For other $p, q \in Q$ and $x \in X, \delta(p, x, q) = 0$. Also, let $A \in L^Q$ such that $A(q) = 1/2$ and $A(p) = 0$. Then $A(q) \bullet \delta(q, x_0, p) = 0 = A(p)$ and $A(p) \bullet \delta(p, x_0, q) = 0 < 1/2 = A(q)$. Thus A is an L-fuzzy subsystem of M. Also, $Supp(A) = \{p\}$. But $s(Supp(A)) = \{p, q\}$. Thus $s(Supp(A)) \neq Supp(A)$. Hence $N = (Supp(A), X, \lambda)$ is not an L-fuzzy subautomaton of M.

Proposition 3.12. *Let A be an L-fuzzy subsystem of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \bullet, e)$ be a monoid without zero divisors. Then $N = (Supp(A), X, \lambda)$ is an L-fuzzy subautomaton of M, where $\lambda = \delta|_{Supp(A) \times X \times Supp(A)}$.*

Proof. Let A be an L-fuzzy subsystem of $M = (Q, X, \delta)$ and $p \in s(Supp(A))$. Then $p \in s(q)$ for some $q \in Supp(A)$. Now, $q \in Supp(A)$ implying that $A(q) > 0$. Also, as $p \in s(q), \exists x \in X$ such that $\delta(q, x, p) > 0$. Thus $0 < A(q) \bullet \delta(q, x, p)$ as $(L, \bullet, e)$ is a monoid without zero divisors, whereby $0 < A(p)$, or that $p \in Supp(A)$. Hence $s(Supp(A)) \subseteq Supp(A)$ and therefore N is an L-fuzzy subautomaton of M. □

Proposition 3.13. *Let A be an L-fuzzy subsystem of an L-fuzzy automaton $M = (Q, X, \delta)$ and $N_t = (A_t, X, \lambda_t)$, where $\lambda_t = \delta|_{A_t \times X \times A_t}$, $t \in [0, 1]$. If $\forall t \in [0, 1]$, N_t is an L-fuzzy subautomaton of M, then A is an L-fuzzy subsystem of M.*

Proof. Let A be an L-fuzzy subsystem of M. Also, let $p, q \in Q$ and $x \in X$. If $A(p) = 0$ or $\delta(p, x, q) = 0$, then nothing is to prove. So, let $A(p) > 0$ and $\delta(p, x, q) > 0$. Then $A(p) \bullet \delta(p, x, q) > 0$. Now, let $A(p) \bullet \delta(p, x, q) = t$. Then $p \in A_t$. Also, as N_t is an L-fuzzy subautomaton of $M, s(A_t) = A_t$. Thus $q \in s(p) \subseteq s(A_t) = A_t$, or that $A(q) \geq t$, i.e., $A(q) \geq A(p) \bullet \delta(p, x, q)$. Hence A is an L-fuzzy subsystem of M. □

Now, we introduce the following concept, which resembles the concept of singly generated fuzzy subsystem introduced in [1].

Definition 3.14. Let (Q, X, δ) be an L-fuzzy automaton, $t \in (0, 1]$ and $q \in Q$. An L-fuzzy subset $q_t X$ of Q is given by

$$(q_t X)(p) = \vee\{t \bullet \delta(q, y, p) : y \in X\}, \forall p \in Q.$$

Proposition 3.15. *Let (Q, X, δ) be an L-fuzzy automaton, $t \in (0, 1]$ and $q \in Q$. Then $q_t X$ is an L-fuzzy subsystem of M.*

Proof. Follows from the fact that $\forall p \in Q, (q_tX)(p) = \vee\{t \bullet \delta(q, y, p) : y \in X\} = c(q_t)$ and $c(q_t)$ being τ_c-closed. □

Proposition 3.16. *Let (Q, X, δ) be an L-fuzzy automaton and the monoid $(L, \bullet, e)$ be without zero divisors. Then $Supp(q_tX) = s(q), \forall t \in (0, 1]$ and $\forall q \in Q$.*

Proof. Let $p \in s(q)$. Then $\delta(q, x, p) > 0$, for some $x \in X$. Thus for $t \in (0, 1], \vee\{t \bullet \delta(q, x, p) : x \in X\} > 0$ (as the monoid $(L, \bullet, e)$ is without zero divisors), whereby $(q_tX)(p) > 0$, or that $p \in Supp(q_tX)$. Hence $s(q) \subseteq Supp(q_tX)$.
Conversely, let $p \in Supp(q_tX)$. Then $\vee\{t \bullet \delta(q, x, p) : x \in X\} > 0$, or that there exists $x \in X$ such that $\delta(q, x, p) > 0$. Thus $p \in s(q)$, whereby $Supp(q_tX) \subseteq s(q)$. □

Remark 3.17. For an L-fuzzy automaton (Q, X, δ), $t \in (0, 1]$ and $q \in Q$, $Supp(q_tX) \neq s(q)$, as the following counter-example shows.

Counter-example 3.18. *For the lattice-ordered monoid L, consider the monoid $(L, \bullet, e)$, where $L = [0, 1], a \bullet b = max(0, a + b - 1), \forall a, b \in L$ and $e = 1$. Consider an L-fuzzy automaton (Q, X, δ), where $Q = X = \{0, 1, 2, ...\}$ and $\delta : Q \times X \times Q \to L$ is given by*

$$\delta(q, 0, p) = \begin{cases} 1 & \textit{if } q = p \\ 0 & \textit{if } q \neq p \end{cases}$$

$\forall q, p \in Q$, and $\delta(q, x_0, p) = 1/2$, for fixed $p, q \in Q$ and for fixed $x_0 \in X(x_0 \neq 0)$. For other $p, q \in Q$ and $x \in X$, $\delta(q, x, p) = 0$. Here $p \in s(q)$. Now, let $t = 1/2$. Then $Supp(q_tX) = \phi$ as $(q_tX)(p) = \vee\{t \bullet \delta(q, x_0, p) : x_0 \in X\} = 0$. Thus $Supp(q_tX) \neq s(q)$.

Inspired from [19], we now introduce the concept of a strong L-fuzzy subsystem of an L-fuzzy automaton.

Definition 3.19. $A \in L^Q$ is called a *strong L-fuzzy subsystem* of an L-fuzzy automaton (Q, X, δ) if $\delta(q, x, p) > 0$, for some $x \in X$, then $A(p) \geq A(q), \forall p, q \in Q$.

Proposition 3.20. *Let (Q, X, δ) be an L-fuzzy automaton and $A \in L^Q$. Define an L-fuzzy subset $\overline{c}(A)$ of Q as:*

$$\overline{c}(A)(q) = \vee\{A(p) : p \in \sigma(q)\}, \forall A \in L^Q, \forall q \in Q.$$

Also, let $A, A_i(i \in J) \in L^Q$. Then
(i) $\overline{c}(\underline{\alpha}) = \underline{\alpha}, \forall \alpha \in L,$
(ii) $A \leq \overline{c}(A),$
(iii) $\overline{c}(\vee\{A_i : i \in J\}) = \vee\{\overline{c}(A_i) : i \in J\}.$

Proof. The proofs are straightforward. □

Remark 3.21. From above definition, it is obvious that $\overline{c}(A) \leq \overline{c}(\overline{c}(A))$, but in general $\overline{c}(A) \neq \overline{c}(\overline{c}(A))$. As, for $q \in Q, \overline{c}(A)(q) = \vee\{A(p) : p \in \sigma(q)\}$. But $\overline{c}(\overline{c}(A))(q) = \vee\{\{A(r) : r \in \sigma(p)\} : p \in \sigma(q)\} \neq \vee\{A(r) : r \in \sigma(q)\} = \overline{c}(A)(q)$ (follows from Remark 2.6, $p \in \sigma(q), r \in \sigma(p) \nRightarrow r \in \sigma(q)$). Thus $\overline{c}(\overline{c}(A)) \neq \overline{c}(A)$.

Proposition 3.22. *Let $(L, \bullet, e)$ be a monoid without zero divisors and (Q, X, δ) be an L-fuzzy automaton. Then for all $A \in L^Q, \overline{c}(\overline{c}(A)) = \overline{c}(A)$.*

Proof. Let $A \in L^Q$ and $q \in Q$. Then $\overline{c}(\overline{c}(A))(q) = \vee\{\overline{c}(A)(p) : p \in \sigma(q)\} = \vee\{\vee\{A(r) : r \in \sigma(p)\} : p \in \sigma(q)\}$. Now, the monoid $(L, \bullet, e)$ being without zero divisors, $p \in \sigma(q), r \in \sigma(p) \Rightarrow r \in \sigma(q)$ (cf., Proposition 2.7). Thus $\overline{c}(\overline{c}(A))(q) = \vee\{A(r) : r \in \sigma(q)\} = \overline{c}(A)(q)$. Hence $\overline{c}(\overline{c}(A)) = \overline{c}(A)$. □

Remark 3.23. Let (Q, X, δ) be an L-fuzzy automaton, where the monoid $(L, \bullet, e)$ is without zero divisors. Then from Propositions 3.20 and 3.22, it is clear that $\overline{c}$ is a Kuratowski saturated L-fuzzy closure operator on Q. Thus $\overline{c}$ induces an L-fuzzy topology on Q, say, $\tau_{\overline{c}}(Q)$ or $\tau_{\overline{c}}$.

Proposition 3.24. *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton, where the monoid $(L, \bullet, e)$ is without zero divisors. Then $A \in L^Q$ is a strong L-fuzzy subsystem of M if and only if A is $\tau_{\overline{c}}$-closed.*

Proof. Let A be a strong L-fuzzy subsystem of an L-fuzzy automaton M and $\delta(p, x, q) > 0$ for some $x \in X$. Then $A(p) \leq A(q), \forall p, q \in Q$, whereby $A(q) \geq \vee\{A(p) : p \in \sigma(q)\}, \forall A \in L^Q, \forall p \in Q$, or that $A(q) \geq \overline{c}(A)(q)$. Also, $A(q) \leq \overline{c}(A)(q), \forall q \in Q$. Thus $\overline{c}(A) = A$. Hence A is a strong L-fuzzy subsystem of M. Converse follows similarly. □

Unlike to the case of L-fuzzy subsystems of an L-fuzzy automaton, the following does not depend on the monoid structure in case of strong L-fuzzy subsystems.

Proposition 3.25. *Let A be a strong L-fuzzy subsystem of an L-fuzzy automaton $M = (Q, X, \delta)$. Then $N = (Supp(A), X, \lambda)$ is an L-fuzzy subautomaton of M, where $\lambda = \delta|_{Supp(A) \times X \times Supp(A)}$*

Proof. Let $p \in s(Supp(A))$. Then $p \in s(q)$ for some $q \in Supp(A)$, or that $\delta(q, x, p) > 0$ for some $x \in X$ such that $A(q) > 0$. As, A is a strong L-fuzzy subsystem, $A(p) \geq A(q)$, whereby $A(p) > 0$, i.e., $p \in Supp(A)$. Thus $s(Supp(A)) \subseteq Supp(A)$. Hence N is an L-fuzzy subautomaton of M. □

Proposition 3.26. *Let A be a strong L-fuzzy subsystem of an L-fuzzy automaton (Q, X, δ) and $N_t = (A_t, X, \lambda_t)$, where $\lambda_t = \delta|_{A_t \times X \times A_t}, t \in [0, 1]$. If $\forall t \in [0, 1], N_t$ is an L-fuzzy subautomaton of M, then A is a strong L-fuzzy subsystem of M.*

Proof. Let $p, q \in Q$ and $x \in X$ such that $\delta(p, x, q) > 0$. Suppose $A(p) > 0$ such that $A(p) = t$. Then $p \in A_t$. Also, as N_t is an L-fuzzy subautomaton of $M, s(A_t) = A_t$. Thus $q \in s(p) \subseteq s(A_t) = A_t$, or that $A(q) \geq t$, i.e., $A(q) \geq A(p)$. Hence A is a strong L-fuzzy subsystem of M. □

Finally, we introduce the following concept of homomorphism between fuzzy automata based on lattice-ordered monoid.

Definition 3.27. Let $M = (Q, X, \delta)$ and $N = (R, X, \lambda)$ be L-fuzzy automata. Then $f : Q \to R$ is called a homomorphism from M to N if

$$\delta(q, x, p) \leq \lambda(f(q), x, f(p)), \forall p, q \in Q, x \in X.$$

The following is an L-fuzzy topological characterization of homomorphism between two fuzzy automata based on lattice-ordered monoid.

Proposition 3.28. *If $f : (Q, X, \delta) \to (R, X, \lambda)$ is a homomorphism between L-fuzzy automata (Q, X, δ) and (R, X, λ), then $f : (Q, \tau_{\underline{c}}(Q)) \to (R, \tau_{\underline{c}}(R))$ is L-fuzzy continuous.*

Proof. Similar to the proof given in [29]. □

4. Conclusions

In this paper, we have tried to studied the concepts of L-fuzzy subsystems and strong L-fuzzy subsystems of fuzzy automata based on lattice ordered monoid. Interestingly, we found that the such concepts for fuzzy automata based on lattice-ordered monoid depend on the associated monoid structure. The obtained results generalize the observations made in [1] and [17].

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