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Interval-valued fuzzy quasi-ideals and bi-ideals of semirings

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Abstract. The interval-valued prime fuzzy ideals (in brevity, the i-v. prime fuzzy ideals) of a semigroup have been recently studied by Kar, Sarkar and Shum [18]. As a continued study of i-v fuzzy ideals, we are going to investigate the properties of i-v fuzzy quasi-ideal and i-v fuzzy bi-ideal of a semiring and then we characterize the regularity and intra-regularity of a semiring in terms of the above i-v fuzzy ideals.

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1. Introduction

Quasi-ideals of rings and semigroups were introduced and investigated by O. Steinfeld ([24], [25], [26]). In 1975, R. A. Good and D. R. Hughes [11] also considered the bi-ideals in semirings. The quasi-ideals are generalization of left ideals and right ideals whereas the bi-ideals are the generalization of quasi-ideals. The properties of fuzzy subquasi semigroup of a quasigroup were investigated by W. A. Dudek in [7]. S. Kar and P. Sarkar considered the fuzzy quasi-ideals and fuzzy bi-ideals of a ternary semigroup in [16].

The notion of i-v fuzzy set was introduced by L. A. Zadeh in 1975 [29]. Later on, I. Grattan-Guiness [12], K. U. Jahn [14] and R. Sambuc [23] studied the i-v fuzzy sets and they regarded this kind of fuzzy sets as a generalization of the ordinary fuzzy set. In fact, i-v fuzzy sets (in short, IVFS) are defined in terms of i-v membership functions.

After the i-v fuzzy sets have been introduced (see [3], [4], [5], [6], [13], [15], [17], [20], [21], [27]), some theories related with i-v fuzzy sets have been developed. There are natural ways to fuzzify various algebraic structures and the approaches have already been extensively studied in the literature. In particular, A. Rosenfeld [22] studied the fuzzy subgroups in 1971. Also, N. Kuroki in 1979 [19] further mentioned the fuzzy semigroups. In 1993, J. Ahsan, K. Saifullah and M. Farid Khan [1] introduced the fuzzy semirings. Recently, many interesting results of semirings have been obtained and given by using the context of fuzzy sets.

In this paper, we first introduce i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of a semiring. Then, we proceed to characterize the regular and intra-regular semirings by using the i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of the semirings. Our study of i-v fuzzy quasi-ideals in this paper is a continued study of our recent work on i-v fuzzy ideals of a semigroup [18].

2. Preliminaries

Definition 2.1 ([10])**.** A non-empty set S together with two binary operations '+' and '$\cdot$' is said to be a semiring if (i) $(S,+)$ is an abelian semigroup; (ii) $(S,\cdot)$ is a semigroup and (iii) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in S$.

Let $(S,+,\cdot)$ be a semiring. If there exists an element '0_S' $\in S$ such that $a+0_S=a=0_S+a$ and $a\cdot 0_S=0_S=0_S\cdot a$ for all $a\in S$; then '0_S' is called the zero element of S.

Throughout this paper, we consider a semiring $(S,+,\cdot)$ with a zero element '0_S'. Unless otherwise stated, a semiring $(S,+,\cdot)$ will be simply denoted by S and the multiplication '$\cdot$' will be denoted by juxtaposition. In this paper, by the product AB of two subsets A and B of a semiring S, we mean the finite sum $\sum_{i=1}^{n} a_ib_i$, for some $a_i\in A$, $b_i\in B$ and $n\in\mathbb{Z}^+$.

Definition 2.2 ([15])**.** An interval number on $[0,1]$, denoted by $\widetilde{a}$, is defined as the closed subinterval of $[0,1]$, where $\widetilde{a}=[a^-,a^+]$ satisfying $0\le a^-\le a^+\le 1$.

For any two interval numbers $\widetilde{a}=[a^-,a^+]$ and $\widetilde{b}=[b^-,b^+]$, we define the followings:

(i) $\widetilde{a}\le\widetilde{b}$ if and only if $a^-\le b^-$ and $a^+\le b^+$.
(ii) $\widetilde{a}=\widetilde{b}$ if and only if $a^-=b^-$ and $a^+=b^+$.
(iii) $\widetilde{a}<\widetilde{b}$ if and only if $\widetilde{a}\ne\widetilde{b}$ and $\widetilde{a}\le\widetilde{b}$.

Note 2.3. *We write $\widetilde{a}\ge\widetilde{b}$ whenever $\widetilde{b}\le\widetilde{a}$ and $\widetilde{a}>\widetilde{b}$ whenever $\widetilde{b}<\widetilde{a}$.*
We denote the interval number $[0,0]$ by $\widetilde{0}$ and $[1,1]$ by $\widetilde{1}$.

Definition 2.4 ([6])**.** Let $\{\widetilde{a_i}: i\in\Lambda\}$ be a family of interval numbers, where $\widetilde{a_i}=[a_i^-,a_i^+]$. Then we define $\sup_{i\in\Lambda}\{\widetilde{a_i}\}=[\sup_{i\in\Lambda}a_i^-,\sup_{i\in\Lambda}a_i^+]$ and $\inf_{i\in\Lambda}\{\widetilde{a_i}\}=[\inf_{i\in\Lambda}a_i^-,\inf_{i\in\Lambda}a_i^+]$.

We denote the set of all interval numbers on $[0,1]$ by $D[0,1]$.
Let us recall the following known definitions.

Definition 2.5 ([28])**.** Let S be a non-empty set. Then a mapping $\mu: S\longrightarrow[0,1]$ is called a fuzzy subset of S.

Definition 2.6 ([29])**.** Let S be a non-empty set. Then, a mapping $\widetilde{\mu} :\longrightarrow D[0,1]$ is called an i-v fuzzy subset of S.

Note 2.7 ([8])**.** *We can write $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in S$, for any i-v fuzzy subset $\widetilde{\mu}$ of a non-empty set S, where μ^- and μ^+ are some fuzzy subsets of S.*

We state below several definitions which will be useful in further study of this paper.

Definition 2.8 ([8])**.** Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two i-v fuzzy subsets of a set $X \neq \varnothing$. Then $\widetilde{\mu_1}$ is said to be subset of $\widetilde{\mu_2}$, denoted by $\widetilde{\mu_1} \subseteq \widetilde{\mu_2}$ if $\widetilde{\mu_1}(x) \leq \widetilde{\mu_2}(x)$ i.e. $\mu_1^-(x) \leq \mu_2^-(x)$ and $\mu_1^+(x) \leq \mu_2^+(x)$, for all $x \in X$ where $\widetilde{\mu_1}(x) = [\mu_1^-(x), \mu_1^+(x)]$ and $\widetilde{\mu_2}(x) = [\mu_2^-(x), \mu_2^+(x)]$.

Definition 2.9 ([15])**.** The interval Min-norm is a function $Min^i : D[0,1] \times D[0,1] \longrightarrow D[0,1]$, defined by :
$Min^i(\widetilde{a}, \widetilde{b}) = [min(a^-, b^-), min(a^+, b^+)]$ for all $\widetilde{a}$, $\widetilde{b} \in D[0,1]$, where $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$.

Definition 2.10 ([8])**.** The interval Max-norm is a function $Max^i : D[0,1] \times D[0,1] \longrightarrow D[0,1]$, defined by : $Max^i(\widetilde{a}, \widetilde{b}) = [max(a^-, b^-), max(a^+, b^+)]$ for all $\widetilde{a}$, $\widetilde{b} \in D[0,1]$, where $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$.

Definition 2.11 ([8])**.** Let $X \neq \varnothing$ be a set and $A \subseteq X$. Then the i-v characteristic function $\widetilde{\chi}_A$ of A is an i-v fuzzy subset of X which is defined as follows :

$$\widetilde{\chi}_A(x) = \begin{cases} \widetilde{1} & \text{when} \quad x \in A. \\ \widetilde{0} & \text{when} \quad x \notin A. \end{cases}$$

Definition 2.12. Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two i-v fuzzy subsets of a non-empty set X. Then we define their intersection and union by $(\widetilde{\mu_1} \cap \widetilde{\mu_2})(x) = Min^i(\widetilde{\mu_1}(x), \ \widetilde{\mu_2}(x))$ and $(\widetilde{\mu_1} \cup \widetilde{\mu_2})(x) = Max^i(\widetilde{\mu_1}(x), \ \widetilde{\mu_2}(x))$ for all $x \in X$.

The following results can be easily observed.

Lemma 2.13. *Let S be a non-empty set and A, B be two subsets of S. Then $\widetilde{\chi}_A \cup \widetilde{\chi}_B = \widetilde{\chi}_{A \cup B}$ and $\widetilde{\chi}_A \cap \widetilde{\chi}_B = \widetilde{\chi}_{A \cap B}$.*

Lemma 2.14 ([9])**.** *Let A and B be two non-empty subsets of a semiring S. Then $\widetilde{\chi}_A \widetilde{\chi}_B = \widetilde{\chi}_{AB}$.*

We first state the definition of a fuzzy point in a semiring S.

Definition 2.15 ([8])**.** Let S be a semiring and $x \in S$. Let $\widetilde{a} \in D[0,1] \setminus \{\widetilde{0}\}$. Then an i-v fuzzy subset $x_{\widetilde{a}}$ of S is called an i-v fuzzy point of S if

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a} & if \ x \ = \ y, \\ \widetilde{0} & otherwise. \end{cases}$$

We now state the definitions of i-v fuzzy left(right) ideals of a semiring.

Definition 2.16 ([9])**.** Let $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring S (i.e. $\widetilde{\mu}(x) \neq \widetilde{0}$ for some $x \in S$). Then $\widetilde{\mu}$ is called an i-v fuzzy left (resp. i-v fuzzy right) ideal of S if the following conditions hold.

(i) $\widetilde{\mu}(x+y) \geq Min^i(\widetilde{\mu}(x),\ \widetilde{\mu}(y))$
(ii) $\widetilde{\mu}(xy) \geq \widetilde{\mu}(y)$ [resp. $\widetilde{\mu}(xy) \geq \widetilde{\mu}(x)$], for all $x, y \in S$.

An i-v fuzzy ideal of a semiring S is a non-empty i-v fuzzy subset of S which is an i-v fuzzy left ideal as well as an i-v fuzzy right ideal of S.

Definition 2.17 ([9]). Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two i-v fuzzy subsets of a semiring S. Then their product, denoted by $\widetilde{\mu_1}\widetilde{\mu_2}$, is defined by :

$$(\widetilde{\mu_1}\widetilde{\mu_2})(x) = \begin{cases} \sup\left\{ \inf\limits_{1\leq i\leq n}\left\{Min^i\Big(\widetilde{\mu_1}(u_i),\ \widetilde{\mu_2}(v_i)\Big)\right\} : x = \sum\limits_{i=1}^{n} u_i v_i; \right. \\ \qquad\qquad \left. u_i, v_i \in S, n \in \mathbb{Z}^+ \right\}; \\ \widetilde{0} \qquad \text{if } x \text{ can not be expressed as } \quad x = \sum\limits_{i=1}^{n} u_i v_i; \ \text{ for any } \quad u_i, v_i \in S. \end{cases}$$

Throughout this paper, we assume that any two interval numbers in $D[0,1]$ are comparable, i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0,1]$, we have either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$.

3. i-v fuzzy quasi-ideals of a semiring

We begin with the following definition of i-v fuzzy quasi-ideal of a semiring. Some properties of the quasi subsemigroups of a quasigroup have already been studied in [7].

Definition 3.1. A non-empty i-v fuzzy subset $\widetilde{\mu}$ of a semiring S is said to be an i-v fuzzy quasi-ideal of S if for any $x, y \in S$, $\widetilde{\mu}(x+y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$ and $\widetilde{\mu}\widetilde{\chi}_S \cap \widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}$.

Lemma 3.2. *For any three i-v fuzzy subsets $\widetilde{\mu_1}, \widetilde{\mu_2}, \widetilde{\mu_3}$ of a semiring S,we have the following properties :*
(i) $\widetilde{\mu_1}(\widetilde{\mu_2} \cup \widetilde{\mu_3}) = (\widetilde{\mu_1}\widetilde{\mu_2}) \cup (\widetilde{\mu_1}\widetilde{\mu_3})$; $(\widetilde{\mu_2} \cup \widetilde{\mu_3})\widetilde{\mu_1} = (\widetilde{\mu_2}\widetilde{\mu_1}) \cup (\widetilde{\mu_3}\widetilde{\mu_1})$
(ii) $\widetilde{\mu_1}(\widetilde{\mu_2} \cap \widetilde{\mu_3}) \subseteq (\widetilde{\mu_1}\widetilde{\mu_2}) \cap (\widetilde{\mu_1}\widetilde{\mu_3})$; $(\widetilde{\mu_2} \cap \widetilde{\mu_3})\widetilde{\mu_1} \subseteq (\widetilde{\mu_2}\widetilde{\mu_1}) \cap (\widetilde{\mu_3}\widetilde{\mu_1})$.

For i-v fuzzy quasi-ideals of a semiring S, we have the following lemmas.

Lemma 3.3. *A non-empty subset A of a semiring S is a quasi-ideal of S if and only if $\widetilde{\chi}_A$ is an i-v fuzzy quasi-ideal of S.*

Lemma 3.4. *Let S be a semiring. Then the following statements hold.*
(i) *Every i-v fuzzy left (or right) ideal of S is an i-v fuzzy quasi-ideal of S.*
(ii) *The intersection of an i-v fuzzy left ideal and an i-v fuzzy right ideal of S is an i-v fuzzy quasi-ideal of S.*
(iii) *If $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of S, then $\widetilde{\chi}_S\widetilde{\mu}$ is an i-v fuzzy left ideal, $\widetilde{\mu}\widetilde{\chi}_S$ is an i-v fuzzy right ideal, $\widetilde{\chi}_S\widetilde{\mu}\widetilde{\chi}_S$ is an i-v fuzzy ideal and $\widetilde{\chi}_S\widetilde{\mu} \cap \widetilde{\mu}\widetilde{\chi}_S$ is an i-v fuzzy quasi-ideal of S.*

Note 3.5. *It can be easily seen that each i-v fuzzy left ideal or an i-v fuzzy right ideal of a semiring S is an i-v fuzzy quasi-ideal of S. But the converse is in general not true. We have the following example.*

Example 3.6. We consider the semiring $S = M_2(\mathbb{N}_0)$ with respect to the usual addition and multiplication of matrices. Suppose that P is the set
$P = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{N}_0) \right\}$. Now we define an i-v fuzzy subset
$\widetilde{\mu} : M_2(\mathbb{N}_0) \longrightarrow D[0,1]$ by

$$\widetilde{\mu}(A) = \begin{cases} [0.8, 0.9] & \text{when} \quad A \in P; \\ [0.3, 0.4] & \text{otherwise.} \end{cases}$$

We can easily check that $\widetilde{\mu}$ is an i-v fuzzy quasi-ideal of S but is not an i-v fuzzy left ideal and an i-v fuzzy right ideal of S either.

We state the following proposition concerning the i-v fuzzy quasi-ideals of a semiring.

Proposition 3.7. *Let $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ be two idempotent i-v fuzzy points of a semiring S. Also let $\widetilde{\mu}$ and $\widetilde{\theta}$ be an i-v fuzzy left ideal and i-v fuzzy right ideal of S, respectively. Then, we deduce the following equalities:*

$x_{\widetilde{a}}\widetilde{\mu} = x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu}$, $\widetilde{\theta} y_{\widetilde{b}} = \widetilde{\chi}_S y_{\widetilde{b}} \cap \widetilde{\theta}$, $x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}} = x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}}$ and so each of these i-v fuzzy subsets is an i-v fuzzy quasi-ideal of S.

Proof. We first prove that $x_{\widetilde{a}}\widetilde{\mu} = x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu}$. Clearly, $x_{\widetilde{a}}\widetilde{\mu} \subseteq x_{\widetilde{a}}\widetilde{\chi}_S$. Also since $\widetilde{\mu}$ is an i-v fuzzy left ideal of S, we have $x_{\widetilde{a}}\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}$. It hence follows that $x_{\widetilde{a}}\widetilde{\mu} \subseteq x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu}$.

For the reverse inclusion, we let $y \in S$. Then, we have
$(x_{\widetilde{a}}x_{\widetilde{a}})(y) = \sup\limits_{y=\sum_{i=1}^{m} p_i q_i} \left\{ \inf\limits_{1\le i\le m} \{Min^i(x_{\widetilde{a}}(p_i), x_{\widetilde{a}}(q_i))\} \right\}$. If $(x_{\widetilde{a}}x_{\widetilde{a}})(y) = \widetilde{a}$, then there exists at least one expression $y = \sum\limits_{i=1}^{m} p_i q_i$, for which
$\inf\limits_{1\le i\le m} \{Min^i(x_{\widetilde{a}}(p_i), x_{\widetilde{a}}(q_i))\} = \widetilde{a}$.

$\Longrightarrow Min^i(x_{\widetilde{a}}(p_i), x_{\widetilde{a}}(q_i)) = \widetilde{a}$ for each $1 \le i \le m$, where $y = \sum\limits_{i=1}^{m} p_i q_i$

$\Longrightarrow p_i = x = q_i$ for each $1 \le i \le m$, where $y = \sum\limits_{i=1}^{m} p_i q_i$.

Now $x_{\widetilde{a}}(y) = (x_{\widetilde{a}}x_{\widetilde{a}})(y)$ implies that $y = x$. Hence, we have $x = \sum\limits_{i=1}^{m} p_i q_i$. Thus we get $x = \sum\limits_{i=1}^{m} x^2$. Let $z \in S$. Then $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z) = Min^i\Big((x_{\widetilde{a}}\widetilde{\chi}_S)(z), \widetilde{\mu}(z)\Big)$. Let T be the set

$$T = \Big\{ \sum_{i=1}^{n} a_i b_i : a_i, b_i \in S; n \in \mathbb{N} \Big\}$$

<u>*CaseI*</u> : If $z \notin T$, then $(x_{\widetilde{a}}\widetilde{\chi}_S)(z) = \widetilde{0}$. Therefore, $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z) = \widetilde{0}$ and also $(x_{\widetilde{a}}\widetilde{\mu})(z) = \widetilde{0}$.

<u>*CaseII*</u> : Let $z \in T$. Then, we have

$$\begin{aligned}(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z) &= Min^i\Big(\sup_{z=\sum_{i=1}^{n} a_i b_i}\Big\{\inf_{1\le i\le n} Min^i(x_{\widetilde{a}}(a_i), \widetilde{\chi}_S(b_i))\Big\}, \widetilde{\mu}(z)\Big)\\ &= Min^i\Big(\sup_{z=\sum_{i=1}^{n} a_i b_i}\Big\{\inf_{1\le i\le n}\{x_{\widetilde{a}}(a_i)\}\Big\}, \widetilde{\mu}(z)\Big).\end{aligned}$$

Now, we let T_1 be the set $T_1 = \Big\{\sum_{i=1}^{n} a_i b_i \in T : a_i = x \text{ for all } 1 \le i \le n\Big\}$. If $z \in T_1$, then, $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z) = Min^i(\widetilde{a}, \widetilde{\mu}(z))$. Again, $z \in T_1 \Longrightarrow z = \sum_{i=1}^{n} x b_i = x\sum_{i=1}^{n} b_i = (\sum_{i=1}^{m} x^2)\sum_{i=1}^{n} b_i = \sum_{i=1}^{m} x(x\sum_{i=1}^{n} b_i)$. Therefore,

$$\begin{aligned}(x_{\widetilde{a}}\widetilde{\mu})(z) &\ge \inf_{1\le i\le m}\Big\{Min^i\Big(x_{\widetilde{a}}(x), \widetilde{\mu}(\sum_{i=1}^{n} x b_i)\Big)\Big\}\\ &= \inf_{1\le i\le m}\Big\{Min^i\Big(\widetilde{a}, \widetilde{\mu}(z)\Big)\Big\} = Min^i(\widetilde{a}, \widetilde{\mu}(z)) = (x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z).\end{aligned}$$

If $z \in T \setminus T_1$, then $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu})(z) = \widetilde{0} = (x_{\widetilde{a}}\widetilde{\mu})(z)$. Thus we get $x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu} \subseteq x_{\widetilde{a}}\widetilde{\mu}$. Consequently, $x_{\widetilde{a}}\widetilde{\mu} = x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\mu}$.

Again, we see that $x_{\widetilde{a}}\widetilde{\mu}$ is an intersection of an i-v fuzzy left ideal and an i-v fuzzy right ideal of S. Hence $x_{\widetilde{a}}\widetilde{\mu}$ is an i-v fuzzy quasi-ideal of S, by Lemma 3.4 (ii).

Similarly, we can prove that $\widetilde{\theta}y_{\widetilde{b}} = \widetilde{\chi}_S y_{\widetilde{b}} \cap \widetilde{\theta}$ and $\widetilde{\theta}y_{\widetilde{b}}$ is an i-v fuzzy quasi-ideal of S, where $\widetilde{\theta}$ is an i-v fuzzy right ideal of S and $y_{\widetilde{b}}$ is an idempotent i-v fuzzy point of S.

Now, $x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}} \subseteq x_{\widetilde{a}}\widetilde{\chi}_S\widetilde{\chi}_S \subseteq x_{\widetilde{a}}\widetilde{\chi}_S$. Also, $x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}} \subseteq \widetilde{\chi}_S\widetilde{\chi}_S y_{\widetilde{b}} \subseteq \widetilde{\chi}_S y_{\widetilde{b}}$. This implies that $x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}} \subseteq x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}}$. To prove the reverse inclusion, let $t \in S$. Since, $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ are both idempotent, we have $x = \sum_{i=1}^{m_1} x^2$ and $y = \sum_{j=1}^{m_2} y^2$ for some $m_1, m_2 \in \mathbb{N}$. If t can not be expressed as $t = \sum_{i=1}^{n_1} a_i b_i$, for any $a_i, b_i \in S$, then $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}})(t) = \widetilde{0} = (x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}})(t)$. Now, Suppose that $t = \sum_{i=1}^{n_1} a_i b_i$, for some $a_i, b_i \in S$. Then

$$\begin{aligned}
&(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}})(t)\\
&= Min^i\Big((x_{\widetilde{a}}\widetilde{\chi}_S)(t), (\widetilde{\chi}_S y_{\widetilde{b}})(t)\Big)\\
&= Min^i\Big(\sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{Min^i(x_{\widetilde{a}}(a_i), \widetilde{\chi}_S(b_i))\}\Big\},\\
&\qquad \sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{Min^i(\widetilde{\chi}_S(a_i), y_{\widetilde{b}}(b_i))\}\Big\}\Big)\\
&= Min^i\Big(\sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{x_{\widetilde{a}}(a_i)\}\Big\}, \sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{y_{\widetilde{b}}(b_i)\}\Big\}\Big).
\end{aligned}$$

Now if $a_i = x$ and $b_i = y$ for each $1 \le i \le n_1$, then $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}})(t) = Min^i\Big(\inf_{1\le i\le n_1}\{x_{\widetilde{a}}(x)\}, \inf_{1\le i\le n_1}\{y_{\widetilde{b}}(y)\}\Big) = Min^i(\widetilde{a}, \widetilde{b})$. Again, $a_i = x$ and $b_i = y$ for each $1 \le i \le n_1$ implies that $t = \sum_{i=1}^{n_1} xy = \sum_{i=1}^{n_1}(\sum_{i=1}^{m_1} x^2)(\sum_{j=1}^{m_2} y^2)$. Then, we deduce that

$$\begin{aligned}
(x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}})(t) &= \sup_{t=\sum_{i=1}^{k} a_i b_i}\Big\{\inf_{1\le i\le k}\{Min^i((x_{\widetilde{a}}\widetilde{\chi}_S)(a_i), y_{\widetilde{b}}(b_i))\}\Big\}\\
&\ge \inf_{1\le i\le n_1}\Big\{Min^i((x_{\widetilde{a}}\widetilde{\chi}_S)(\sum_{i=1}^{m_1} x^2), y_{\widetilde{b}}(\sum_{j=1}^{m_2} y^2))\Big\}\\
&\ge \inf_{1\le i\le n_1}\Big\{Min^i\Big(\inf_{1\le i\le m_1} Min^i(x_{\widetilde{a}}(x), \widetilde{\chi}_S(x)), y_{\widetilde{b}}(y)\Big)\Big\}\\
&= Min^i(\widetilde{a}, \widetilde{b}) = (x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}})(t).
\end{aligned}$$

Now let us consider the case where $a_i \ne x$ or, $b_i \ne y$ for some $1 \le i \le n_1$. Then $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}})(t)$

$$\begin{aligned}
&= Min^i\Big(\sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{x_{\widetilde{a}}(a_i)\}\Big\}, \sup_{t=\sum_{i=1}^{n_1} a_i b_i}\Big\{\inf_{1\le i\le n_1}\{y_{\widetilde{b}}(b_i)\}\Big\}\Big)\\
&= \widetilde{0} \le (x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}})(t).
\end{aligned}$$

This implies that $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}}) \subseteq (x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}})$. Consequently, we get that $(x_{\widetilde{a}}\widetilde{\chi}_S \cap \widetilde{\chi}_S y_{\widetilde{b}}) = x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}}$. Being an intersection of i-v fuzzy left ideal and i-v fuzzy right ideal, $x_{\widetilde{a}}\widetilde{\chi}_S y_{\widetilde{b}}$ is an i-v fuzzy quasi-ideal of S. □

Definition 3.8. Let $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring S. The intersection of all i-v fuzzy left ideals of S containing $\widetilde{\mu}$ is said to be the i-v fuzzy left ideal of S generated by $\widetilde{\mu}$ and it is denoted by $(\widetilde{\mu})_l$. The i-v fuzzy right ideal $(\widetilde{\mu})_r$ and i-v fuzzy quasi-ideal $(\widetilde{\mu})_q$ of S, generated by $\widetilde{\mu}$ can be defined similarly.

Definition 3.9. Let $\widetilde{\mu}$ be an i-v fuzzy subset of a semiring S. We define an i-v fuzzy subset $<\widetilde{\mu}>$ of S by $<\widetilde{\mu}>(x) = \sup\Big\{\inf\limits_{1\leq i\leq n}\{\widetilde{\mu}(a_i)\} : x = \sum\limits_{i=1}^{n} a_i, a_i \in S; n \in \mathbb{N}\Big\}$, for all $x \in S$.

Lemma 3.10. *Let $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring S. Then* (i) $(\widetilde{\mu})_l = <\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>$, (ii) $(\widetilde{\mu})_r = <\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S>$ *and* (iii) $(\widetilde{\mu})_q = <\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$.

Proof. (i) We first prove that $<\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>$ is an i-v fuzzy left ideal of S containing $\widetilde{\mu}$. Let $x = \sum\limits_{i=1}^{m} a_i$ and $y = \sum\limits_{j=1}^{n} b_j$ for some $a_i, b_j \in S$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then

$$<\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>(x+y)$$
$$= \sup\Big\{\inf_{1\leq i\leq m_1}\{(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})(c_i)\} : x+y = \sum_{i=1}^{m_1} c_i\Big\}$$
$$\geq \sup\Big\{Min^i\Big(\inf_{1\leq i\leq m}(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})(a_i), \inf_{1\leq j\leq n}(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})(b_j)\Big) : x = \sum_{i=1}^{m} a_i, y = \sum_{j=1}^{n} b_j\Big)\Big\}$$
$$\geq Min^i\Big(<\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>(x), <\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>(y)\Big).$$

Now $\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}) = \widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu} = \widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}$. This implies that $<\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})> \subseteq <\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>$. Thus $\widetilde{\chi}_S<\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}> \subseteq <\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})> \subseteq <\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>$. Therefore, $<\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>$ is an *i.v.* fuzzy left ideal of S and clearly, it contains $\widetilde{\mu}$. Let $z \in S$ and $IFL(S)$ be the set of all i-v fuzzy left ideals of S. Then $(\widetilde{\mu})_l(z) = \Big(\bigcap\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}\Big)(z) = \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}(z) \leq <\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>(z)$, since $<\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>$ is an i-v fuzzy left ideal of S containing $\widetilde{\mu}$. Thus we obtain that $(\widetilde{\mu})_l \subseteq <\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}>$. Again $(\widetilde{\mu})_l(z) = \Big(\bigcap\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}\Big)(z) = \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}(z) \geq \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\mu}(z) = \widetilde{\mu}(z)$. Also, $(\widetilde{\mu})_l(z) = \bigcap\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}(z) = \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}\widetilde{\theta}(z) \geq \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}(\widetilde{\chi}_S\widetilde{\theta})(z) \geq \inf\limits_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFL(S)}(\widetilde{\chi}_S\widetilde{\mu})(z) = (\widetilde{\chi}_S\widetilde{\mu})(z)$. This shows that $(\widetilde{\mu})_l(z) \geq Max^i\Big(\widetilde{\mu}(z), (\widetilde{\chi}_S\widetilde{\mu})(z)\Big) = (\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})(z)$. Therefore $(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}) \subseteq (\widetilde{\mu})_l$ which implies that $<(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})> \subseteq <(\widetilde{\mu})_l> = (\widetilde{\mu})_l$. Hence, we get that $(\widetilde{\mu})_l = <\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>$.

(ii) Proof of this part is similar to (i).

(iii) We first prove $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$ is an i-v fuzzy quasi-ideal of S containing $\widetilde{\mu}$. We have $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})> = <(\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S)\cap(\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})> = <\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S> \cap <\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>$. Now, $<\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S>$ and $<\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}>$ are i-v fuzzy left ideal and i-v fuzzy right ideal of S containing $\widetilde{\mu}$ respectively. Therefore, $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$ is an intersection of i-v fuzzy left ideal and an i-v fuzzy right ideal of S respectively and clearly, it contains $\widetilde{\mu}$. Thus, $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$ is an i-v fuzzy quasi-ideal of S containing $\widetilde{\mu}$. Suppose that $IFQ(S)$ denotes the set of all i-v fuzzy quasi-ideals of S. Let $x\in S$. Then $(\widetilde{\mu})_q(x)=(\bigcap_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}\widetilde{\theta})(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}\widetilde{\theta}(x)\leq<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>(x)$, since $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$ is an i-v fuzzy quasi-ideal of S containing $\widetilde{\mu}$. This implies that $(\widetilde{\mu})_q\subseteq<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$. Again, $(\widetilde{\mu})_q(x)=(\bigcap_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}\widetilde{\theta})(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}\widetilde{\theta}(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}(\widetilde{\mu}\cup\widetilde{\theta})(x)\geq\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}(\widetilde{\mu}\cup(\widetilde{\theta}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\theta}))(x)\geq\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFQ(S)}(\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu}))(x)=(\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu}))(x)$. So, $\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})\subseteq(\widetilde{\mu})_q$. This shows that $<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>\subseteq<(\widetilde{\mu})_q>=(\widetilde{\mu})_q$. Thus we obtain that $(\widetilde{\mu})_q=<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$. □

Definition 3.11 ([10]). An element 'a' of a semiring S is said to be regular if there exists an element $x\in S$ such that $a=axa$. A semiring S is said to be regular if its every element is regular.

The following theorem is known in regular semirings.

Theorem 3.12 ([9]). *A semiring S is regular if and only if $\widetilde{\mu}\widetilde{\theta}=\widetilde{\mu}\cap\widetilde{\theta}$ for any i-v fuzzy right ideal $\widetilde{\mu}$ and i-v fuzzy left ideal $\widetilde{\theta}$ of S.*

Theorem 3.13. *The following statements are equivalent in a semiring S.*

(i) *S is regular.*

(ii) *For each i-v fuzzy right ideal $\widetilde{\mu}$ and i-v fuzzy left ideal $\widetilde{\theta}$ of S, $\widetilde{\mu}\widetilde{\theta}=\widetilde{\mu}\cap\widetilde{\theta}$.*

(iii) *For each i-v fuzzy right ideal $\widetilde{\mu}$ and each i-v fuzzy left ideal $\widetilde{\theta}$ of S, a) $\widetilde{\mu}^2=\widetilde{\mu}$, b) $\widetilde{\theta}^2=\widetilde{\theta}$, c) $\widetilde{\mu}\widetilde{\theta}$ is an i-v fuzzy quasi-ideal of S.*

(iv) *The set $IFQ(S)$ of all i-v fuzzy quasi-ideals of S forms a regular semigroup with respect to the usual product of i-v fuzzy subsets of S.*

(v) *Each i-v fuzzy quasi-ideal $\widetilde{\eta}$ of S satisfies $\widetilde{\eta}=\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta}$.*

The statements (iii)(a) and (iii)(b) imply that each i-v fuzzy quasi-ideal $\widetilde{\eta}$ of S can be written as the intersection of an i-v fuzzy left ideal and an i-v fuzzy right ideal of S, since it satisfies $\widetilde{\eta}=\widetilde{\chi}_S\widetilde{\eta}\cap\widetilde{\eta}\widetilde{\chi}_S$.

The proof of this theorem is straightforward. We hence omit the proof.

In the following theorem, we study the type of i-v fuzzy quasi-ideals in a regular semiring S.

Theorem 3.14. *The following statements are equivalent in a semiring S.*

(i) *$\widetilde{\mu}\widetilde{\theta}=\widetilde{\mu}\cap\widetilde{\theta}\subseteq\widetilde{\theta}\widetilde{\mu}$ for any i-v fuzzy right ideal $\widetilde{\mu}$ and i-v fuzzy left ideal $\widetilde{\theta}$ of S.*

(ii) *$IFQ(S)$ forms an idempotent semigroup with respect the usual product of i-v fuzzy subsets of S.*

(iii) *$\widetilde{\eta}=\widetilde{\eta}^2$ for any i-v fuzzy quasi-ideal $\widetilde{\eta}$ of S.*

Proof. (i) $\Longrightarrow$ (ii) : Suppose that (i) hold. Then it follows from Theorem 3.13 that $IFQ(S)$ forms a regular semigroup with respect to the usual product of the i-v fuzzy subsets of S. It remains to prove that $IFQ(S)$ is idempotent. Let $\widetilde{\eta}\in IFQ(S)$.

Then, by Theorem 3.13, we get that $\widetilde{\eta} = \widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta}$. Thus, we obtain that :

$$\begin{aligned}
\widetilde{\eta} &= \widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta}\\
&= (\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta})\widetilde{\chi}_S(\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta})\\
&= \widetilde{\eta}\widetilde{\chi}_S(\widetilde{\eta}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\eta})\widetilde{\chi}_S\widetilde{\eta} \quad \text{(since, by Theorem 3.13, } \widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\eta} \subseteq \widetilde{\chi}_S\widetilde{\eta} = \widetilde{\chi}_S\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta} \subseteq \widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\eta})\\
&\subseteq \widetilde{\eta}\widetilde{\chi}_S(\widetilde{\chi}_S\widetilde{\eta}\widetilde{\eta}\widetilde{\chi}_S)\widetilde{\chi}_S\widetilde{\eta} \quad \text{(by our assumption)}\\
&= (\widetilde{\eta}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\eta})(\widetilde{\eta}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\eta})\\
&= (\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta})(\widetilde{\eta}\widetilde{\chi}_S\widetilde{\eta}) = \widetilde{\eta}^2.
\end{aligned}$$

This shows that $\widetilde{\eta} \subseteq \widetilde{\eta}^2$. Now $\widetilde{\eta}^2 \subseteq \widetilde{\chi}_S\widetilde{\eta}$ and as well as $\widetilde{\eta}^2 \subseteq \widetilde{\eta}\widetilde{\chi}_S$ imply that $\widetilde{\eta}^2 \subseteq \widetilde{\chi}_S\widetilde{\eta} \cap \widetilde{\eta}\widetilde{\chi}_S \subseteq \widetilde{\eta}$, since $\widetilde{\eta}$ is an i-v fuzzy quasi-ideal of S. Hence $\widetilde{\eta} = \widetilde{\eta}^2$. Thus $IFQ(S)$ forms an idempotent semigroup with respect to the usual product of i-v fuzzy subsets of S.

(ii) $\Longrightarrow$ (iii) : This is just a restriction.

(iii) $\Longrightarrow$ (i) : Let $\widetilde{\eta} = \widetilde{\eta}^2$ for any i-v fuzzy quasi-ideal $\widetilde{\eta}$ of S. Let $\widetilde{\mu}$ and $\widetilde{\theta}$ be an i-v fuzzy right ideal and an i-v fuzzy left ideal of S respectively. Then $\widetilde{\mu}\widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\chi}_S \subseteq \widetilde{\mu}$ as well as, $\widetilde{\mu}\widetilde{\theta} \subseteq \widetilde{\chi}_S\widetilde{\theta} \subseteq \widetilde{\theta}$. This implies that $\widetilde{\mu}\widetilde{\theta} \subseteq \widetilde{\mu}\cap\widetilde{\theta}$. Now being an intersection of an i-v fuzzy right ideal and an i-v fuzzy left ideal of S, $\widetilde{\mu}\cap\widetilde{\theta}$ is an i-v fuzzy quasi-ideal of S. Hence, we have $\widetilde{\mu}\cap\widetilde{\theta} = (\widetilde{\mu}\cap\widetilde{\theta})^2 = (\widetilde{\mu}\cap\widetilde{\theta})(\widetilde{\mu}\cap\widetilde{\theta}) \subseteq \widetilde{\mu}\widetilde{\theta}$. Similarly, $(\widetilde{\mu}\cap\widetilde{\theta}) \subseteq \widetilde{\theta}\widetilde{\mu}$. Thus we have proved that $\widetilde{\mu}\widetilde{\theta} = \widetilde{\mu}\cap\widetilde{\theta} \subseteq \widetilde{\theta}\widetilde{\mu}$. □

4. Interval-valued fuzzy bi-ideals of a semiring :

Definition 4.1. A non-empty i-v fuzzy subset $\widetilde{\mu}$ of a semiring S is said to be an i-v fuzzy bi-ideal of S if for any $x, y, z \in S, \widetilde{\mu}(x+y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$, $\widetilde{\mu} \circ \widetilde{\mu} \subseteq \widetilde{\mu}$ and $\widetilde{\mu}(xyz) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(z))$.

We characterize the i-v fuzzy bi-ideals of a semiring in the following lemma.

Lemma 4.2. *A non-empty i-v fuzzy subset $\widetilde{\mu}$ of a semiring S is an i-v fuzzy bi-ideal of S if and only if $\widetilde{\mu}(x+y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$ for any $x, y \in S$ and $\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}$.*

In the following proposition, we state the relation between i-v fuzzy quasi-ideal and i-v fuzzy bi-ideal of a semiring.

Proposition 4.3. *Every i-v fuzzy quasi-ideal of a semiring S is also an i-v fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\mu}$ be an i-v fuzzy quasi-ideal of a semiring S. Then $\widetilde{\mu}(x+y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$, for any $x, y \in S$. Now $\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}\widetilde{\chi}_S\widetilde{\chi}_S \subseteq \widetilde{\mu}\widetilde{\chi}_S$. Also, $\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu}$. Hence, we get $\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu} \cap \widetilde{\mu}\widetilde{\chi}_S$. Since, $\widetilde{\mu}$ is an i-v fuzzy quasi-ideal of S, it follows that $\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\chi}_S\widetilde{\mu} \cap \widetilde{\mu}\widetilde{\chi}_S \subseteq \widetilde{\mu}$. Consequently, $\widetilde{\mu}$ is an i-v fuzzy bi-ideal of S. □

We note that the converse of the above Proposition does not hold in general.

Definition 4.4. Let $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring S. Then the i-v fuzzy bi-ideal of S generated by $\widetilde{\mu}$ is denoted by $(\widetilde{\mu})_b$ and is defined as the intersection of all i-v fuzzy bi-ideals of S containing $\widetilde{\mu}$.

Lemma 4.5. *Let $\widetilde{\mu}$ be a non-empty i-v fuzzy subset of a semiring S. Then, we have the following equality.*
$(\widetilde{\mu})_b = <\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$.

Proof. We first prove that $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$ is an $i.v.$ fuzzy bi-ideal of S, containing $\widetilde{\mu}$. Similar to the proof given in Lemma 3.10 (i), we can show that $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>(x+y)\geq Min^i\Big(<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>(x), <\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>(y)\Big)$, for any $x, y\in S$. Now, we easily deduce that

$$\begin{aligned}
&(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\\
&=(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})(\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}^2\cup\widetilde{\chi}_S\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\\
&\subseteq(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})(\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu})\\
&\subseteq(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})(\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\chi}_S\widetilde{\mu})\\
&=(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\widetilde{\chi}_S\widetilde{\mu}\\
&=\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}^2\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\\
&\subseteq\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu}\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\\
&\subseteq\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\subseteq(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}).
\end{aligned}$$

Therefore, $<(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})>\subseteq<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$. This shows that $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})>\widetilde{\chi}_S<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>\subseteq<(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\widetilde{\chi}_S(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})>$ $\subseteq<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$. Consequently, $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$ is an i-v fuzzy bi-ideal of S and clearly, it contains $\widetilde{\mu}$. Suppose that $IFB(S)$ denote the set of all i-v fuzzy bi-ideals of S. Let $x\in S$. Then, we have $(\widetilde{\mu})_b(x)=\Big(\bigcap_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}\widetilde{\theta}\Big)(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}\widetilde{\theta}(x)\leq<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>(x)$, since $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$ is an i-v fuzzy bi-ideal of S, containing $\widetilde{\mu}$. Thus, $(\widetilde{\mu})_b\subseteq<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$. Again, we have $(\widetilde{\mu})_b(x)=\Big(\bigcap_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}\widetilde{\theta}\Big)(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}\widetilde{\theta}(x)=\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\theta})(x)$ (since, $\widetilde{\mu}\subseteq\widetilde{\theta}$, and $\widetilde{\theta}$ is an $i.v$ fuzzy bi-ideal of S, it follows that $\widetilde{\mu}^2=\widetilde{\mu}\widetilde{\mu}\subseteq\widetilde{\theta}\widetilde{\theta}\subseteq\widetilde{\theta}$) $\geq\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\theta}\widetilde{\chi}_S\widetilde{\theta})(x)\geq\inf_{\widetilde{\mu}\subseteq\widetilde{\theta}\in IFB(S)}(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})(x)=(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})(x)$. Thus, we obtain that $(\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\subseteq(\widetilde{\mu})_b$. This implies that $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>$ $\subseteq<(\widetilde{\mu})_b>=(\widetilde{\mu})_b$. Hence, $<\widetilde{\mu}\cup\widetilde{\mu}^2\cup\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}>=(\widetilde{\mu})_b$. □

For i-v fuzzy bi-ideals of a semiring, we have the following Proposition.

Proposition 4.6. *The product of an i-v fuzzy bi-ideal and an i-v fuzzy sub-semiring of a semiring S is still an i-v fuzzy bi-ideal of S.*

The following corollaries are easy consequence of the above Proposition.

Corollary 4.7. *The product of two i-v fuzzy bi-ideals of a semiring is again an i-v fuzzy bi-ideal of S.*

Corollary 4.8. *The product of two i-v fuzzy quasi-ideals of a semiring is an i-v fuzzy bi-ideal of S.*

In the following theorem, we state some properties of i-v fuzzy quasi-ideals of a regular semiring.

Theorem 4.9. *Let S be a regular semiring. The following properties of an i-v fuzzy quasi-ideal of S hold.*

(i) *Each i-v fuzzy quasi-ideal $\widetilde{\mu}$ of S satisfies $\widetilde{\mu} = \widetilde{\theta} \cap \widetilde{\eta} = \widetilde{\theta}\widetilde{\eta}$, where $\widetilde{\theta} = (\widetilde{\mu})_r$ and $\widetilde{\eta} = (\widetilde{\mu})_l$.*

(ii) *Each i-v fuzzy quasi-ideal $\widetilde{\mu}$ of S satisfies $\widetilde{\mu}^2 = \widetilde{\mu}^3$.*

(iii) *Each i-v fuzzy bi-ideal of S is an i-v fuzzy quasi-ideal of S.*

(iv) *Each i-v fuzzy bi-ideal of a two-sided ideal T of S is an i-v fuzzy quasi-ideal of S.*

Proof. (i) In a regular semiring S, each i-v fuzzy quasi-ideal $\widetilde{\mu}$ of S satisfies $\widetilde{\mu} = \widetilde{\chi}_S\widetilde{\mu} \cap \widetilde{\mu}\widetilde{\chi}_S$, by Theorem 3.13. Hence, it suffices to prove that $(\widetilde{\mu})_l = \widetilde{\chi}_S\widetilde{\mu}$ and $(\widetilde{\mu})_r = \widetilde{\mu}\widetilde{\chi}_S$. Now, we deduce the followings:

$$\begin{aligned}
\widetilde{\chi}_S\widetilde{\mu} &\subseteq < \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > \\
&= < \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} >< \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > \quad \text{(since, in a regular semiring } S, \\
&\qquad \widetilde{\mu_1}^2 = \widetilde{\mu_1}, \text{ where, } \widetilde{\mu_1} \text{ is an i-v fuzzy left ideal of } S, \text{ by Theorem 3.13)} \\
&\subseteq < \widetilde{\mu}^2 \cup \widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu}\widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} > \\
&\subseteq < \widetilde{\chi}_S\widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > \quad \text{(since, by Theorem 3.14, } \widetilde{\mu} = \widetilde{\mu}^2) \\
&= < \widetilde{\chi}_S\widetilde{\mu} > = \widetilde{\chi}_S\widetilde{\mu}.
\end{aligned}$$

Thus, we obtain $\widetilde{\chi}_S\widetilde{\mu} \subseteq < \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > \subseteq \widetilde{\chi}_S\widetilde{\mu}$. So, $\widetilde{\chi}_S\widetilde{\mu} = < \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > = (\widetilde{\mu})_l$. Similarly, we can get that $\widetilde{\mu}\widetilde{\chi}_S = (\widetilde{\mu})_r$. Therefore, $\widetilde{\mu} = \widetilde{\chi}_S\widetilde{\mu} \cap \widetilde{\mu}\widetilde{\chi}_S = < \widetilde{\mu} \cup \widetilde{\chi}_S\widetilde{\mu} > \cap < \widetilde{\mu} \cup \widetilde{\mu}\widetilde{\chi}_S > = (\widetilde{\mu})_l \cap (\widetilde{\mu})_r = (\widetilde{\mu})_r(\widetilde{\mu})_l$, by Theorem 3.12.

(ii) Let $\widetilde{\mu}$ be an i-v fuzzy quasi-ideal of S. Then by Theorem 3.14, it follows that $\widetilde{\mu}^2$ is a i-v fuzzy quasi-ideal of S, since S is regular. Then by Theorem 3.13, we have $\widetilde{\mu}^2 = \widetilde{\mu}^2\widetilde{\chi}_S\widetilde{\mu}^2 = \widetilde{\mu}(\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu})\widetilde{\mu} = \widetilde{\mu}\widetilde{\mu}\widetilde{\mu} = \widetilde{\mu}^3$.

(iii) Let $\widetilde{\mu}$ be an i-v fuzzy bi-ideal of S. Then $\widetilde{\mu}\widetilde{\chi}_S \cap \widetilde{\chi}_S\widetilde{\mu} = \widetilde{\mu}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu} \subseteq \widetilde{\mu}$ (since $\widetilde{\mu}$ is an i-v fuzzy bi-ideal of S). Thus $\widetilde{\mu}$ is an i-v fuzzy quasi-ideal of S.

(iv) Suppose that $\widetilde{\mu_1}$ be an i-v fuzzy bi-ideal of a two sided ideal T of S. Let $t \in T \subseteq S$. Since S is regular, there exist $u \in S$ such that $t = tut$. This implies that $t = t(utu)t$. Since T is a two-sided ideal of S, $utu \in T$ and hence T is also regular. Now,

$$\begin{aligned}
&\widetilde{\mu_1}\widetilde{\chi}_S \cap \widetilde{\chi}_S\widetilde{\mu_1} \\
&= (\widetilde{\mu_1}\widetilde{\chi}_T\widetilde{\mu_1})\widetilde{\chi}_S \cap \widetilde{\chi}_S(\widetilde{\mu_1}\widetilde{\chi}_T\widetilde{\mu_1}) \\
&\qquad \text{(follows from the regularity of } T \text{ and Theorem 3.13)} \\
&= (\widetilde{\mu_1}\widetilde{\chi}_T)(\widetilde{\mu_1}\widetilde{\chi}_S) \cap (\widetilde{\chi}_S\widetilde{\mu_1})(\widetilde{\chi}_T\widetilde{\mu_1}) \\
&\subseteq (\widetilde{\mu_1}\widetilde{\chi}_S)(\widetilde{\chi}_T\widetilde{\chi}_S) \cap (\widetilde{\chi}_S\widetilde{\chi}_T)(\widetilde{\chi}_S\widetilde{\mu_1}) \\
&= \widetilde{\mu_1}(\widetilde{\chi}_S\widetilde{\chi}_T\widetilde{\chi}_S) \cap (\widetilde{\chi}_S\widetilde{\chi}_T\widetilde{\chi}_S)\widetilde{\mu_1} \\
&\subseteq \widetilde{\mu_1}\widetilde{\chi}_T \cap \widetilde{\chi}_T\widetilde{\mu_1} \quad \text{(since, } T \text{ is a two sided ideal of } S \text{ implies that } \widetilde{\chi}_T \\
&\qquad\qquad \text{is an i-v fuzzy two sided ideal of } S) \\
&\subseteq \widetilde{\mu_1} \quad \text{(since, } T \text{ is regular, } \widetilde{\mu_1} \text{ is also an i-v fuzzy quasi-ideal of } T).
\end{aligned}$$

Consequently, $\widetilde{\mu_1}$ is an i-v fuzzy quasi-ideal of S. □

Definition 4.10 ([2]). An element 'x' of a semiring S is said to be intra-regular if there exist $a_i, b_i \in S$ such that $x = \sum_{i=1}^{m} a_i x^2 b_i$. A semiring S is said to be intra-regular if its every element is intra-regular.

In the following theorem, we characterize the intra-regular semirings.

Theorem 4.11 ([2]). *A semiring S is intra-regular if and only if $L \cap R \subseteq LR$, for any left ideal L and right ideal R of S.*

Theorem 4.12. *A semiring S is intra-regular if and only if $\widetilde{\mu} \cap \widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\theta}$, for any i-v fuzzy left ideal $\widetilde{\mu}$ and i-v fuzzy right ideal $\widetilde{\theta}$ of S.*

Proof. Let S be an intra-regular semiring. Let $\widetilde{\mu}$ and $\widetilde{\theta}$ be an *i.v.* fuzzy left ideal and an i-v fuzzy right ideal of S respectively. Suppose that $x \in S$. Since, S is intra-regular, there exist $a_i, b_i \in S$ such that $x = \sum_{i=1}^{n} a_i x^2 b_i$. So, $x = \sum_{i=1}^{n} (a_i x)(x b_i)$. Then, we deduce that

$$\begin{aligned}
(\widetilde{\mu}\widetilde{\theta})(x) &= \sup\Big\{ \inf_{1 \leq i \leq k} Min^i(\widetilde{\mu}(p_i), \widetilde{\theta}(q_i)) : x = \sum_{i=1}^{k} p_i q_i;\ p_i, q_i \in S \Big\} \\
&\geq \inf_{1 \leq i \leq n} \{Min^i(\widetilde{\mu}(a_i x), \widetilde{\theta}(x b_i))\} \\
&\geq \inf_{1 \leq i \leq n} \{Min^i(\widetilde{\mu}(x), \widetilde{\theta}(x))\} \\
&\qquad \text{(since, } \widetilde{\mu} \text{ is an i-v fuzzy left ideal and } \widetilde{\theta} \text{ is an i-v fuzzy right ideal of } S) \\
&= Min^i(\widetilde{\mu}(x), \widetilde{\theta}(x)) \\
&= (\widetilde{\mu} \cap \widetilde{\theta})(x).
\end{aligned}$$

Thus, we obtain $\widetilde{\mu} \cap \widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\theta}$.

Conversely, suppose that $\widetilde{\mu} \cap \widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\theta}$, for any i-v fuzzy left ideal $\widetilde{\mu}$ and i-v fuzzy right ideal $\widetilde{\theta}$ of S. Let L and R be a left ideal and a right ideal of S respectively. Then, by our assumption, we have $\widetilde{\chi}_L \cap \widetilde{\chi}_R \subseteq \widetilde{\chi}_L \widetilde{\chi}_R$. This implies that $\widetilde{\chi}_{L \cap R} \subseteq \widetilde{\chi}_{LR}$, by Lemma 2.13 and Lemma 2.14. Thus, we have shown that $L \cap R \subseteq LR$. Hence, S is an intra-regular semiring, by Theorem 4.11. □

Now we state the main theorem. This theorem is a characterization theorem of a regular and intra-regular semiring S in terms of their i-v fuzzy quasi-ideal and i-v fuzzy bi-ideal of S.

Theorem 4.13. *Let S be a semiring. Then the following statements are equivalent.*

(i) *S is regular and intra-regular.*
(ii) *Every i-v fuzzy quasi-ideal of S is idempotent.*
(iii) *Every i-v fuzzy bi-ideal of S is idempotent.*
(iv) *$\widetilde{\mu} \cap \widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\theta}$ for all i-v fuzzy quasi-ideals $\widetilde{\mu}$ and $\widetilde{\theta}$ of S.*
(v) *$\widetilde{\mu} \cap \widetilde{\theta} \subseteq \widetilde{\mu}\widetilde{\theta}$ for every i-v fuzzy quasi-ideal $\widetilde{\mu}$ and i-v fuzzy bi-ideal $\widetilde{\theta}$ of S.*

(vi) $\widetilde{\mu}\cap\widetilde{\theta}\subseteq\widetilde{\mu}\widetilde{\theta}$ *for every i-v fuzzy bi-ideal* $\widetilde{\mu}$ *and i-v fuzzy quasi-ideal* $\widetilde{\theta}$ *of* S.
(vii) $\widetilde{\mu}\cap\widetilde{\theta}\subseteq\widetilde{\mu}\widetilde{\theta}$ *for all i-v fuzzy bi-ideals* $\widetilde{\mu}$ *and* $\widetilde{\theta}$ *of* S.

Proof. (i) $\Longrightarrow$ (vii) : Let (i) hold and $x\in S$. Since S is regular, there exists $a\in S$ such that $x=xax$. So we can write $x=xaxax$ ....(1). Again since S is intra-regular, there exist $a_i,b_i\in S$ such that $x=\sum_{i=1}^{m}a_ix^2b_i$, where $m\in\mathbb{N}$. Then from (1), we have $x=xa(\sum_{i=1}^{m}a_ix^2b_i)ax=\sum_{i=1}^{m}(xaa_ix)(xb_iax)$. Now let $\widetilde{\mu}$ and $\widetilde{\theta}$ be two i-v fuzzy bi-ideals of S. Then, the following conditions hold :

$$\begin{aligned}(\widetilde{\mu}\widetilde{\theta})(x)&=\sup\Big\{\inf_{1\le i\le n}\{Min^i(\widetilde{\mu}(p_i),\widetilde{\theta}(q_i))\}:x=\sum_{i=1}^{n}p_iq_i;p_i,q_i\in S\Big\}\\&\ge\inf_{1\le i\le m}\{Min^i(\widetilde{\mu}(xaa_ix),\widetilde{\theta}(xb_iax))\}\\&\ge\inf_{1\le i\le m}\Big\{Min^i\Big(Min^i(\widetilde{\mu}(x),\widetilde{\mu}(x)),Min^i(\widetilde{\theta}(x),\widetilde{\theta}(x))\Big)\Big\}\\&\qquad\text{(since }\widetilde{\mu}\text{ and }\widetilde{\theta}\text{ are i-v fuzzy bi-ideals of }S)\\&=Min^i(\widetilde{\mu}(x),\widetilde{\theta}(x))\\&=(\widetilde{\mu}\cap\widetilde{\theta})(x).\end{aligned}$$

Consequently, $\widetilde{\mu}\cap\widetilde{\theta}\subseteq\widetilde{\mu}\widetilde{\theta}$.
(vii) $\Longrightarrow$ (vi) : This implication is clear since each i-v fuzzy quasi-ideal of S is also an i-v fuzzy bi-ideal of S.
(vi) $\Longrightarrow$ (v) : Suppose that (vi) holds. Let $\widetilde{\mu}$ be an i-v fuzzy quasi-ideal and $\widetilde{\theta}$ be an i-v fuzzy bi-ideal of S. Then $\widetilde{\mu}$ is also an i-v fuzzy bi-ideal of S. Now, by our assumption, we have $\widetilde{\mu}\cap(\widetilde{\theta})_q\subseteq\widetilde{\mu}(\widetilde{\theta})_q=\widetilde{\mu}<\widetilde{\theta}\cup(\widetilde{\theta}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\theta})>$ ....(2). As $\widetilde{\theta}\widetilde{\chi}_S$ is an i-v fuzzy right ideal of S, it is an i-v fuzzy quasi-ideal as well as an i-v fuzzy bi-ideal of S. Again $\widetilde{\chi}_S\widetilde{\theta}$ is an i-v fuzzy left ideal and hence an i-v fuzzy quasi-ideal of S. Thus, by our assumption, we conclude that $\widetilde{\theta}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\theta}\subseteq\widetilde{\theta}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\theta}\subseteq\widetilde{\theta}\widetilde{\chi}_S\widetilde{\theta}\subseteq\widetilde{\theta}$, since $\widetilde{\theta}$ is an i-v fuzzy bi-ideal of S. Then by (2), we have $\widetilde{\mu}\cap(\widetilde{\theta})_q\subseteq\widetilde{\mu}<\widetilde{\theta}\cup\widetilde{\theta}>\subseteq<\widetilde{\mu}\widetilde{\theta}>=\widetilde{\mu}\widetilde{\theta}$. Thus, $\widetilde{\mu}\cap\widetilde{\theta}\subseteq\widetilde{\mu}\cap(\widetilde{\theta})_q\subseteq\widetilde{\mu}\widetilde{\theta}$.
(v) $\Longrightarrow$ (iv) : It is clear since each i-v fuzzy quasi-ideal of S is also an i-v fuzzy bi-ideal of S.
(iv) $\Longrightarrow$ (iii) : Suppose that (iv) holds. Let $\widetilde{\mu}$ be an i-v fuzzy bi-ideal of S. Now, by our assumption, we have $\widetilde{\mu}\subseteq(\widetilde{\mu})_q=(\widetilde{\mu})_q\cap\widetilde{\mu}_q\subseteq(\widetilde{\mu})_q(\widetilde{\mu})_q=<\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})><\widetilde{\mu}\cup(\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu})>$ ....(3).
Finally, by our assumption, we have $\widetilde{\mu}\widetilde{\chi}_S\cap\widetilde{\chi}_S\widetilde{\mu}\subseteq\widetilde{\mu}\widetilde{\chi}_S\widetilde{\chi}_S\widetilde{\mu}\subseteq\widetilde{\mu}\widetilde{\chi}_S\widetilde{\mu}\subseteq\widetilde{\mu}$ since $\widetilde{\mu}$ is an i-v fuzzy bi-ideal of S. Hence, from (3), it follows that $\widetilde{\mu}\subseteq<\widetilde{\mu}><\widetilde{\mu}>\subseteq<\widetilde{\mu}^2>=\widetilde{\mu}^2$. Again, since $\widetilde{\mu}$ is an i-v quasi-ideal of S, it follows that $\widetilde{\mu}^2\subseteq\widetilde{\mu}$. Consequently, we have $\widetilde{\mu}=\widetilde{\mu}^2$.
(iii) $\Longrightarrow$ (ii) : This part is clear since each i-v fuzzy quasi-ideal of S is also an i-v fuzzy bi-ideal of S.

(ii) $\Longrightarrow$ (i) : This implication follows from Theorem in 3.14, Theorem 3.12 and Theorem 4.12. □

5. Conclusions

We have characterized regular and intra-regular semiring in terms of i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals of a semiring. So this paper helps us to realize that we can study different properties of semirings and even some other algebraic structures from the view of i-v fuzzy set theory. For example, as a continuation of this paper we shall study the k-regularity and k-intra-regularity of a semiring in terms of i-v fuzzy k-quasi ideal and i-v fuzzy k-bi-ideal of semirings.

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