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# Uncertain impulse response of imprecisely defined half order mechanical system

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Abstract. This paper investigates the numerical solution of imprecisely defined fractional order dynamic system, subjected to an unit impulse load. A mechanical spring mass system having fractional damping of order 1/2 with fuzzy initial condition is taken into consideration. Fuzziness appearing in the initial conditions is modelled through different types of convex normalised fuzzy sets viz. triangular, trapezoidal and Gaussian fuzzy numbers. Homotopy Perturbation Method (HPM) is used with fuzzy based approach to obtain the uncertain impulse response. Numerical examples related to this approach are solved by symbolic computations. Obtained results are depicted in term of plots to show the efficiency and powerfulness of the methodology.

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## 1. Introduction

Structural design and analysis plays a vital role for the structural safety. Most of the structures fail due to the poor design. In the design process the system parameters involved such as mass, geometry, material properties, external loads, or boundary conditions are considered as crisp or defined exactly. But, rather than the particular (crisp) value we may have only the vague, imprecise and incomplete information about the variables and parameters, which are uncertain in nature. These are arises due to errors in measurement, observations, experiment, applying different operating conditions or for maintenance induced error, etc. These uncertainties can be modelled through probabilistic, interval and fuzzy theory.

In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions. If the structural parameters and the external load are modeled as random variables with known probability density functions, the response of the structure can be predicted using the theory of probability and stochastic processes as studied by Elishakoff [22]. Also the probabilistic concept is already well established for the extension of the deterministic finite element method towards assessment of uncertainty. This has led to a number of probabilistic and stochastic finite element procedures (Halder and Mohadevan [26]; Antonio and Hoff Bauer [6]). Unfortunately, probabilistic methods may not able to deliver reliable results at the required precision without sufficient experimental data. It may be due to the probability density functions involved in it. As such in the recent decades, interval analysis and fuzzy theory are becoming powerful tools for many real life applications. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices.

Interval computations was first introduced by Moore [40]. He also studied various aspects of interval analysis along with applications in [41]. If only incomplete information is available, it is possible to establish the minimum and maximum favorable response of the structures using interval analysis or convex models (Ganzerli and Pantelides [24]; Ben-Haim and Elishakoff [15]). Moreover structural analysis with interval parameters using interval based approach has been studied by various authors ( Muhanna and Mullen [44]; Rao and Berke [52]; Qui et al. [49]). Zhang [69] using interval finite element method for uncertain reliability assessment of structures. An interesting method is proposed by Chen et al. [19] for computing the upper and lower bounds on frequencies of structures with interval parameters. Modal analysis of structures with uncertain-but-bounded parameters via interval analysis is investigated by Sim et al. [58]. Interval analysis for vibrating systems is discussed by Dimarogonas [20]. The uncertainty behaviour in mechanics problems is explained by Muhanna and Mullen [44] through interval -based-approach in an excellent way. Moens and Vandepitte [37] applied an interval finite element approach for the calculation of envelope frequency response functions. Qui and Wang [50] proposed some solution theorems for the standard eigenvalue problem of structures with interval parameters. Interval eigenvalue analysis for structures with interval parameters is studied by Chena et al. [18] using interval finite element method. Gao [25] analysed natural frequency and mode shape of structures for both random and interval parameter using random and interval factor method. Truss structure is used for the analysis. Recently Bounds for the stationary stochastic response of truss structures with interval parameters are explained by Muscolino and Sofi [46].

Fuzzy set theoretical concept was developed by Zadeh [68] which is further used in the uncertain analysis of structures in a wide range. As discussed above, if the structural parameters and the external loads are described in imprecise terms, then fuzzy theory can be applied. Valliappan and Pham [64] applied fuzzy logic for the numerical modelling of engineering problems. An optimization algorithm is developed by Munck et al. [45] for fuzzy properties based on response surface for the calculation of fuzzy envelope and fuzzy response functions of models. Fuzzy structural analysis using $\alpha-$level optimization is excellently studied by Moller et al. [39]. The transformation method has been applied for the simulation and analysis of structural systems with uncertain parameters by Hanss [30]. Also an important book is written by Hanss [27] in which applications of fuzzy arithmetic into engineering problems are described. Fuzzy behavior of mechanical systems with uncertain boundary conditions is investigated by Chekri et al. [17]. Nonlinear membership function for fuzzy optimization of mechanical and structural systems is discussed in Dhingra et al. [21]. Reuter and Schirwitz [54] have developed the cost-effectiveness of fuzzy analysis. Fuzzy arithmetical approach for comprehensive modelling and analysis of uncertain systems is applied to the simulation of automotive crash in structural dynamics as well as to the simulation of landslide failure in geotechnical science and engineering by Hanss and Turrin [29]. In both applications, epistemic uncertainties are considered which arise from some lack of knowledge, from simplification in modelling as well as from subjectivity in implementation. Rama Rao et al. [51] investigated the transient response of structures with uncertain structural parameters. Very recently Farkas et al. [23] presented the optimisation study of a vehicle bumper subsystem with fuzzy parameters in a systematic manner.

Recently various generalized model of uncertainty have been applied to finite element method to solve the structural problems with fuzzy parameters. As such a few papers that are related to fuzzy FEM are discussed here. Fuzzy finite element approach is applied to describe structural systems with imprecisely defined parameters in an excellent way by Rao and Sawyer [53]. Verhaeghe et al. [65] discussed the fuzzy finite element analysis technique to describe the static analysis of structures which is based on interval computation. Both fuzzy static and dynamic analysis of structures are explained by Akpan et al. [3] using fuzzy finite element approach. Vertex method and VAST software is used in it for the fuzzy finite element analysis. Also Akpan, et al. [2] derived fuzzy finite element method for smart structures. Fuzzy finite element method is formulated by Muhanna and Mullen [43] for mechanics problems. Hanss and Willner [28] used fuzzy arithmetical approach for the solution of finite element problems with fuzy parameters. Recently Balu and Rao [9, 10] investigated the structural problems with fuzzy parameters. They have used High Dimensional Model Representation (HDMR) along with FEM for the analysis. Also Balu and Rao [11] explained both static and dynamic responses of structures using FFEM with HDMR. Various solution methods have been proposed for the solution of fully fuzzy system of linear equations and applied in structural mechanics problems by Skalna et al. [59]. Behera et al. [14] developed a method to find finite element solution of a stepped rectangular bar in presence of fuzziness in material properties. Morales et al. [42] used finite element method for active vibration control of uncertain structures using fuzzy design. This work provides a tool for studying the influence of uncertainty propagation on both stability and performance of a vibration control system.

Also in recent years, fractional order differential equations have been used to model physical and engineering problems. Since, it is too difficult to obtain the exact solution of fractional differential equation so, one may need a reliable and efficient numerical technique for the solution of fractional differential equations. Many important works have been reported regarding fractional calculus in the last few decades. Relating to this field several excellent books have also been written by different authors representing the scope and various aspects of fractional calculus such as in [48, 47, 36, 57, 35] (Podlubny [48], Oldham and Spanier [47], Miller and Ross

[36], Samko et al. [57] and Kiryakov [35]). These books also give an extensive review on fractional derivative and fractional differential equations which help the reader in understating the basic concepts of fractional calculus. Further many authors have developed various methods to solve fractional ordinary and partial differential equations and integral equations of physical systems. Suarez and Shokooh [60] used an eigenvector expansion method for the solution of a mechanical spring-mass system containing fractional derivatives and the results obtained are found quite satisfactory. The same type of problem is also studied by Lixia and Agrawal [67] using a numerical technique when the damping factor is defined as fractional. Also recently Behera and Chakraverty [13] and Chakraverty and Behera [16] studied the numerical solution fractionally damped beam and spring mass system respectively using HPM.

Most of the literature deals with fuzzy or interval finite element method for the uncertain dynamic analysis structures to obtain the vibration characteristics. Not much work has been carried to determine the uncertain structural response of an imprecisely defined structural system. No work has been carried out when uncertainty has been taken into consideration for the structural system when damping factor is defined as fractional. As both fractional derivative and fuzzy analysis plays an important role in the structural modelling and design for the structural safety hence an attempt has been made to combined the both for a better reliable analysis. Some recent useful contributions on the theory of fuzzy differential equations and fuzzy fractional differential equations may be seen in [12, 61, 62, 63, 66, 33, 34, 38, 41, 1, 7, 8, 56, 4, 5].

In the present analysis, HPM [31, 32] is used to compute the uncertain dynamic response of a single degree-of-freedom spring-mass fractionally damped system subjected to an unit impulse load where the initial condition is defined as uncertain. Uncertainty present in the initial condition is defined in term of different types of fuzzy numbers viz. triangular, trapezoidal and Gaussian fuzzy number. The order of the fractional derivative of the damping factor is taken as 1/2. Although the formulation and the results are presented for a single degree-of-freedom model only, the approach presented here may easily be extended in a straight forward manner to a multi degree-of-freedom model. In the following sections preliminaries are first given. Next, implementations of HPM for fuzzy fractional dynamic system with unit impulse load are discussed. Lastly Numerical examples and conclusions are given.

## 2. Preliminaries

In this section, we present some notations, definitions and preliminaries which are used further in this paper [61, 70, 48, 47, 36, 57, 55, 35].

**Definition 2.1.** (Fuzzy number) A fuzzy number $\tilde{U}$ is convex normalised fuzzy set $\tilde{U}$ of the real line $R$ such that $\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall x \in R\}$ where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

**Definition 2.2.** (Triangular fuzzy number) A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line $R$ such that

- **i:** There exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ ($x_0$ is called the mean value of $\tilde{U}$),

where $\mu_{\tilde{U}}$ is called the memebership function of the fuzzy set.

**ii:** $\mu_{\tilde{U}}(x)$ is piecewise continuous.

Let us consider an arbitrary triangular fuzzy number $\tilde{U} = (a, b, c)$ as depicted in Fig. 1(i). The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be define as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, x \leq a \\ \frac{x-a}{b-a}, a \leq x \leq b \\ \frac{c-x}{c-b}, b \leq x \leq c \\ 0, x \geq c \end{cases}$$

The triangular fuzzy number $\tilde{U} = (a, b, c)$ can be represented with an ordered pair of functions through $\gamma-$ cut approach viz. $[\underline{u}(\gamma), \bar{u}(\gamma)] = [(b-a)\gamma + a, -(c-b)\gamma + c]$ where, $\gamma \in [0, 1]$

**Definition 2.3.** Trapezoidal fuzzy number

Again, let us consider an arbitrary trapezoidal fuzzy number $\tilde{U} = (a, b, c, d)$ as depicted in Fig. 1(ii). The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be interpreted as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, x \leq a \\ \frac{x-a}{b-a}, a \leq x \leq b \\ 1, b \leq x \leq c \\ \frac{d-x}{d-c}, c \leq x \leq d \\ 0, x \geq d \end{cases}$$

The trapezoidal fuzzy number $\tilde{U} = (a, b, c, d)$ can be represented with an ordered pair of functions through $\gamma-$ cut approach i.e $[\underline{u}(\gamma), \bar{u}(\gamma)] = [(b-a)\gamma+a, -(d-c)\gamma+d]$ where,$\gamma \in [0, 1]$.

**Definition 2.4.** (Gaussian fuzzy number) Let us now define an arbitrary asymmetrical Gaussian fuzzy number, $\tilde{U} = (r, \sigma_l, \sigma_r)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} \exp[-(x-r)^2/2\sigma_l^2] for x \leq r \\ \exp[-(x-r)^2/2\sigma_r^2] for x \geq r \end{cases} \qquad \forall x \in R$$

where, the modal value is denoted as $r$ and $\sigma_l, \sigma_r$ denote the left-hand and right-hand spreads (fuzziness) corresponding to the Gaussian distribution. For symmetric Gaussian fuzzy number the left-hand and right-hand spreads are equal i.e. $\sigma_l = \sigma_r = \sigma$. So the symmetric Gaussian fuzzy number may be written as $\tilde{U} = (r, \sigma, \sigma)$ and corresponding membership function may be defined as $\mu_{\tilde{U}}(x) = \exp\{-\beta(x-r)^2\} \forall x \in R$ where, $\beta = 1/2\sigma^2$. The symmetric Gaussian fuzzy number in parametric can be represented as

$$\tilde{U} = [\underline{u}(\gamma), \bar{u}(\gamma)] = \left[ r - \sqrt{-\frac{(\log_e \gamma)}{\beta}}, r + \sqrt{-\frac{(\log_e \gamma)}{\beta}} \right]$$

where,$\gamma \in [0, 1]$.

For all the above type of fuzzy numbers the left and right bound of the fuzzy numbers satisfies the following requirements

**i:** $\underline{u}(\gamma)$ is a bounded left continuous non-decreasing function over $[0, 1]$.

**ii:** $\bar{u}(\gamma)$ is a bounded right continuous non-increasing function over $[0, 1]$.

**iii:** $\underline{u}(\gamma) \leq \bar{u}(\gamma)$,$0 \leq \gamma \leq 1$.

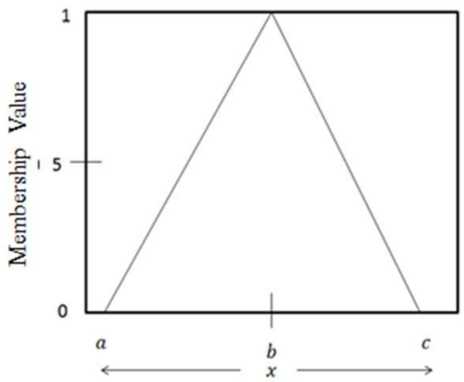



Fig. 1(i) Triangular fuzzy Number

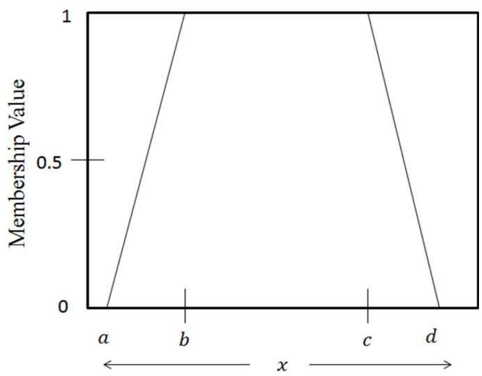



Fig. 1(ii) Trapezoidal fuzzy number

**Definition 2.5.** (Fuzzy arithmetic) For any two arbitrary fuzzy numbers $\tilde{x} = [\underline{x}(\gamma), \bar{x}(\gamma)]$, $\tilde{y} = [\underline{y}(\gamma), \bar{y}(\gamma)]$ and scalar $k$ , the fuzzy arithmetic is defined as follows,

**i:** $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\gamma) = \underline{y}(\gamma)$ and $\bar{x}(\gamma) = \bar{y}(\gamma)$

**ii:** $\tilde{x} + \tilde{y} = [\underline{x}(\gamma) + \underline{y}(\gamma), \bar{x}(r) + \bar{y}(\gamma)]$

**iii:** $\tilde{x} - \tilde{y} = [\underline{x}(\alpha) - \overline{y}(\alpha), \overline{x}(\alpha) - \underline{y}(\alpha)]$

**iv:** $\tilde{x} \times \tilde{y} = \begin{bmatrix} \min\left(\underline{x}(\gamma) \times \underline{y}(\gamma), \underline{x}(\gamma) \times \bar{y}(\gamma), \bar{x}(\gamma) \times \underline{y}(\gamma), \bar{x}(\gamma) \times \bar{y}(\gamma)\right), \\ \max\left(\underline{x}(\gamma) \times \underline{y}(\gamma), \underline{x}(\gamma) \times \bar{y}(\gamma), \bar{x}(\gamma) \times \underline{y}(\gamma), \bar{x}(\gamma) \times \bar{y}(\gamma)\right) \end{bmatrix}$

**v:** $k\tilde{x} = \left\{ \begin{array}{l} [k\overline{x}(\gamma), k\underline{x}(\gamma)], k < 0 \\ [k\underline{x}(\gamma), k\overline{x}(\gamma)], k \geq 0 \end{array} \right]$

**Lemma 2.6** ([66]). *If $\tilde{u}(t) = (x(t), y(t), z(t))$ is a fuzzy triangular number valued function and if $\tilde{u}$ is Hukuhara differentiable, then $\tilde{u}' = (x', y', z')$.*

*By using this property, we intend to solve the fuzzy initial value problem*

$$\begin{cases} \tilde{x}' = f(t, \tilde{x}) \\ \tilde{x}(t_0) = \tilde{x}_0 \end{cases}$$

*where,* $\tilde{x}_0 = (\underline{x}_0, x_0^c, \bar{x}_0) \in R, \tilde{x}(t) = (\underline{u}, u^c, \bar{u}) \in R$ *and* $f : [t_0, t_0 + a] \times R \to R, f(t, (\underline{u}, u^c, \bar{u})) = (\underline{f}(t, \underline{u}, u^c, \bar{u}), f^c(t, \underline{u}, u^c, \bar{u}), \overline{f}(t, \underline{u}, u^c, \bar{u}))$.
*We can translate this into the following system of ordinary differential equations as below:*

$$\begin{cases} \underline{u} = \underline{f}(t, \underline{u}, u^c, \bar{u}), \\ u^c = f^c(t, \underline{u}, u^c, \bar{u}) \\ \bar{u} = \overline{f}(t, \underline{u}, u^c, \bar{u}) \\ \underline{u}(0) = \underline{x}_0, u^c(0) = x_0^c, \bar{u}(0) = \bar{x}_0 \end{cases}$$

**Definition 2.7.** (Riemann-Liouville fractional integral) There are several definitions of fractional integral. The most commonly used definition is of Riemann-Liouville and Caputo [48]. The Riemann-Liouville integral operator $J^\alpha$ of order $\alpha \geq 0$, is defined by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau, t > 0.$$

**Definition 2.8.** (Caputo derivative) The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t)$$
$$= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\alpha+1-m}}, & m - 1 < \alpha < m, m \in N \\ \frac{d^m}{dt^m} f(t), & \alpha = m, m \in N \end{cases}$$

where, the parameter $\alpha$ is the order of the derivative and it is allowed to be real or even complex. In this paper, only real and positive $\alpha$ will be considered. For the Caputo's derivative we have

$$D^\alpha C = 0, \ C \text{ is a constatnt}$$

$$D^\alpha t^\beta = \begin{cases} 0, & (\beta \leq \alpha - 1) \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta > \alpha - 1) \end{cases}$$

Similar to integer-order differentiation, Caputo's fractional differentiation is linear operation:

$$D^\alpha (\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t),$$

where $\lambda, \mu$ are constants and satisfies so called Leibnitz rule:

$$D^\alpha (g(t) f(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} g^{(k)}(t) D^{\alpha-k} f(t),$$

if $f(\tau)$ is continuous in $[0, t]$ and $g(\tau)$ has $n + 1$ continuous derivative in $[0, t]$.

## 3. Application of HPM [31, 32] to fuzzy fractional dynamic systems

In this section homotopy perturbation method [31, 32] is applied to solve a fuzzy fractional single-degree of freedom spring-mass-damper system. Here the damping factor is defined as fraction and the initial conditions are taken as fuzzy that is in term of various types of fuzzy numbers to find the fuzzy displacements. The said problem is described by the following differential equation as

$$mD^2\tilde{x}(t;\gamma) + cD^{\alpha}\tilde{x}(t;\gamma) + k\tilde{x}(t;\gamma) = f(t) \tag{3.1}$$

where,$m$, $c$ and $k$ represent the mass, damping and stiffness coefficients respectively. $f(t)$ is the externally applied force, and $D^{\alpha}\tilde{x}(t;\gamma)$ for $0 < \alpha < 1$, is the derivative of order $\alpha$ (as define in Definition 2.7) of the fuzzy displacement function $\tilde{x}(t;\gamma) = [\underline{x}(t;\gamma), \bar{x}(t;\gamma)]$. Here $\tilde{x}(t;\gamma)$ is represented by $\gamma-$cut form of fuzzy displacements. Although the coefficient $\alpha$ (known as the memory parameter), may take any value between 0 to 1, the value 1/2 has been adopted here for this study because it has been shown that it describes the frequency dependence of the damping materials quite satisfactorily in the crisp fractional dynamic systems [60, 67]. Fuzzy initial displacements $\tilde{x}(0)$ and initial velocity $v(0) = \dot{\tilde{x}}(0)$ are taken as triangular, trapezoidal and Gaussian fuzzy number respectively for Cases 1 to 3 as depicted in Table 1.

Table 1. Data for fuzzy initial conditions

| Fuzzy initial Conditions | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $\tilde{x}(0)$ | $(-0.1, 0, 0.1)$ | $(-0.1, -0.050, 0.05, 0.1)$ | $(0, 0.1, 0.1)$ |
| $v(0) = \dot{\tilde{x}}(0)$ | $(-0.1, 0, 0.1)$ | $(-0.1, -0.050, 0.05, 0.1)$ | $(0, 0.1, 0.1)$ |

Through $\gamma-$ cut approach fuzzy initial condition for Cases 1 to 3 as given in Table 1 are now expressed in Table 2.

Table 2. $\gamma-$ cut representations of fuzzy initial conditions

| Fuzzy initial Conditions | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $\tilde{x}(0;\gamma)$ | $[0.1\gamma - 0.1,$ $0.1 - 0.1\gamma]$ | $[0.05\gamma - 0.1,$ $0.1 - 0.05\gamma]$ | $[-0.1\sqrt{-2\log_e(\gamma)},$ $0.1\sqrt{-2\log_e(\gamma)}]$ |
| $v(0;\gamma) = \dot{\tilde{x}}(0;\gamma)$ | $[0.1\gamma - 0.1,$ $0.1 - 0.1\gamma]$ | $[0.05\gamma - 0.1,$ $0.1 - 0.05\gamma]$ | $[-0.1\sqrt{-2\log_e(\gamma)},$ $0.1\sqrt{-2\log_e(\gamma)}]$ |

The fuzzy fractionally damped dynamic system (3.1) may be written as

$$D^2\tilde{x}(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{x}(t;\gamma) + \frac{k}{m}\tilde{x}(t;\gamma) = \frac{f(t)}{m} \tag{3.2}$$

According to HPM, we may construct a simple homotopy for an embedding parameter $p \in [0,1]$ as follows

$$\begin{aligned}(1-p)\left(D^2\tilde{X}(t;\gamma) - D^2\tilde{x}_0(t;\gamma)\right)\\ +p\left(D^2\tilde{X}(t;\gamma) + \tfrac{c}{m}D^{1/2}\tilde{X}(t;\gamma) + \tfrac{k}{m}\tilde{X}(t;\gamma) - \tfrac{f(t)}{m}\right) = 0\end{aligned} \tag{3.3}$$

or

$$D^2\tilde{X}(t;\gamma) - D^2\tilde{x}_0(t;\gamma) + p\left(D^2\tilde{x}_0(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}(t;\gamma) + \frac{k}{m}\tilde{X}(t;\gamma) - \frac{f(t)}{m}\right) = 0 \tag{3.4}$$

In the changing process from 0 to 1, for $p = 0$, Eq. (3.3) or (3.4) gives $D^2\tilde{X}(t;\gamma) - D^2\tilde{x}_0(t;\gamma) = 0$ and for $p = 1$, we have the original system $D^2\tilde{X}(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}(t;\gamma) + \frac{k}{m}\tilde{X}(t;\gamma) - \frac{f(t)}{m} = 0$. This is called deformation in topology. $\left(D^2\tilde{X}(t;\gamma) - D^2\tilde{x}_0(t;\gamma)\right)$ and $\left(D^2\tilde{X}(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}(t;\gamma) + \frac{k}{m}\tilde{X}(t;\gamma) - \frac{f(t)}{m}\right)$ are called homotopic.

Next, we can assume the solution of Eq. (3.3) or (3.4) as a power series expansion in $p$ as

$$\tilde{X}(t;\gamma) = \tilde{X}_0(t;\gamma) + p\tilde{X}_1(t;\gamma) + p^2\tilde{X}_2(t;\gamma) + p^3\tilde{X}_3(t;\gamma) + \cdots, \tag{3.5}$$

where $\tilde{X}_i(t;\gamma), \mathrm{i} = 0,1,2,\ldots$ are functions yet to be determined. Substituting Eq. (3.5) into Eq. (3.3) or (3.4), and equating the terms with the identical power of $p$ we can obtain a series of equations of the form

$$\begin{aligned}
p^0 &: D^2\tilde{X}_0(t;\gamma) - D^2\tilde{x}_0(t;\gamma) = 0\\
p^1 &: D^2\tilde{X}_1(t;\gamma) + D^2\tilde{x}_0(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_0(t;\gamma) + \frac{k}{m}\tilde{X}_0(t;\gamma) - \frac{f(t)}{m} = 0\\
p^2 &: D^2\tilde{X}_2(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_1(t;\gamma) + \frac{k}{m}\tilde{X}_1(t;\gamma) = 0\\
p^3 &: D^2\tilde{X}_3(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_2(t;\gamma) + \frac{k}{m}\tilde{X}_2(t;\gamma) = 0\\
p^4 &: D^2\tilde{X}_4(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_3(t;\gamma) + \frac{k}{m}\tilde{X}_3(t;\gamma) = 0\\
p^5 &: D^2\tilde{X}_5(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_4(t;\gamma) + \frac{k}{m}\tilde{X}_4(t;\gamma) = 0\\
p^6 &: D^2\tilde{X}_6(t;\gamma) + \frac{c}{m}D^{1/2}\tilde{X}_5(t;\gamma) + \frac{k}{m}\tilde{X}_5(t;\gamma) = 0
\end{aligned} \tag{3.6}$$

and so on.

Applying the operator $L_{tt}^{-1}$ (the inverse operator of $D^2 = \frac{d^2}{dt^2}$ ) on both sides of Eq. (3.6) one may get the approximate solution $\tilde{x}(t;\gamma) = \lim\limits_{p\to 1}\tilde{X}(t;\gamma)$ which can be expressed as

$$\tilde{x}(t;\gamma) = \tilde{X}_0(t;\gamma) + \tilde{X}_1(t;\gamma) + \tilde{X}_2(t;\gamma) + \tilde{X}_3(t;\gamma) + \cdots \tag{3.7}$$

Now the above expression can be equivalently written as follows

$$[\underline{x}(t;\gamma), \bar{x}(t;\gamma)] = \sum_{n=0}^{\infty} \tilde{X}_n(t;\gamma)$$

Using Lemma 2.1 one may have the lower and upper bounds of the solution in parametric form are given respectively as $\underline{x}(t;\gamma) = \sum_{n=0}^{\infty} \underline{X}_n(t;\gamma)$ and $\bar{x}(t;\gamma) = \sum_{n=0}^{\infty} \bar{X}_n(t;\gamma)$. The series obtained by HPM converges very rapidly and only few terms are required to get the approximate solutions. The proof may be found in [62, 63].

## 4. Response analysis

In this section we have consider the response subject to a unit impulsive load viz. $f(t) = \delta(t)$, where $\delta(t)$ is the unit impulse function. Various type of simulation using HPM has been made with different type of fuzzy initial conditions for Cases 1 to 3 as given in Table 1 or 2.

Case 1: By using HPM for triangular fuzzy initial conditions we have

$$\underline{X}_0(t;\gamma) = 0.1\gamma - 0.1$$

$$\overline{X}_0(t;\gamma) = 0.1 - 0.1\gamma$$

$$\underline{X}_1(t;\gamma) = -(0.1\gamma - 0.1)\frac{k}{m}\frac{t^2}{2} + \frac{t}{m}$$

$$\overline{X}_1(t;\gamma) = -(0.1 - 0.1\gamma)\frac{k}{m}\frac{t^2}{2} + \frac{t}{m}$$

$$\underline{X}_2(t;\gamma) = (0.1\gamma - 0.1)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)} + \frac{k}{m}\frac{t^4}{\Gamma(5)}\right) - \frac{c}{m^2}\frac{t^{5/2}}{\Gamma(7/2)} - \frac{k}{m^2}\frac{t^3}{\Gamma(4)}$$

$$\overline{X}_2(t;\gamma) = (0.1 - 0.1\gamma)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)} + \frac{k}{m}\frac{t^4}{\Gamma(5)}\right) - \frac{c}{m^2}\frac{t^{5/2}}{\Gamma(7/2)} - \frac{k}{m^2}\frac{t^3}{\Gamma(4)}$$

$$\underline{X}_3(t;\gamma) = -(0.1\gamma - 0.1)\frac{k}{m}\left(\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)} + \frac{2kc}{m^2}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)\frac{c^2}{m^3}\frac{t^4}{\Gamma(5)} + \frac{2kc}{m^3}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^3}\frac{t^5}{\Gamma(6)}$$

$$\overline{X}_3(t;\gamma) = -(0.1 - 0.1\gamma)\frac{k}{m}\left(\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)} + \frac{2kc}{m^2}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)\frac{c^2}{m^3}\frac{t^4}{\Gamma(5)} + \frac{2kc}{m^3}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^3}\frac{t^5}{\Gamma(6)}$$

and so on.

In the similar manner higher order approximation may be obtained as discussed above. Substituting these in Eq. (3.7) we may get the approximate solution of $\tilde{x}(t)$. Accordingly, the general solution may be written as

(4.1)

$$\underline{x}(t;\gamma) = (0.1\gamma - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right) + \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.1\gamma - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)$$

and

(4.2)

$$\bar{x}(t;\gamma) = (0.1 - 0.1\gamma)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.1 - 0.1\gamma)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)$$

Case 2: Applying HPM for trapezoidal fuzzy initial condition the obtained general solution can be represented as

$$\underline{x}(t;\gamma) = (0.05\gamma - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$\begin{aligned}(4.3)\qquad &= (0.05\gamma - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)\\ &+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)\end{aligned}$$

$$\overline{x}(t;\gamma) = (0.1 - 0.05\gamma)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$\begin{aligned}(4.4)\qquad &= (0.1 - 0.05\gamma)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)\\ &+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)\end{aligned}$$

Case 3: Similarly for Gaussian fuzzy initial condition one may have the general solution by using HPM as

$$\underline{x}(t;\gamma) = -0.1\sqrt{-2\log_e(\gamma)}\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$
$$+ \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

(4.5)

$$= -0.1\sqrt{-2\log_e(\gamma)}\left(1-\tfrac{k}{m}\left(\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2(r+1)}E^r_{3/2,r/2+3}\left(\tfrac{-c}{m}t^{3/2}\right)\right)\right)$$
$$+\tfrac{1}{m}\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\tfrac{-c}{m}t^{3/2}\right)$$

$$\bar{x}(t;\gamma) = 0.1\sqrt{-2\log_e(\gamma)}\left(1-\tfrac{k}{m}\left(\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\tfrac{-c}{m}\right)^j\tfrac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$
$$+\tfrac{1}{m}\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\tfrac{-c}{m}\right)^j\tfrac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$(4.6)\quad \begin{aligned}&= 0.1\sqrt{-2\log_e(\gamma)}\left(1-\tfrac{k}{m}\left(\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2(r+1)}E^r_{3/2,r/2+3}\left(\tfrac{-c}{m}t^{3/2}\right)\right)\right)\\&+\tfrac{1}{m}\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!}\left(\tfrac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\tfrac{-c}{m}t^{3/2}\right)\end{aligned}$$

For Cases 1 and 3 fuzzy initial conditions for $\gamma = 1$ converts to nominal (crisp) initial values that means the lower and upper bounds of the fuzzy initial values are equal. So it is interesting to note that the solution obtained using HPM for $\gamma = 1$ is exactly same with the analytical crisp solution as given in [48] for the same crisp initial conditions. For impulse response, that is when $f(t) = \delta(t)$ the analytic solution may be obtained from [48] as

$$(4.7)\qquad x(t) = \frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)$$

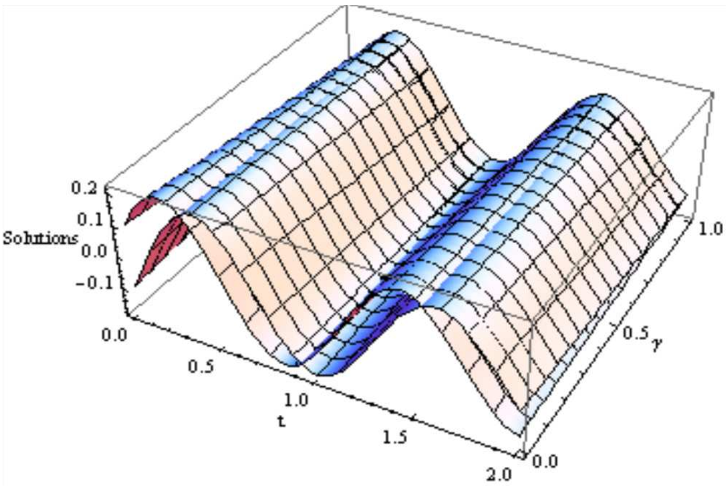
In Eqs. (4.1) to (4.6) and for Eq. (4.7),$E^r_{\lambda,\mu}(y)$ is called the Mittage-Leffler function [48] of two parameters $\lambda$ and $\mu$. Where

$$E^r_{\lambda,\mu}(y) \equiv \frac{d^r}{dy^r}E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty}\frac{(j+r)!y^j}{j!\Gamma(\lambda j+\lambda r+\mu)}, (r = 0, 1, 2, \ldots)$$

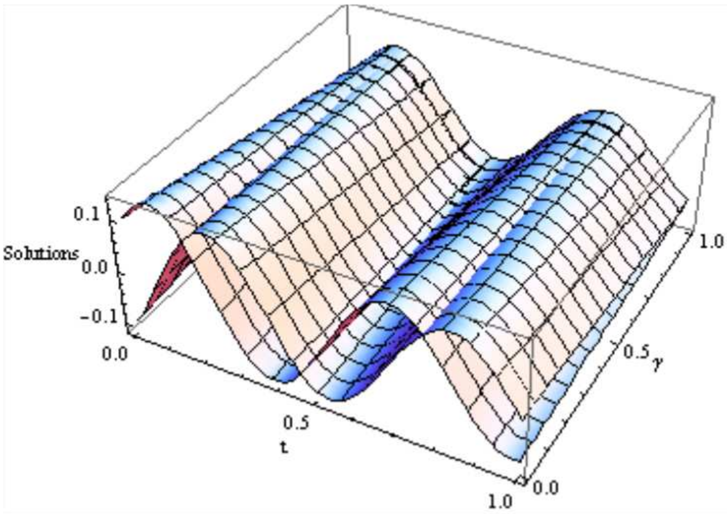
## 5. Numerical results and discussions

As discussed above, unit impulse load have been considered for the present analysis with various fuzzy initial conditions. Obtained results are depicted in term of plots. Depending upon the values of natural frequency $\omega_n$ and damping ratio $\eta$ different cases have been studied. First the numerical values of the natural frequency $\omega_n = 5\,rad/s$ and damping ratio $\eta = 0.05$ are taken. Next, natural frequency $\omega_n = 10\,rad/s$ with damping ratio $\eta = 0.05$ with unit impulse load is considered for the oscillation. With these parametric values fuzzy displacements are obtained for triangular, trapezoidal and Gaussian fuzzy initial conditions as discussed in Section 4 (with respect to time) and are depicted in Figs. 2 to 4 respectively. Also one can see from Figs. 2 and 4 for Cases 1 and 3 that lower and upper bounds of the fuzzy displacements coincide for $\gamma = 1$, this is because the fuzzy initial conditions converts in this case to crisp one. It is worth mentioning that the results obtained for all the cases are strong fuzzy solution.

It may be noted that if the initial condition is crisp then we may have the initial condition as $v(0) = v'(0) = 0$ . Here the initial condition has been taken as fuzzy with an idea that the condition may actually be uncertain viz. it may be due to error in observation or experiment etc. where we may model uncertainty in terms of fuzzy triangular, trapezoidal and Gaussian membership function. As such this will force the governing differential equation as a whole as uncertain. So, naturally the outcome or the output (result) must be uncertain. This way we may have the actual essence of the uncertainty in response which may benefit the engineers to understand the safety of the system in a better way.

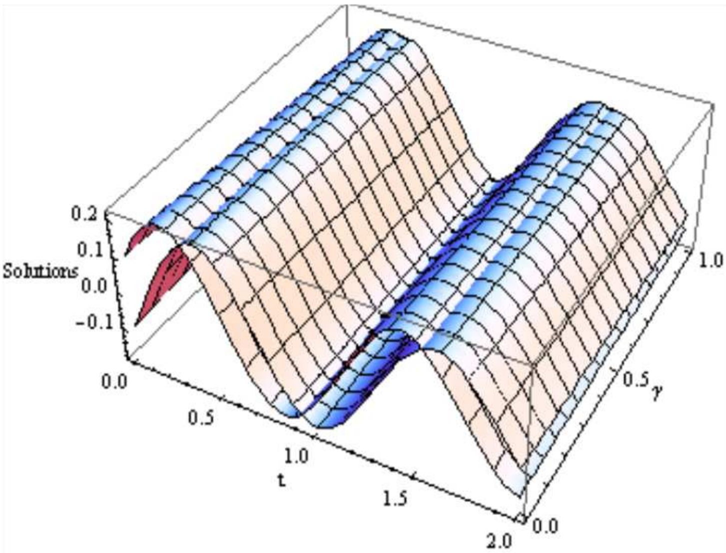



2 (i)

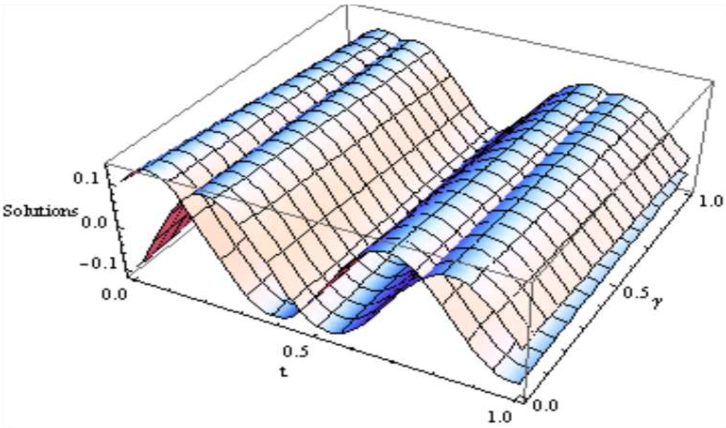



2 (ii)

Fig. 2 Triangular fuzzy response subject to unit impulse load for Case 1 with natural frequency (i) $\omega_n = 5\,rad/s$ (ii)$\omega_n = 10\,rad/s$ and damping ratio $\eta = 0.05$




3(i)




3(ii)

Fig. 3 Trapezoidal fuzzy response subject to unit impulse load for Case 1 with natural frequency (i) $\omega_n = 5\,rad/s$ (ii)$\omega_n = 10\,rad/s$ and damping ratio $\eta = 0.05$

## 6. Conclusions

Homotopy perturbation method with fuzzy based approach has successfully been applied to obtain the uncertain solution of an imprecisely defined fractionally damped spring-mass mechanical system subject to an unit impulse load, where the fraction derivative is considered as of order 1/2. For uncertain impulse responses triangular, trapezoidal and Gaussian fuzzy initial conditions are chosen to illustrate the method. The presented method may very well be used for other type of fuzzy number also. This method is found to be efficient for computing approximate solution bounds of uncertain differential equation for fractional order because only a few terms are required for the convergence. As, the present study able to deliver the lower and upper bounds of the uncertain response of spring-mass mechanical system, hence it is promising that it may directly be applied for other engineering problems too.

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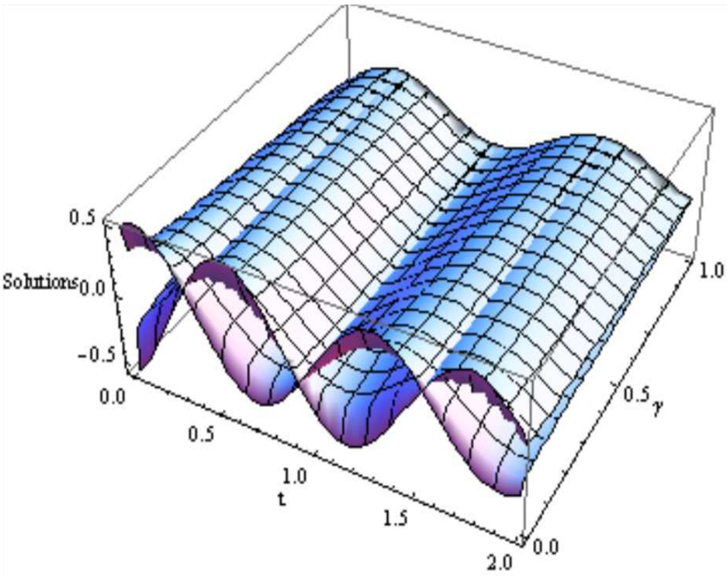
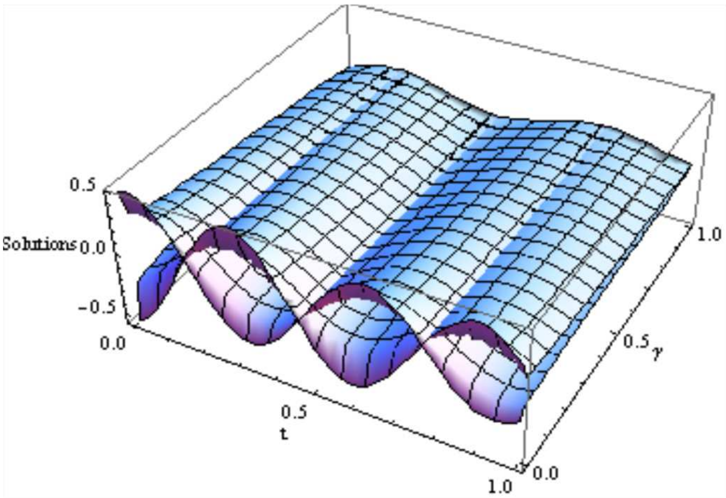



4 (i)




4(ii)

Fig. 4 Gaussian fuzzy response subject to unit impulse load for Case 1 with natural frequency (i) $\omega_n = 5\,rad/s$ (ii)$\omega_n = 10\,rad/s$ and damping ratio $\eta = 0.05$

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