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Intuitionistic fuzzy generalized normed spaces

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Abstract. The aim of this paper is to study the generalization of the intuitionistic fuzzy normed spaces such as intuitionistic fuzzy 2-normed space. In this structure, we have discussed the intuitionistic fuzzy 2-continuity and intuitionistic fuzzy 2-boundedness. Also, we have introduced the intuitionistic fuzzy ψ-2-normed space which is a generalization of intuitionistic fuzzy 2-normed space. We have discussed some results in this new set up.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh[19] in 1965. After the pioneer work of Zadeh, many researchers have extended this concept in various branches of mathematics and introduced new theories like fuzzy set theory [9], fuzzy group theory, fuzzy differential equation, fuzzy topology, fuzzy metric spaces [2, 5, 7], fuzzy normed spaces [14] etc. We are especially interested in theory of fuzzy normed spaces and their generalizations. Atanassov[3] introduced the concept of intuitionistic fuzzy sets which is further studied by Coker[4]. Park[13] has introduced the concept of intuitionistic fuzzy metric space. Saadati and Park[14] coined the notion of intuitionistic fuzzy normed space. Hee Won Kang, Jeong-Gon Lee, Kul Hur[8] studied some fundamental properties of intuitionistic fuzzy mapping. Certainly, there are some situations where the ordinary norm does not work and the concept of intuitionistic fuzzy norm seems to be more suitable. A lot of works have been done in intuitionistic fuzzy normed spaces see in [17],[11],[12],[14], [6], [18].

Recently, M. Mursaleen[10] defined the new structure intuitionistic fuzzy 2-normed space and studied some basic results of normed linear spaces. In this paper, we have studied the continuity and boundedness in intuitionistic fuzzy 2-normed spaces. T.K. Samanta and Sumit Mohinta[15] have introduced the concept of intuitionistic fuzzy ψ-normed space and discussed continuity and boundedness in this structure. We have coined the concept of intuitionistic fuzzy ψ-2-normed space which is generalization of intuitionistic fuzzy 2-normed space. It shall provide more suitable framework to deal with the inexactness of the norm or 2-norm in some situations.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1 ([16])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Example 2.2. Two typical examples of continuous t-norms are

$$a * b = ab \quad \text{and} \quad a * b = \min\{a, b\}.$$

Definition 2.3 ([16])**.** A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a) $\diamond$ is associative and commutative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Example 2.4. Two typical examples of continuous t-conorms are

$$a \diamond b = \min\{a + b, 1\} \quad \text{and} \quad a \diamond b = \max\{a, b\}.$$

M. Amini and R. Saadati studied some properties of t-norm in [1].

Definition 2.5 ([14])**.** The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,

(a) $\mu(x,t) + \nu(x,t) \leq 1$;
(b) $\mu(x,t) > 0$;
(c) $\mu(x,t) = 1$ if and only if $x = 0$;
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(e) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$;
(f) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;
(h) $\nu(x,t) < 1$;
(i) $\nu(x,t) = 0$ if and only if $x = 0$;
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+t, y+s)$;
(l) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.6. Let $(V, \|\cdot\|)$ be normed space over $\mathbb{F}$. Denote $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}, \forall a, b \in [0,1]$ and let μ_0 and ν_0 be fuzzy sets on $V \times (0,\infty)$ defined as follows $\mu_0(x,t) = \dfrac{t}{t+\|x\|}$, $\nu_0(x,t) = \dfrac{\|x\|}{t+\|x\|}$, for all $t \in \mathbb{R}^+$. Then $(V, \mu_0, \nu_0, *, \diamond)$ is an intuitionistic fuzzy normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.7 ([10])**.** Let V be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on V is a function $\|\cdot,\cdot\| : V \times V \to \mathbb{R}$ which satisfies, for every $x, y, z \in V$

(a) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x,y\| = \|y,x\|$;
(c) $\|\alpha x, y\| = |\alpha|\|x,y\|$;
(d) $\|x, y+z\| \leq \|x,y\| + \|y,z\|$.

The pair $V, \|\cdot,\cdot\|$ is then called a 2-normed space.

As an example of a 2-normed space take $V = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula $\|x,y\| = |x_1y_2 - x_2y_1|, x = (x_1,x_2), y = (y_1,y_2)$.

Definition 2.8 ([10])**.** The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times V \times (0,\infty)$ satisfying the following conditions. For every $x, y, z \in V$ and $s, t > 0$,

(a) $\mu(x,y,t) + \nu(x,y,t) \leq 1$;
(b) $\mu(x,y,t) > 0$;
(c) $\mu(x,y,t) = 1$ if and only if x and y are linearly dependent;
(d) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(e) $\mu(x,y,t) * \mu(x,z,s) \leq \mu(x, y+z, t+s)$;
(f) $\mu(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(g) $\lim_{t\to\infty} \mu(x,y,t) = 1$ and $\lim_{t\to 0} \mu(x,y,t) = 0$;
(h) $\mu(x,y,t) = \mu(y,x,t)$;
(i) $\nu(x,y,t) < 1$;
(j) $\nu(x,y,t) = 0$ if and only if x and y are linearly dependent;
(k) $\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(l) $\nu(x,y,t) \diamond \nu(x,z,s) \geq \nu(x, y+z, t+s)$;
(m) $\nu(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(n) $\lim_{t\to\infty} \nu(x,y,t) = 0$ and $\lim_{t\to 0} \nu(x,y,t) = 1$,
(o) $\nu(x,y,t) = \nu(y,x,t)$.

In this case $(\mu,\nu)_2$ is called an intuitionistic fuzzy 2-norm on V. We denote it by $(\mu,\nu)_2$.

Example 2.9 ([10])**.** Let $(V, \|\cdot,\cdot\|)$ be 2-normed space over F and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$, for all $a, b \in [0,1]$ and every $t > 0$, consider $\mu(x,y,t) =$

$\dfrac{t}{t+\|x,y\|}$, $\nu(x,y,t) = \dfrac{\|x,y\|}{t+\|x,y\|}$. Then $(V,\mu,\nu,*,\diamond)$ is an intuitionistic fuzzy 2-normed space.

Definition 2.10 ([10]). Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy 2-normed space and let $r \in (0,1), t > 0$ and $x \in X$. The set $B(x,r,t) = \{y \in V : \mu(y-x,z,t) > 1-r,\ \ \nu(y-x,z,t) < r, \forall z \in V\}$ is called the open ball with center x and radius r with respect to t.

Definition 2.11 ([10]). Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy 2-normed space. A set $U \subset V$ is said to an open set if each of its points is the centre of some open ball contained in U. The open set in an intuitionistic fuzzy 2-normed space $(V,\mu,\nu,*,\diamond)$ is denoted by $\mathbb{U}$.

Definition 2.12 ([10]). Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy 2-normed space. A sequence $\{x_n\}$ in V is said to be Cauchy if for each $r > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, z, t) > 1 - r$ and $\nu(x_n - x_m, z, t) < r$ for all $n, m \geq n_0$ and for all $z \in V$.

Definition 2.13 ([10]). Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy 2-normed space. A sequence $\{x_k\}$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy 2-norm $(\mu,\nu)_2$, if for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, z, t) > 1 - \epsilon$ and $\nu(x_k - L, z, t) < \epsilon$ for all $k \geq k_0$ and for all $z \in V$.

Definition 2.14 ([15]). Let ψ be a function defined on the real field $\mathbb{R}$ into itself satisfying the following properties;

(a) $\psi(-t) = \psi(t)$ for all $t \in \mathbb{R}$
(b) $\psi(1) = 1$
(c) ψ is strictly increasing and continuous on $(0,\infty)$
(d) $\lim_{\alpha\to 0}\psi(\alpha) = 0$ and $\lim_{\alpha\to\infty}\psi(\alpha) = \infty$.

Example 2.15 ([15]). Consider $\psi(\alpha) = |\alpha|; \psi(\alpha) = |\alpha|^p, p \in \mathbb{R}^+;\ \psi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}, n \in \mathbb{N}^+$. The function ψ allows us to generalize fuzzy metric and normed space.

Definition 2.16 ([15]). The five-tuple $(V,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy ψ-normed space if V is a vector space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ,ν are fuzzy sets on $V \times (0,\infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,

(a) $\mu(x,t) + \nu(x,t) \leq 1$;
(b) $\mu(x,t) > 0$;
(c) $\mu(x,t) = 1$ if and only if $x = 0$;
(d) $\mu(\alpha x,t) = \mu(x, \frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;
(f) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(g) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to 0}\mu(x,t) = 0$;
(h) $\nu(x,t) < 1$;
(i) $\nu(x,t) = 0$ if and only if $x = 0$;
(j) $\nu(\alpha x,t) = \nu(x, \frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+t, y+s)$;

(l) $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous;
(m) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case (μ,ν) is called an intuitionistic fuzzy ψ-norm.

3. Intuitionistic fuzzy 2-normed space

Theorem 3.1. *In an intuitionistic fuzzy 2-normed space $(V,\mu,\nu,*,\diamond)$, if*
$$\{x_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2} x \text{ and } \{y_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2} y,$$
then $\{x_n+y_n\}_{n=1}^{\infty}$ is convergent to $x+y$. In other word, if $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy 2-normed space then the addition is continuous in $(V,\mu,\nu,*,\diamond)$.

Theorem 3.2. *In an intuitionistic fuzzy 2-normed space $(V,\mu,\nu,*,\diamond)$, if $\lambda_n,\lambda\in\mathbb{R}^+$, $\lambda_n\to\lambda$ as $n\to\infty$ and $\{x_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2} x$ as $n\to\infty$ then $\{\lambda_n x_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2}\lambda x$. In other word, if $(V,\mu,\nu,*,\diamond)$ be an IF-2-NS then the scalar multiplication is continuous in $(V,\mu,\nu,*,\diamond)$.*

Proof. The proof of theorems (3.1) and (3.2) directly follows from definitions. □

Lemma 3.3. *Let $\{x_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2} x$ as $n\to\infty$ in intuitionistic fuzzy 2-normed space $(V,\mu,\nu,*,\diamond)$. Then for every $t>0$ as $n\to\infty$,*
$$\mu(x_n,z,t)\to\mu(x,z,t),\qquad \nu(x_n,z,t)\to\nu(x,z,t). \tag{3.1}$$

Proof. Let $\{x_n\}_{n=1}^{\infty}\to^{(\mu,\nu)_2} x$ as $n\to\infty$ in $(V,\mu,\nu,*,\diamond)$. Then $t>0$, $\forall k\in\mathbb{N}^+$,
$$\begin{aligned}\mu(x_n,z,t)&=\mu(x_n-x+x,z,\frac{t}{k+1}+\frac{kt}{k+1})\\ &\geq\mu(x_n-x,z,\frac{t}{k+1})*\mu(x,z,\frac{kt}{k+1})\\ &\to 1*\mu(x,z,\frac{kt}{k+1}),(n\to\infty)\\ &=\mu(x,z,\frac{kt}{k+1}),\end{aligned}$$
so $\underline{lim}_{n\to\infty}\mu(x_n,z,t)\geq\mu(x,z,\frac{kt}{k+1}),(k=1,2,\cdots)$.

Letting $k\to+\infty$ yields that,
$$\underline{lim}_{n\to\infty}\mu(x_n,z,t)\geq\mu(x,z,t). \tag{3.2}$$

On the other hand, for all $k\in\mathbb{N}^+$, $\mu(x-x_n,z,\frac{1}{k+1})\to 1>\frac{k}{k+1}>0$, as $n\to\infty$. So there exists an N such that, $\mu(x-x_n,z,\frac{1}{k+1})>\frac{k}{k+1},(\forall n>N)$. Thus, $\forall n>N$ and $\forall t>0$, we have
$$\mu(x_n,z,t)*\frac{k}{k+1}\leq\mu(x_n,z,t)*\mu(x-x_n,z,\frac{1}{k+1})\leq\mu(x,z,t+\frac{1}{k+1}).$$
Thus,
$$\mu(x_n,z,t)*\frac{k}{k+1}\leq\mu(x,z,t+\frac{1}{k+1}),(\forall n>N).$$

Hence,

$$\overline{\lim}_{n\to\infty}\mu(x_n, z, t) * \frac{k}{k+1} \leq \mu(x, z, t + \frac{1}{k+1}).$$

for all $k = 1, 2, 3, \cdots$. Letting $k \to +\infty$ yields that

$$\overline{\lim}_{n\to\infty}\mu(x_n, z, t) \leq \mu(x, z, t). \tag{3.3}$$

Now (3.2),(3.3) implies that $\lim_{n\to\infty} \mu(x_n, z, t) = \mu(x, z, t)$.

Similarly, we get, $\lim_{n\to\infty} \nu(x_n, z, t) = \nu(x, z, t)$. The proof is completed. □

Theorem 3.4. *In an intuitionistic fuzzy 2-normed space $(V, \mu, \nu, *, \diamond)$, the mappings $\mu, \nu : V \times V \times (0, \infty) \to [0, 1]$ are continuous.*

Proof. Let $x \in V$ and $t > 0$ with $(x_n, z, t_n) \to (x, z, t)$ as $n \to \infty$ in $V \times V \times (0, \infty)$. Then $x_n \to^{(\mu,\nu)_2} x$ as $n \to \infty$ in V and $t_n \to t$ as $n \to \infty$ in $(0, \infty)$. Thus, for every $\delta > 0$ such that $\delta < \min\{\frac{t}{2}, 1\}$, there is $n_0 \in N$ such that for all $n \geq n_0$,

$$t - \delta < t_n < t + \delta \quad \textit{and} \quad \mu(x - x_n, z, \delta) > 1 - \delta, \quad \nu(x - x_n, z, \delta) < \delta. \tag{3.4}$$

Hence, for all $n \geq n_0$, we see from (3.4)

$$\begin{aligned} \mu(x_n, z, t_n) &\geq \mu(x_n, z, t - \delta) \\ &= \mu(x_n - x + x, z, \delta + t - 2\delta) \\ &\geq \mu(x_n - x, z, \delta) * \mu(x, z, t - 2\delta) \\ &\geq (1 - \delta) * \mu(x, z, t - 2\delta) \end{aligned}$$

and

$$\begin{aligned} \nu(x_n, z, t_n) &\leq \nu(x_n, z, t - \delta) \\ &= \nu(x_n - x + x, z, \delta + t - 2\delta) \\ &\leq \nu(x_n - x, z, \delta) \diamond \nu(x, z, t - 2\delta) \\ &\leq \delta \diamond \nu(x, z, t - 2\delta). \end{aligned}$$

Thus, for all $n \geq n_0$, $\mu(x_n, z, t_n) \geq (1 - \delta) * \mu(x, z, t - 2\delta)$ and $\nu(x_n, z, t_n) \leq \delta \diamond \nu(x, z, t - 2\delta)$. This shows that

$$\underline{\lim}_{n\to\infty}\mu(x_n, z, t_n) \geq (1 - \delta) * \mu(x, z, t - 2\delta) \tag{3.5}$$

and

$$\overline{\lim}_{n\to\infty}\nu(x_n, z, t_n) \leq \delta \diamond \nu(x, z, t - 2\delta). \tag{3.6}$$

Letting $\delta \to 0^+$, in (3.5), (3.6) yields that

$$\underline{\lim}_{n\to\infty}\mu(x_n, z, t_n) \geq 1 * \mu(x, z, t) = \mu(x, z, t) \tag{3.7}$$

and

$$\overline{\lim}_{n\to\infty}\nu(x_n, z, t_n) \leq 0 \diamond \nu(x, z, t) = \nu(x, z, t). \tag{3.8}$$

On the other hand, when $n \geq n_0$. It follows from Lemma (3.3) that

$$\mu(x_n, z, t_n) \leq \mu(x_n, z, t + \delta) \to \mu(x, z, t + \delta) \quad as \quad n \to \infty,$$

and

$$\nu(x_n, z, t_n) \geq \nu(x_n, z, t - \delta) \to \nu(x, z, t - \delta) \quad as \quad n \to \infty.$$

Hence,

$$\overline{\lim}_{n\to\infty}\mu(x_n, z, t_n) \leq \mu(x, z, t + \delta) \tag{3.9}$$

and

$$\underline{\lim}_{n\to\infty}\nu(x_n, z, t_n) \geq \nu(x, z, t - \delta). \tag{3.10}$$

Letting $\delta \to 0^+$, in (3.9) and (3.10) yields that

$$\overline{\lim}_{n\to\infty}\mu(x_n, z, t_n) \leq \mu(x, z, t) \quad \text{and} \quad \underline{\lim}_{n\to\infty}\nu(x_n, z, t_n) \geq \nu(x, z, t).$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} \mu(x_n, z, t_n) \leq \mu(x, z, t) \quad \text{and} \quad \lim_{n\to\infty} \nu(x_n, z, t_n) \geq \nu(x, z, t).$$

Therefore, the mappings $\mu, \nu : V \times V \times (0, \infty) \to [0, 1]$ are continuous. □

Definition 3.5. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond)$ is said to be intuitionistic fuzzy 2-bounded (shortly,IF-2-B) if there exist constants $h, k \in \mathbb{R} - \{0\}$ such that, $\mu'(Tx, z, t) \geq \mu(hx, z, t)$ and $\nu'(Tx, z, t) \leq \nu(kx, z, t)$ for every $x, z(nonzero) \in V$ and for every $t > 0$.

Theorem 3.6. *Suppose that $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *, \diamond)$ are intuitionistic fuzzy 2-normed spaces over $\mathbb{F}$ with*

(*a*) $1 \geq a \geq c \geq 0$ *and* $1 \geq b \geq c \geq 0$ *implies* $a * b \geq c$
(*b*) $0 \leq a \leq c \leq 1$ *and* $0 \leq b \leq c \leq 1$ *implies* $a \diamond b \leq c$.

If linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond)$ are (IF-2-B) intuitinistic fuzzy 2-bounded, then $T_1 + T_2$ and $cT(c \in \mathbb{F})$ are also IF-2-B.

Proof. Since, linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond)$ are IF-2-B there exists $k_1, k_2, h_1, h_2 \in \mathbb{R}^+$ such that,

$$\mu'(T_1x, z, t) \geq \mu(h_1x, z, t) \quad and \quad \nu'(T_1x, z, t) \geq \nu(k_1x, z, t),$$

$$\mu'(T_2x, z, t) \geq \mu(h_2x, z, t) \quad and \quad \nu'(T_2x, z, t) \geq \nu(k_2x, z, t),$$

for every $x, z(nonzero) \in V$ and for every $t > 0$. Put $h = \max\{h_1, h_2\}$ and $k = \max\{k_1, k_2\}$ then for every $x \in V$, we have

$$\begin{aligned}
\mu^{'}((T_1 + T_2)x, z, t) &= \mu^{'}(T_1x + T_2x, z, t) \\
&\geq \mu^{'}(T_1x, z, \frac{t}{2}) * \mu^{'}(T_2x, z, \frac{t}{2}) \\
&\geq \mu(h_1x, z, \frac{t}{2}) * \mu(h_2x, z, \frac{t}{2}) \\
&= \mu(x, z, \frac{t}{2h_1}) * \mu(x, z, \frac{t}{2h_2}) \\
&\geq \mu(x, z, \frac{t}{2h}) * \mu(x, z, \frac{t}{2h}) \\
&\geq \mu(x, z, \frac{t}{3h}) \\
&= \mu(3hx, z, t).
\end{aligned}$$

Similarly, $\nu^{'}((T_1 + T_2)x, z, t) \leq \mu(3hx, z, t)$. Hence $T_1 + T_2$ is IF-2-B. Similarly, we can prove that cT is IF-2-B. □

Definition 3.7. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ is said to be intuitionistic fuzzy 2-continuous shortly,(IF-2-C) at a point $x \in V$ if $x_n \to^{(\mu,\nu)_2} x$ as $n \to \infty$ in $(V, \mu, \nu, *, \diamond)$ implies that $Tx_n \to^{(\mu^{'},\nu^{'})_2} Tx$ as $n \to \infty$ in $(V, \mu^{'}, \nu^{'}, *, \diamond)$. An operator T is said to be IF-2-C if it is intuitionistic fuzzy 2-continuous everywhere.

Definition 3.8. A map $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ is said to be intuitionistic fuzzy 2-continuous shortly,(IF-2-C) at a point $x_0 \in V$ if for any given $\epsilon > 0$, $\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for all $x \in V$,

$$\mu(x - x_0, z, \delta) > \beta \Rightarrow \mu^{'}(Tx - Tx_0, z, \epsilon) > \alpha,$$

$$\nu(x - x_0, z, \delta) < 1 - \beta \Rightarrow \nu^{'}(Tx - Tx_0, z, \epsilon) < 1 - \alpha.$$

Remark 3.9. The above definitions (3.7) and (3.8) are equivalent.

Theorem 3.10. *Suppose that $(V, \mu, \nu, *, \diamond)$ and $(V, \mu^{'}, \nu^{'}, *, \diamond)$ are intuitionistic fuzzy 2-normed spaces over $\mathbb{F}$. If linear operators $T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ are IF-2-C, then $c_1T_1 + c_2T_2$ is IF-2-C for all scalars $c_1, c_2 \in \mathbb{F}$.*

Proof. Let $x_n \to^{(\mu,\nu)_2} x$ as $n \to \infty$ in $(V, \mu, \nu, *, \diamond)$. Since, linear operators $T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ are IF-2-C, we get,

$$T_1x_n \to^{(\mu^{'},\nu^{'})_2} T_1x, \quad T_2x_n \to^{(\mu^{'},\nu^{'})_2} T_2x \;\; as \;\; n \to \infty$$

in $(V, \mu^{'}, \nu^{'}, *, \diamond)$, by (3.1),(3.2), we get for all $c_1, c_2 \in \mathbb{F}$

$$c_1(T_1x_n) + c_2(T_2x_n) \to^{(\mu^{'},\nu^{'})_2} c_1(T_1x) + c_2(T_2x) \;\; as \;\; n \to \infty$$

which gives

$$(c_1T_1 + c_2T_2)x_n \to^{(\mu^{'},\nu^{'})_2} (c_1T_1 + c_2T_2)x \;\; as \;\; n \to \infty.$$

Hence, $c_1T_1 + c_2T_2$ is intuitinistic fuzzy 2-continuous. □

Definition 3.11. A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ is said to be strongly intuitionistic fuzzy 2-continuous at a point $x \in V$ if for any given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$,

$$\mu^{'}(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \ \ and \ \ \nu^{'}(Tx - Tx_0, z, \epsilon) < \nu(x - x_0, z, \delta).$$

A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V^{'}, \mu^{'}, \nu^{'}, *, \diamond)$ is strongly $IF - 2 - C$, if T is strongly $IF - 2 - C$ at each point of V.

Theorem 3.12. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V^{'}, \mu^{'}, \nu^{'}, *, \diamond)$ is strongly $IF - 2 - C$, then it is $IF - 2 - C$, but converse is not true.*

Proof. Let a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ be strongly $IF-2-C$. Let $x_0 \in V$ if for any given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $x \in V$,

$$\mu^{'}(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \text{ and } \nu^{'}(Tx - Tx_0, z, \epsilon) < \nu(x - x_0, z, \delta).$$

Let $\{x_n\}$ be a sequence in V such that $\{x_n\} \to x_0$ for all $t > 0$ then

$$\lim_{n\to\infty} \mu(x - x_0, z, t) = 1 \text{ and } \lim_{n\to\infty} \nu(x - x_0, z, t) = 0.$$

Thus we see that

$$\mu^{'}(Tx_n - Tx_0, z, \epsilon) \geq \mu(x_n - x_0, z, \delta) \text{ and } \nu^{'}(Tx_n - Tx_0, z, \epsilon) < \nu(x_n - x_0, z, \delta).$$

which implies that

$$\lim_{n\to\infty} \mu^{'}(Tx_n - Tx_0, z, \epsilon) = 1 \text{ and } \lim_{n\to\infty} \nu^{'}(Tx_n - Tx_0, z, \epsilon) = 0$$

which gives $Tx_n \to Tx_0$ in $(V, \mu^{'}, \nu^{'}, *, \diamond)$. Hence, T is $IF - 2 - C$. Conversely, we provide example, which is $IF - 2 - C$ but not strongly $IF - 2 - C$. □

Example 3.13. Let $(V = \mathbb{R}, ||\cdot, \cdot||)$ be 2-normed space over $\mathbb{F}$. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$. Let $\mu, \nu, \mu^{'}, \nu^{'}$ are fuzzy sets on $V \times V \times (0, \infty)$ defined by $\mu(x, z, t) = \dfrac{t}{t + ||x, z||}$, $\nu(x, z, t) = \dfrac{\|x, z\|}{t + \|x, z\|}$ and
$\mu^{'}(x, z, t) = \dfrac{t}{t + k\|x, z\|}$,
$\nu^{'}(x, z, t) = \dfrac{k\|x, z\|}{t + k\|x, z\|}$, for all $t \in \mathbb{R}^+$ and $k > 0$.
In short,
$$\mu(Y, t) = \frac{t}{t + \|Y\|}, \nu(Y, t) = \frac{\|Y\|}{t + \|Y\|} \quad \text{and } \mu^{'}(Y, t) = \frac{t}{t + k\|Y\|}, \nu^{'}(Y, t) = \frac{k\|Y\|}{t + k\|Y\|}.$$

Let us now define, $T(Y) = \frac{Y^4}{1+Y^2}$ for all $Y \in V$. Let $Y_0 \in V$ and $\{Y_k\}$ be a sequence in V such that $\{Y_k\} \to Y_0$ in $(V, \mu, \nu, *, \diamond)$,

i.e. for all $t > 0, \lim\limits_{k\to\infty} \mu(Y_k - Y_0, t) = 1$ and $\lim\limits_{k\to\infty} \mu(Y_k - Y_0, t) = 0$,

$$\Rightarrow \lim_{k\to\infty} \frac{\|t\|}{t + \|Y_n - Y_0\|} = 1 \text{ and } \lim_{k\to\infty} \frac{\|Y_n - Y_0\|}{t + \|Y_n - Y_0\|} = 0$$
$\Rightarrow \lim_{k\to\infty}\{\|Y_k - Y_0\|\} = 0.$

Now for all $t > 0$,

$$\mu^{'}(TY_n - TY_0, t) = \frac{t}{t + k\|TY_n - TY_0\|} = \frac{t}{t + k\|\frac{Y_n^4}{1+Y_n^2} - \frac{Y_0^4}{1+Y_0^2}\|}$$

$$\Rightarrow \lim_{n\to\infty} \mu^{'}(TY_n - TY_0, t) = 1.$$

Similarly, we get, $\lim_{n\to\infty} \nu^{'}(TY_n - TY_0, t) = 0$. Thus T is $IF - 2 - C$.

Let $\epsilon > 0$ be given. Then

$$\mu^{'}(TY - TY_0, \epsilon) \geq \mu(Y - Y_0, \delta)$$
$$\Rightarrow \frac{\epsilon\|1 + Y^2\|\|1 + Y_0^2\|}{\epsilon\|1 + Y^2\|\|1 + Y_0^2\| + k\|Y - Y_0\|\|(Y + Y_0)(Y^2 + Y_0^2) + Y^2Y_0^2(Y + Y_0)\|}$$
$$\geq \frac{\delta}{\delta + \|Y - Y_0\|}$$

and

$$\nu^{'}(TY - TY_0, \epsilon) \leq \nu(Y - Y_0, \delta)$$
$$\Rightarrow \frac{k\|Y - Y_0\|\|(Y + Y_0)(Y^2 + Y_0^2) + Y^2Y_0^2(Y + Y_0)\|}{\epsilon + \|1 + Y^2\|\|1 + Y_0^2 + k\|\|Y - Y_0\|\|(Y + Y_0)(Y^2 + Y_0^2) + Y^2Y_0^2(Y + Y_0)\|}$$
$$\leq \frac{\delta}{\delta + \|Y - Y_0\|}$$

So,

$$k\delta\|Y - Y_0\|\|Y + Y_0\|\|Y^2 + Y_0^2 + Y^2Y_0^2\| \leq \epsilon\|1 + Y^2\|\|1 + Y_0^2\|\|Y - Y_0\| \tag{3.11}$$

$$\Rightarrow \delta \leq \frac{\epsilon\|1 + Y^2\|\|1 + Y_0^2\|}{k\|Y - Y_0\|\|Y^2 + Y_0^2 + Y^2Y_0^2\|} (\text{for} \quad Y \neq Y_0). \tag{3.12}$$

We see that T is IF-2-C at Y_0 if there exists $\delta > 0$ satisfying (3.11) for all $Y \neq Y_0$. Let $\delta_1 = \inf \frac{\|t + Y^2\|\|t + Y_0^2\|}{\|Y - Y_0\|\|Y^2 + Y_0^2 + Y^2Y_0^2\|}$ where the infimum is taken over all Y, where, $Y \neq Y_0$. Then $\delta = \frac{\epsilon}{k}\delta_1$ satisfies (3.11). But $\delta_1 = 0$ which is impossible. Hence, T is not strongly IF-2-C.

Theorem 3.14. *$(V, \mu, \nu, *, \diamond)$,$(V, \mu^{'}, \nu^{'}, *, \diamond)$ are IF-2-NS and $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ be a linear operators, then the following conditions are equivalent.*

(a) T is intuitionistic fuzzy 2-bounded (IF-2-B).

(b) If there exist constants $h, k \in \mathbb{R} - \{0\}$ such that,$\mu^{'}(Tx, z, t) \geq \mu(hx, z, t)$ and $\nu^{'}(Tx, z, t) \leq \nu(kx, z, t)$ for every x, z(nonzero) $\in V$ and for every $t > 0$.

(c) T is intuitionistic fuzzy 2-continuous at some point $x_0 \in V$.

(d) T is intuitionistic fuzzy 2-continuous (IF-2-C).

Proof. $(a) \Leftrightarrow (b)$ Obviously, result holds by definition (3.5).

$(c) \Leftrightarrow (d)$ Suppose, T is intuitionistic fuzzy 2-continuous at some point $x_0 \in V$. Let $\{x_n\} \to^{(\mu,\nu)_2} x$ as $n \to \infty$ in $(V, \mu, \nu, *, \diamond)$. By theorems (3.1) and (3.2), we see that,

$$(x_n - x) + x_0 \to^{(\mu,\nu)_2} x_0 \quad as \quad n \to \infty.$$

$$\therefore \quad T((x_n - x) + x_0) \to^{(\mu',\nu')_2} Tx_0 \quad as \quad n \to \infty.$$

Since T is a linear, we obtain that

$$(Tx_n - Tx + Tx_0) \to^{(\mu',\nu')_2} Tx_0 \quad as \quad n \to \infty$$

implies that $Tx_n \to^{(\mu',\nu')_2} Tx_0$. By definition (3.7), we conclude that, T is IF-2-C. Obviously, converse hold.
$(a) \Leftrightarrow (d)$ It follows from [10] and converse holds by definitions (3.5), (3.8). □

Definition 3.15. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy-2-normed space. A subset D of V is said to compact if any sequence in D has a subsequence converging to an element of D.

Theorem 3.16. *Let $T : (V_1, \mu_1, \nu_1, *, \diamond) \to (V_2, \mu_2, \nu_2, *, \diamond)$ be a mapping and D be a compact subset of V_1. If T is a IF-2-C on V_1 then $T(D)$ is a compact subset of V_2.*

Proof. Let y_n be a sequence in $T(D)$ then for each n there exist $x_n \in D$ such that $T(x_n) = y_n$. Since D is a compact there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ and $x_0 \in D$ such that $\{x_{n_k}\} \to^{(\mu_1,\nu_1)_2} x_0$ in $(V_1, \mu_1, \nu_1, *, \diamond)$. Since T is an intuitionistic fuzzy-2-continuous at x_0. By definition (3.7)

$$\{x_{n_k}\} \to^{(\mu_1,\nu_1)_2} x_0 \quad \Rightarrow \quad T\{x_{n_k}\} \to^{(\mu_2,\nu_2)_2} T(x_0) \quad \Rightarrow \quad \{y_{n_k}\} \to^{(\mu_2,\nu_2)_2} y_0,$$

for some $y_0 \in T(D)$ such that $T(x_0) = y_0$ implies that $T(D)$ is compact subset of V_2. □

4. Intuitionistic fuzzy ψ-2-normed space

Definition 4.1. The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy ψ-2-normed space, if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times V \times (0, \infty)$ satisfying the following conditions. For every $x, y, z \in V$ and $s, t > 0$,

- (a) $\mu(x, y, t) + \nu(x, y, t) \leq 1$;
- (b) $\mu(x, y, t) > 0$;
- (c) $\mu(x, y, t) = 1$ if and only if x and y are linearly dependent;
- (d) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
- (e) $\mu(x, y, t) * \mu(x, z, s) \leq \mu(x, y + z, t + s)$;
- (f) $\mu(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
- (g) $\lim_{t\to\infty} \mu(x, y, t) = 1$ and $\lim_{t\to 0} \mu(x, y, t) = 0$;
- (h) $\mu(x, y, t) = \mu(y, x, t)$;
- (i) $\nu(x, y, t) < 1$;
- (j) $\nu(x, y, t) = 0$ if and only if x and y are linearly dependent;
- (k) $\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{\psi(\alpha)})$ for each $\alpha \neq 0$;
- (l) $\nu(x, y, t) \diamond \nu(x, z, s) \geq \nu(x, y + z, t + s)$;
- (m) $\nu(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
- (n) $\lim_{t\to\infty} \nu(x, y, t) = 0$ and $\lim_{t\to 0} \nu(x, y, t) = 1$;
- (o) $\nu(x, y, t) = \nu(y, x, t)$.

In this case $(\mu, \nu)_2$ is called an intuitionistic fuzzy ψ-2-norm on V.

Definition 4.2. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A sequence $\{x_n\}$ is said to be convergent to $x \in V$ with respect to the intuitionistic fuzzy ψ-2-norm $(\mu, \nu)_2$, if for every $r > 0$ and $t > 0$, $r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$ for all $n \geq n_0$ and for all $z \in V$.

Definition 4.3. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A sequence $\{x_n\}$ in V is said to be Cauchy if for each $r > 0$ and each $t > 0$,$r \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, z, t) > 1 - r$ and $\nu(x_n - x_m, z, t) < r$ for all $n, m \geq n_0$ and for all $z \in V$.

Definition 4.4. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space and let $r \in (0, 1), t > 0$ and $x \in X$.The set $B(x, r, t) = \{y \in V : \mu(y - x, z, t) > 1 - r, \ \ \nu(y - x, z, t) < r, \forall z \in V\}$ is called the open ball with center x and radius r with respect to t.

Definition 4.5. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-2-normed space. A set $U \subset V$ is said to an open set if each of its points is the centre of some open ball contained in U. The open set in an intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ is denoted by $\mathbb{U}$.

Theorem 4.6. *In intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$. A sequence $\{x_n\}$ converges to x if and only if $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$.*

Proof. Fix $t > 0$, Suppose $\{x_n\}$ converges to x in IF ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ then for a given r, $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$. Thus $1 - \mu(x_n - x, z, t) > r$ and $\nu(x_n - x, z, t) < r$, hence, $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$.
Conversely, if for each $t > 0$, $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$ then for every $r \in (0, 1)$, there exists an integer n_0 such that $1 - \mu(x_n - x, z, t) > r$ and $\nu(x_n - x, z, t) < r$, $\forall n \geq n_0$. Hence, $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$. Thus, $\{x_n\}$ converges to x in IF ψ-2-normed space $(V, \mu, \nu, *, \diamond)$. $\square$

Theorem 4.7. *The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$.*

Proof. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$.

$$\lim_{n\to\infty} x_n = x = \begin{cases} \lim_{n\to\infty} \mu(x_n - x, z, t) = 1, \\ \lim_{n\to\infty} \nu(x_n - x, z, t) = 0. \end{cases}$$

$$\lim_{n\to\infty} x_n = y = \begin{cases} \lim_{n\to\infty} \mu(x_n - y, z, t) = 1, \\ \lim_{n\to\infty} \nu(x_n - y, z, t) = 0. \end{cases}$$

$$\begin{aligned}\nu(x-y,z,s+t) &= \nu(x-x_n+x_n-y,z,s+t)\\ &\geq \nu(x-x_n,z,s)\diamond\nu(x_n-y,z,t)\\ &= \nu(x_n-x,z,\frac{s}{\psi(-1)})\diamond\nu(x_n-y,z,t)\\ &= \nu(x_n-x,z,\frac{s}{\psi(1)})\diamond\nu(x_n-y,z,t)\\ &= \nu(x_n-x,z,s)\diamond\nu(x_n-y,z,t).\end{aligned}$$

As $n\to\infty$ we have, $\nu(x-y,z,s+t)=0\Rightarrow x=y$. Thus, The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy ψ-2-normed space $(V,\mu,\nu,*,\diamond)$. □

Theorem 4.8. *In IF ψ-2-NS $(V,\mu,\nu,*,\diamond)$. Every convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence in IF ψ-2-NS $(V,\mu,\nu,*,\diamond)$ with $\lim_{n\to\infty}x_n=x$. Let $r\in(0,1)$, $t,s>0$ then there exist an integer $n_0\in N$ such that $\mu(x_n-x,z,s)>1-r$ and $\nu(x_n-x,z,s)<r$. For $n,p\in\mathbb{N}$

$$\begin{aligned}\mu(x_{n+p}-x_n,z,s+t) &= \mu(x_{n+p}-x+x-x_n,z,s+t)\\ &\geq \mu(x_{n+p}-x,z,s)*\mu(x-x_n,z,t)\\ &= \mu(x_{n+p}-x,z,s)*\mu(x_n-x,z,\frac{t}{\psi(-1)})\\ &= \mu(x_{n+p}-x,z,s)*\mu(x_n-x,z,t)\\ &> (1-r)*(1-r)\\ &= (1-r),\forall n\geq n_0.\end{aligned}$$

Similarly,

$$\begin{aligned}\nu(x_{n+p}-x_n,z,s+t) &= \nu(x_{n+p}-x+x-x_n,z,s+t)\\ &\leq \nu(x_{n+p}-x,z,s)\diamond\nu(x-x_n,z,t)\\ &= \nu(x_{n+p}-x,z,s)\diamond\nu(x_n-x,z,\frac{t}{\psi(-1)})\\ &= \nu(x_{n+p}-x,z,s)\diamond\nu(x_n-x,z,t)\\ &< r\diamond r\\ &= r,\forall n\geq n_0.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in IF ψ-2-NS $(V,\mu,\nu,*,\diamond)$. □

Theorem 4.9. *In IF ψ-2-NS $(V,\mu,\nu,*,\diamond)$. A sequence $\{x_n\}$ is a Cauchy sequence if and only if $\mu(x_{n+p}-x,z,t)\to 1$ and $\nu(x_{n+p}-x,z,t)\to 0$ as $n\to\infty$.*

Proof. Fix $t>0$, Suppose $\{x_n\}$ is a Cauchy sequence in IF ψ-2-normed space $(V,\mu,\nu,*,\diamond)$ then for a given $r\in(0,1)$ there exists an integer $n_0\in N$ such that $\mu(x_{n+p}-x_n,z,t)>1-r$ and $\nu(x_{n+p}-x_n,z,t)<r$. Thus $1-\mu(x_{n+p}-x_n,z,t)>r$ and $\nu(x_{n+p}-x_n,z,t)<r$, hence, $\mu(x_{n+p}-x_n,z,t)\to 1$ and $\nu(x_{n+p}-x_n,z,t)\to 0$ as $n\to\infty$.

Conversely, if for each $t>0$, $\mu(x_{n+p}-x_n,z,t)\to 1$ and $\nu(x_{n+p}-x_n,z,t)\to 0$ as $n\to\infty$ then for every r, $r\in(0,1)$, there exists an integer n_0 such that $1-\mu(x_{n+p}-$

$x_n, z, t) > r$ and $\nu(x_{n+p} - x_n, z, t) < r$, $\forall n \geq n_0$. Hence, $\mu(x_{n+p} - x_n, z, t) > 1 - r$ and $\nu(x_{n+p} - x_n, z, t) < r$. Thus, $\{x_n\}$ is a Cauchy sequence in IF ψ-2-normed space $(V, \mu, \nu, *, \diamond)$. □

Definition 4.10. An intuitionistic fuzzy ψ-2-normed space $(V, \mu, \nu, *, \diamond)$ is said to be complete if every Cauchy sequence in IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$ is convergent.

Theorem 4.11. *Let $(V, \mu, \nu, *, \diamond)$ be a IF ψ-2-NS. A sufficient condition for the IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$ to be complete is that every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ has a convergent subsequence.*

Proof. Let $\{x_n\}_n$ be a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ and $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$ that converges to $x \in V$ and $s, t, s + t > 0$, Since $\{x_n\}_n$ is a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$, We have for $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that $\mu(x_n - x_k, z, s) > 1 - r$ and $\nu(x_n - x_k, z, s) < r$, $\forall n, k \geq n_0$. Again, since $\{x_{n_k}\}$ converges to x. We have $\mu(x_{n_k} - x, z, t) > 1 - r$ and $\nu(x_{n_k} - x, z, t) < r$, $\forall n, k \geq n_0$

$$\begin{aligned}
\mu(x_n - x, z, s + t) &= \mu(x_n - x_{n_k} + x_{n_k}, z, s + t) \\
&\geq \mu(x_n - x_{n_k}, z, s) * \mu(x_{n_k} - x, z, t) \\
&> (1 - r) * (1 - r) \\
&= (1 - r), \forall n \geq n_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu(x_n - x, z, s + t) &= \nu(x_n - x_{n_k} + x_{n_k}, z, s + t) \\
&\leq \mu(x_n - x_{n_k}, z, s) \diamond \mu(x_{n_k} - x, z, t) \\
&< r \diamond r \\
&= r, \forall n \geq n_0.
\end{aligned}$$

Thus $\{x_n\}_n$ converges to x in $(V, \mu, \nu, *, \diamond)$. Hence IF ψ-2-NS $(V, \mu, \nu, *, \diamond)$ is complete. □

Remark 4.12. Straightforwardly, we get the results (3.1), (3.2), (3.3), (3.4), (3.6), (3.10), (3.12), (3.14), (3.16) are also holds in INFψ-2-NS.

Theorem 4.13. *Every intuitionistic fuzzy ψ-2-normed space is intuitionistic fuzzy 2-normed space, converse is not true.*

Proof. Let $(V, \mu, \nu, *, \diamond)$ be a IF ψ-2-NS. By definition (2.14), take $\psi(\alpha) = |\alpha|$ then definition (2.8) implies $(V, \mu, \nu, *, \diamond)$ be a IF 2-NS. Conversely, let $(V, \mu, \nu, *, \diamond)$ be a IF 2-NS. If $\psi(\alpha) \neq |\alpha|$ then definitions (4.1 and 2.14) implies $(V, \mu, \nu, *, \diamond)$ is not a IF ψ-2-NS. □

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