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On connectedness in ditopological texture spaces

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Abstract. The notion of a texture space, under the name of fuzzy structure, was introduced by L. M. Brown in [3], as a means of representing a lattice of fuzzy sets as a lattice of crisp subsets of some base set. The notion of connectedness in ditopological texture spaces was initiated by Diker in [10]. In this paper some additional properties related this notion have obtained. Also, we introduce the notion of ditopological texture subspaces and study some of their properties. Some new types of connectedness in ditopological texture spaces namely, locally connectedness, totally disconnectedness and extremely disconnectedness have investigated. Therefore, we show that the sum of ditopological texture spaces is totally disconnected if and only if each ditopological texture space is totally disconnected.

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1. Introduction

The notion of a texture space, under the name of fuzzy structure, was introduced by L. M. Brown in [3]. The motivation for the study of texture spaces is that they allow us to represent, for instance, classical fuzzy sets, L-fuzzy sets [11], intuitionistic fuzzy sets [1] and intuitionistic sets [9], as lattices of crisp subsets of some base set S. A detailed analysis of this relation between texture spaces and lattices of fuzzy sets of various kinds may be found in [5, 7, 9]. The concept of a ditopology on a texture space is introduced in [4]. Under the relation mentioned above a ditopology corresponds in a natural way to a fuzzy topology, but in general ditopological texture spaces may be regarded as natural generalizations of both topological spaces and bitopological spaces [12]. Indeed bitopological concepts underlay the definition of compactness, co-compactness, stability and co-stability given in [2], and also of dicover compactness, co-compactness and bicompactness [9]. The notion of connectedness in ditopological texture spaces was introduced in [10].

2. Preliminaries

This section contains the notions which are needed in the sequel. For more details see [3, 4, 10, 12, 13, 14].

Definition 2.1 ([3]). Let X be a set. Then $L \subseteq P(X)$ is called texturing of X and X is said to be textured by L if L separates the points of X, i.e $\forall\, x_1, x_2 \in X$, $x_1 \neq x_2$, $\exists\, A \in L$ such that $x_1 \in A$ and $x_2 \notin A$, complete, completely distributive lattice with respect to inclusion, which contains X, ϕ, and for which arbitrary meet coincides with intersection and finite joins coincide with unions. The pair (X,L) is then known as a texture.
The internal definition of textural concepts are expressed using the p-sets and q-sets. That is, for each $x \in X$ the sets $p_x = \bigcap\{A \in L : x \in A\}$ and $q_x = \bigvee\{A \in L : x \notin A\}$.
A surjection $\sigma : L \to L$ is called a complementation if $\sigma^2(A) = A\ \forall A \in L$ and $A \subseteq B$ in L implies $\sigma(B) \subseteq \sigma(A)$. A texture with a complementation is said to be complemented.
We now recall the definition of a dichotomous topology (or ditopology for short) on a texture given in [3].

Example 2.2 ([3]). (1) For any set X, $(X, P(X))$ is the discrete texture space.
(2) Let $X = (0, 1]$ and $L = \{(0, r] : r \in X\}$. Then (X, L) is a texture space.
(3) Let $X = [0, 1]$ and $L = \{[0, r) : r \in X\} \cup \{[0, r] : r \in X\}$. Then (X, L) is a texture space.

Definition 2.3 ([4]). (L, τ, K) is called a ditopological texture space on X if
(1) $\tau \subseteq L$ satisfies
  (a) $X,\ \phi \in \tau$,
  (b) $G_1,\ G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and
  (c) $G_i \in \tau$, $i \in I \Rightarrow \bigvee_{i\in I} G_i \in \tau$, and
(2) $K \subseteq L$ satisfies
  (a) $X,\ \phi \in K$,
  (b) $F_1,\ F_2 \in K \Rightarrow F_1 \cup F_2 \in K$, and
  (c) $F_i \in K$, $i \in I \Rightarrow \bigwedge_{i\in I} F_i \in K$.

The elements of τ are called open and those of K are called closed. We refer to τ as the topology and to K as the cotopology of (τ,K). In general there is no a priori relation between τ and K , but if σ is a complementation on (X, L), and τ, k are related by the relation $K = \sigma(\tau)$, then we call (τ, K) a complemented ditopology on (X, L, σ).

Finally, let $Z \subseteq X$. Then the closure of Z is the set $[Z] = \bigcap\{F \in K : Z \subseteq F\}$, the interior of Z is $]Z[= \bigvee\{G \in \tau : G \subseteq Z\}$, the exterior of Z is $ext(Z) = \bigvee\{G \in \tau : G \cap Z = \phi\}$ and Z is called dense in X if $[Z] = X$. Also, if $A \nsubseteq F\ \forall F \in K - \{X\}$, we say A is co-dense.

Example 2.4 ([6]). (1) For any texture (X, L), a ditopology (τ, K) with $\tau = L$ is called discrete, and one with $K = L$ is called co-discrete.

(2) For any texture (X, L), a ditopology (τ, K) with $\tau = \{X, \phi\}$ is called indiscrete, and one with $K = \{X, \phi\}$ is called co-indiscrete.

(3) For any topology τ on X, (τ, τ'), $\tau' = \{X - G : G \in \tau\}$, is a complemented ditopology on the usual(crisp) set structure $(X, P(X), \sigma_X)$ of X, where $\sigma_X : P(X) \to P(X)$ defined by $\sigma_X(A) = A'$ where $A' = X - A \; \forall A \in P(X)$.

(4) For any bitopological space (X, τ_1, τ_2), (τ_1, τ_2') is a ditopology on$(X, P(X))$.

(5) Let $X = [0, 1]$, $L = \{[0, r) : r \in X\} \cup \{[0, r] : r \in X\}$, $\tau = \{[0, r) : r \in X\} \cup \{X\}$, $K = \{[0, r] : r \in X\} \cup \{\phi\}$ and let σ be a complementation $\sigma([0, r) = [0, 1-r]$, $\sigma([0, r] = [0, 1-r)$, then we have (X, L, τ, K, σ) is a complemented ditopological texture space.

Definition 2.5 ([6])**.** (1) Let (τ, K) be a ditopology on (X, L). Then $\sigma \subseteq \tau$ is a subbase of τ if every element of τ is a supremum of finite intersections of sets of σ, while a subset σ of $K$is a subbase of K if every element of K is the intersection of finite unions of sets of σ.

(2) A subset β of τ is called a base of τ if every set in τ can be written as a join of sets in β, while a subset β of K is a base of K if every set in K can be written as an intersection of sets in β.
Hence a subbase of τ is a subset of τ, the set of finite intersections of which is a base of τ, while a subbase of $K$is a subset of K, the set of finite unions of which is a base of K. In the case of a complemented ditopology, the complementation will clearly carry a base (subbase) of τ into a base (subbase) of K, and conversely.

Definition 2.6 ([8])**.** Let (X, L) be a texture space, $Y \in L$ and $L_Y = \{Y \cap A : A \in L\}$. Then (Y, L_Y) is called the induced texture on Y or the principal texture of (X, L) on Y.

Definition 2.7 ([8])**.** Suppose that we are given a family X_α, $\alpha \in \Lambda$ of pairwise disjoint sets, i.e $X_\alpha \cap X_{\alpha'} = \phi$ for $\alpha \neq \alpha'$, let (X_α, L_α), $\alpha \in \Lambda$, be textures. Consider the set $X = \bigcup_{\alpha \in \Lambda} X_\alpha$ and the family $L = \{A \subseteq X : A \cap X_\alpha \in L_\alpha, \alpha \in \Lambda\}$ which defines a texturing of X, then the pair (X, L) is called the sum of textures (X_α, L_α), $\alpha \in \Lambda$.

Definition 2.8 ([10])**.** Let$(X$,$L)$ be a texture space and $\phi \neq Z \subseteq X$. $\{A, B\} \subseteq P(X)$ is said to be a partition of Z if $A \cap Z \neq \phi$, $Z \nsubseteq B$ and $A \cap Z = B \cap Z$. Here we may interchange the roles of A and B. Indeed, if {A,B} is a partition of Z, then we also have $B \cap Z \neq \phi$ and $Z \nsubseteq A$.

Definition 2.9 ([10])**.** Let (L, τ, K) be a ditopological texture space on X and $Z \subseteq X$. Z is said to be connected if there exists no partition $\{G, F\}$ of Z with $G \in \tau$ and $F \in K$.

Theorem 2.10 ([10])**.** *Let (X, L, τ, K) be a ditopological texture space, then X is connected if and only if $\tau \cap K = \{X, \phi\}$.*

Theorem 2.11 ([10])**.** *Let $\{Z_i : i \in I\}$ be a family of connected subsets in L with $\bigcap_{i \in I} Z_i \neq \phi$, then $\bigvee_{i \in I} Z_i$ is also connected.*

Theorem 2.12 ([10])**.** *Let $Z \subseteq X$ be a connected set, $Z \subseteq A \subseteq [Z]$ and $ext(Z) \cap A = \phi$. Then A is also connected.*

3. More properties on connectedness

In this section we introduce some new additional properties of connectedness in the ditopological texture spaces.

Definition 3.1. Nonempty subsets $A,\ B$ of a ditopological texture space (X, L, τ, K) are said to be separated sets if $A\cap[B]^\tau = B\cap[A]^K = \phi$. Here we may interchange the roles of A and B. Indeed, if $A,\ B$ are separated sets, then $B \cap [A]^\tau = A \cap [B]^K = \phi$ (where $[A]^\tau$ is the closure of A w.r t τ and $[B]^K$ is the closure of B w.r t K).

Theorem 3.2. *Let $A \subseteq C,\ B \subseteq D$ and $C,\ D$ are proper separated subsets of a ditopological texture space (X, L, τ, K). Then $A,\ B$ are separated sets.*

Proof. Since $A \subseteq C$, then $[A]^K \subseteq [C]^K$, hence $B\cap[A]^K \subseteq D\cap[A]^K \subseteq D\cap[C]^K = \phi$. Then $[A]^K \cap B = \phi$. By a similar way we obtain $[B]^\tau \cap A = \phi$, then $A,\ B$ are separated sets. □

Theorem 3.3. *Let A and B be separated sets s.t $A \cup B \in \tau \cap K$, then $A(resp.B) \in \tau \cap K$.*

Proof. Suppose that $A,\ B$ are separated sets s.t $A \cup B \in \tau \cap K$, then $[A \cup B]^\tau \in \tau'$. Since $[B]^\tau \in \tau'$, then $([B]^\tau)' \in \tau$, hence $(A\cup B)\cap([B]^\tau)' \in \tau$. Then $(A\cap([B]^\tau)')\cup(B\cap ([B]^\tau)' \in \tau$, hence $A \in \tau$. Since $A \cup B \in K$ and $[A]^K \in K$, then $(A\cup B)\cap[A]^K \in K$. Then $(A \cap [A]^K) \cup (B \cap [A]^K) \in K$, hence $A \in K$, so $A \in \tau \cap K$. By a similar way we can obtain $B \in \tau \cap K$. □

Definition 3.4. A set which is τ-open as well as K-closed is said to be clopen.

Theorem 3.5. *Let (X, L, τ, K) be a ditopological texture space, then the following are equivalent:*

(1) *X is connected.*
(2) *X has no a partition $\{A, B\} \subseteq P(X)$ with $A \in \tau$ and $B \in K$.*
(3) *There is no proper subset A of X which is clopen.*
(4) *X can not be expressed as a union of two nonempty disjoint subsets $A,\ B$ of X with $A \in \tau$ and $B \in K'$.*
(5) *X can not be expressed as a union of two nonempty disjoint subsets $A,\ B$ of X with $A \in \tau'$ and $B \in K$.*
(6) *X can not be expressed as a union of two separated subsets $A,\ B$ of X.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Immediate from Theorem 2.10.

(3) $\Rightarrow$ (4) Suppose that $X = A \cup B$ for some $A \in \tau$ and $B \in K'$ s.t $A \cap B = \phi$. Then $A = B' \in \tau \cap K$, which is a contradiction with (3).

(4) $\Rightarrow$ (5) Suppose that $X = A \cup B$ for some $A \in \tau'$ and $B \in K$ s.t $A \cap B = \phi$. Then $A' \in \tau$, $B' \in K'$, $A = B'$, $A'\cap B' = \phi$ and $X = A'\cup B'$, which is a contradiction with (4).

(5) $\Rightarrow$ (6) Suppose that $X = A \cup B$ for some two separated subsets $A,\ B$ of X, then $X = A \cup B$, $A \cap B = \phi$, $A = B'$ and $B = A'$. Since $A \cap [B]^\tau = \phi$, then $[B]^\tau \subseteq A' = B$, but $B \subseteq [B]^\tau$, then $B = [B]^\tau$, i.e $B \in \tau'$. By a similar way we obtain $A \in K$. Hence X is a union of two nonempty disjoint subsets $A,\ B$ of X with $B \in \tau'$ and $A \in K$, which is a contradiction with (5).

(6) $\Rightarrow$ (1) Suppose that X is disconnected. Then $\exists$ $\{A, B\} \subseteq P(X)$ with $(A, B) \in \tau \times K$, which is a partition of X, hence $A = B \in \tau \cap K$, $A' \in \tau'$, $A \cap [A']^{\tau} = \phi$, $[A]^K \cap A' = \phi$ and $A \cup A' = X$. Hence X is a union of two separated subsets A, A' of X, which is a contradiction with (6). □

Theorem 3.6. *Let Z be a subset of a ditopological texture space (X, L, τ, K). Then Z is disconnected if and only if $\exists$ two nonempty disjoint τ-open and K-open or (τ-closed and K-closed) proper subsets A, B of X s.t $Z \cap A \neq \phi$, $Z \cap B \neq \phi$, $(Z \cap A) \cap (Z \cap B) = \phi$ and $(Z \cap A) \cup (Z \cap B) = Z$. In this case we say that $A \cup B$ forms a disconnection of Z.*

Proof. Immediate. □

Corollary 3.7. *The union of two nonempty separated subsets of a ditopological texture space (X, L, τ, K) is disconnected.*

Proof. Immediate by Theorem 3.6. □

Proposition 3.8. *$A \cup B$ is a disconnection of a subset Z in a ditopological texture space (X, L, τ, K) if and only if A is τ-open and B is K-open or (Ais τ-closed and B is K-closed) proper subsets of X s.t $Z \cap A \neq \phi$, $Z \cap B \neq \phi$, $A \cap B \subseteq Z'$ and $Z \subseteq A \cup B$.*

Proof. Suppose that $A \cup B$ is a disconnection of a subset Z in a ditopological texture space $(X, L, \tau, K) \Leftrightarrow$ A is τ-open and B is K-open proper subsets of X s.t $Z \cap A \neq \phi$, $Z \cap B \neq \phi$, $(Z \cap A) \cap (Z \cap B) = \phi$ and $(Z \cap A) \cup (Z \cap B) = Z$ by Theorem 3.6 $\Leftrightarrow$ A is τ-open and B is K-open proper subsets of X s.t $Z \cap A \neq \phi$, $Z \cap B \neq \phi$, $Z \cap (A \cap B) = \phi$ and $Z \cap (A \cup B) = Z$ $\Leftrightarrow$ A is τ-open and B is K-open proper subsets of X s.t $Z \cap A \neq \phi$, $Z \cap B \neq \phi$, $A \cap B \subseteq Z'$ and $Z \subseteq A \cup B$. □

Theorem 3.9. *Let Z be a clopen connected subset of a ditopological texture space (X, L, τ, K), which has a nonempty intersection with both a τ-open set A and its complement A'. Then Z has a non empty intersection with the boundary of A, where $b(A) = [A]^K -]A[^{\tau}$.*

Proof. Suppose that $Z \cap b(A) = \phi$. Since $Z = X \cap Z$ and $X = A \cup A'$, then $Z = (A \cup A') \cap Z = (A \cap Z) \cup (A' \cap Z)$ and $(A \cap Z) \cap [A' \cap Z]^{\tau} \subseteq (A \cap Z) \cap ([A']^{\tau} \cap [Z]^{\tau}) = (A \cap Z) \cap [A']^{\tau} \subseteq ([A]^K \cap Z) \cap [A']^{\tau} = ([A]^K \cap [A']^{\tau}) \cap Z = b(A) \cap Z = \phi$. By a similar way $(A' \cap Z) \cap [A \cap Z]^K = \phi$, then A, A' is a separation of Z, which is a contradiction with the connectedness of Z. □

Corollary 3.10. *Let A be a clopen proper subset of a connected ditopological texture space (X, L, τ, K), then $b(A) \neq \phi$*

Proof. Immediate from Theorem 3.9. □

Theorem 3.11. *Let $\{Z_i : i \in I\}$ be a family of connected subsets of a ditopological texture space (X, L, τ, K) s.t one of the members of the family intersects every other member. Then $Z = \bigvee_{i \in I} Z_i$ is connected.*

Proof. Let $Z_{i0} \in \{Z_i : i \in I\}$, such that $Z_{i0} \cap Z_i \neq \phi$ $\forall i \in I$. Then $Z_{i0} \bigvee Z_i$ is connected $\forall i \in I$ by Theorem 2.11. Also if $Z_{i1}, Z_{i2} \in \{Z_i : i \in I\}$, then we have $Z_{i0} \cap Z_{i1} \neq \phi$, $Z_{i0} \cap Z_{i2} \neq \phi$ and $(Z_{i0} \vee Z_{i1}) \cap (Z_{i0} \bigvee Z_{i2}) = Z_{i0} \bigvee (Z_{i1} \cap Z_{i2})$, $Z_{i0} \neq \phi$. Then $(Z_{i0} \bigvee Z_{i1}) \cap (Z_{i0} \bigvee Z_{i2}) \neq \phi$ $\forall i_1 \neq i_2$, hence the collection $\{Z_{i0} \vee Z_i : i \in I\}$ is a collection of a connected subsets of X, which having a non-empty intersection. So $Z = \bigvee_{i \in I} Z_i$ is connected by Theorem 2.11. □

Definition 3.12. Let (X, L, τ, K) be a ditopological texture space,and let $Z \subseteq X$ with $x \in Z$. Then the component of Z w.r.t x is the maximal of all connected subsets of Z containing the point x and denoted by $C(Z, x)$, i.e

$$C(Z, x) = \bigvee\{Y \subseteq Z : x \in Y,\ Y \text{ is connected}\}.$$

Theorem 3.13. *Every component of a ditopological texture space (X, L, τ, K) is a maximal connected subset of X.*

Proof. Immediate from Definition 3.12. □

Theorem 3.14. *Let (X, L, τ, K) be a ditopological texture space. Then:*
(1) *Each point in X is contained in exactly one component of X.*
(2) *Any two components w.r.t two different points of X are either disjoint or identical.*

Proof. Immediate from Theorem 3.13. □

Corollary 3.15. *X is connected if and only if X is a component on X.*

Theorem 3.16. *Every clopen connected subset of a ditopological texture space (X, L, τ, K) is a component of X.*

Proof. Suppose that Z be a clopen connected subset of a ditopological texture space (X, L, τ, K). If $Z = \phi$, then we are done. If $Z \neq \phi$, let $x \in Z$, then Z is a connected set containing x, but the component $C(X, x)$ of X w.r.t x is the largest connected set containing x, hence $Z \subseteq C(X, x)$. Now we want show that $C(X, x) \subseteq Z$ i.e $C(X, x) \cap Z' = \phi$. Conversely, suppose that $C(X, x) \cap Z' \neq \phi$. Since $x \in Z \cap C(X, x)$, then $Z \cap C(X, x) \neq \phi$. Since $Z \in \tau \cap K$, then $Z' \in \tau'$, $Z \in K$, $(C(X, x) \cap Z) \cap (C(X, x) \cap Z') = \phi$ and $(C(X, x) \cap Z) \cup (C(X, x) \cap Z') = C(X, x)$, hence $Z \cup Z'$ is a disconnection of $C(X, x)$ by Theorem 3.6, which is a contradiction with the connectedness of $C(X, x)$. Hence $C(X, x) \subseteq Z$. This completes the proof. □

Theorem 3.17. *The set of all distinct components of a ditopological texture space (X, L, τ, K) partition the set X.*

Proof. Immediate from Theorem 3.14. □

Definition 3.18. A ditopological texture space (X, L, τ, K) is said to be locally connected at a point $x \in X$ if and only if every open subset of X containing x contains a connected open set containing x.If X is locally connected at each of its points, then it is said to be locally connected.

Theorem 3.19. *Every connected space is a locally connected space, but the converse is not true in general.*

Proof. Suppose that (X, L, τ, K) be a connected ditopological texture space, then $\tau \cap K = \{X, \phi\}$, hence $\forall\ x \in X\ \exists\ X \in \tau$ which is connected set, $x \in X \subseteq X$. Then X is locally connected. On the other hand, the ditopological texture space $(P(X), P(X))$ on X, which is containing more than one point, is locally connected and disconnected. □

Theorem 3.20. *The component of a locally connected ditopological texture space is an open set.*

Proof. Let (X, L, τ, K) be a locally connected ditopological texture space, $x \in X$ and C be a component of X w.r.t x. Since (X, L, τ, K) is a locally connected space, therefore every open set containing x contains a connected open set G containing x, but C is the largest connected set containing x, then $x \in G \subseteq C$, i.e C is a τ-nbd of x. Then C is a τ-nbd of each of its points, this means that C is an open set. □

Definition 3.21. A ditopological texture space (X, L, τ, K) is said to be totally disconnected if and only if $\forall x, y \in X$ s.t $x \neq y\ \exists$ non empty disjoint clopen proper subsets A, B of X s.t $x \in A$ and $y \in B$.

Theorem 3.22. *Every totally disconnected ditopological texture space is disconnected.*

Proof. Immediate. □

Remark 3.23. The converse of Theorem 3.22 is not true in general, for the following example,
let $X = \{a, b, c\}$, $L = \{X, \phi, \{a\}, \{a, c\}, \{a, b\}\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $K = \{X, \phi, , \{a\}, \{a, c\}\}$. Then (X, L, τ, K) is disconnected but not totally disconnected.

Theorem 3.24. *The components of a totally disconnected ditopological texture space* (X, L, τ, K) *are the singleton subsets of* X.

Proof. Suppose that Y be a subset of a totally disconnected ditopological texture space (X, L, τ, K), which containing more than one point of X. Let $y_1, y_2 \in Y \subseteq X$ s.t $y_1 \neq y_2$. Since X is totally disconnected, then $\exists$ a non empty disjoint clopen proper subsets A, B of X s.t $y_1 \in A$ and $y_2 \in B$. Clearly $\{A, A\}$ is a partition of Y, then Y is disconnected set, but the components are connected sets, hence no subset of X containing more than one point can be a component of X. □

4. Ditopological texture subspaces

In this section we introduce the notion of a ditopological texture subspaces and study some of its properties.

Definition 4.1. Let (X, L, τ, K) be a ditopological texture space, and (Y, L_Y) be the principal texture of (X, L) for $Y \in L$. Then (Y, L_Y, τ_Y, K_Y) is called a subspace of the ditopological texture space (X, L, τ, K), where $\tau_Y = \{Y \cap G : G \in \tau\}$ and $K_Y = \{Y \cap F : F \in K\}$.
Note that if $Y \in \tau$, then Y is said to be an open subspace and if $Y \in K$, then Y is said to be a closed subspace.

Theorem 4.2.
(1) *If σ is a subbase of τ (K), then $\sigma_Y = \{Y \cap A : A \in \sigma\}$ is a subbase of τ_Y (K_Y).*
(2) *If β is a base of τ (K), then $\beta_Y = \{Y \cap A : A \in \beta\}$ is a base of τ_Y (K_Y).*

Proof. (1) Suppose that $Z \in \sigma_Y$, then there exists $A \in \sigma \subseteq \tau$, such that $Z = Y \cap A \in \sigma_Y$, hence $Z \in \tau_Y$. Also if $A \in \tau_Y$, then $A = Y \cap G$ for some $G \in \tau$, in which $G = \bigvee(\bigcap_{i=1}^{n} S_i)$, $S_i \in \sigma$, hence $A = Y \cap (\bigvee(\bigcap_{i=1}^{n} S_i)) = \bigvee(Y \cap (\bigcap_{i=1}^{n} S_i)) = \bigvee(\bigcap_{i=1}^{n}(Y \cap S_i))$, $Y \cap S_i \in \sigma_Y$, where(X, L) is a texture space. Then every element of τ_Y is a supremum of finite intersections of sets of σ_Y.
(2) By a similar way. □

Theorem 4.3. *Let (Y, L_Y, τ_Y, K_Y) be a subspace of a ditopological texture space (X, L, τ, K), and $A \subseteq Y$. Then*
(1) *A is τ_Y-open if and only if $A = Y \cap G$, for some τ-open set G.*
(2) *A is K_Y-closed if and only if $A = Y \cap F$, for some K-closed set F.*

Proof. Immediate by Definition 4.1. □

Theorem 4.4. *Let (Y, L_Y, τ_Y, K_Y) be a subspace of a ditopological texture space (X, L, τ, K) and $A \subseteq Y$. Then*
(1) *$Cl_Y(A) = [A] \cap Y$.*
(2) *$]A[\subseteq Int_Y(A)$.*
(3) *$ext_Y(A) = Y \cap ext(A)$.*

Proof. (1) Since $[A] \cap Y$ is K_Y-closed set contains A, then $Cl_Y(A) \subseteq [A] \cap Y$. Also, since $Cl_Y(A)$ is K_Y-closed, then $Cl_Y(A) = Y \cap F$, for some $F \in K$, then $A \subseteq Cl_Y(A) = Y \cap F$, so $[A] \subseteq F$ and $[A] \cap Y \subseteq F \cap Y = Cl_Y(A)$.
(2) Suppose that $x \in]A[\subseteq A \subseteq Y$. Then $x \in \bigvee\{G \in \tau : G \subseteq A\}$, so $x \in \bigvee\{G \cap Y \in \tau_Y : G \cap Y \subseteq A\}$, hence $x \in Int_Y(A)$, so $]A[\subseteq Int_Y(A)$.
(3) $ext_Y(A) = \bigvee\{Y \cap G \in \tau_Y : (Y \cap G) \cap A = \phi\} = \bigvee\{Y \cap G \in \tau_Y : G \cap A = \phi\} = Y \cap (\bigvee\{G \in \tau : G \cap A = \phi\} = Y \cap ext(A)$. □

Remark 4.5. The equality in Theorem 4.4(2) is not hold in general, for the following example.
Let $X = \{a, b, c\}$, $L = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\} = P(X)$, $\tau = \{X, \phi, \{b\}, \{a, b\}\}$, $K = L$, and $Y = \{a, c\}$, then $L_Y = \{Y, \phi, \{a\}, \{c\}\}$, $\tau_Y = \{Y, \phi, \{a\}\}$, $K_Y = L_Y$. Let $\{a\} \subseteq Y$, then $]\{a\}[= \phi$ and $Int_Y(\{a\}) = \{a\}$.
Note that the equality in Theorem 4.4(2) holds for all subsets of Y if and only if Y is τ-open. Indeed, if $x \in Int_Y(A)$. Then $x \in \bigvee\{G \cap Y \in \tau_Y : G \cap Y \subseteq A\}$. Since $Y \in \tau$, then $x \in]A[$.

Theorem 4.6. *Let (Y, L_Y, τ_Y, K_Y) be a subspace of a ditopological texture space (X, L, τ, K).*
(1) *Every τ_Y open set is τ open set if and only if $Y \in \tau$.*
(2) *Every K_Y closed set is K closed set if and only if $Y \in K$.*

Proof. (1) Suppose that every τ_Y open set is τ open set, then $Y \in \tau_Y \subseteq \tau$. Conversely if $Y \in \tau$ and $A \subseteq Y$ is τ_Y-open, then $A = Y \cap G$, for some $G \in \tau$, but $Y \in \tau$, hence $A \in \tau$.
(2) By a similar way. □

Corollary 4.7. *Let (Y, L_Y, τ_Y, K_Y) be an open subspace of a ditopological texture space (X, L, τ, K) and $A \subseteq Y$. Then A is τ_Y open set if and only if it is τ open set.*

Proof. Immediate by Theorem 4.6. □

Corollary 4.8. *Let (Y, L_Y, τ_Y, K_Y) be a closed subspace of a ditopological texture space (X, L, τ, K) and $A \subseteq Y$. Then A is K_Y closed set if and only if it is K closed set .*

Proof. Immediate by Theorem 4.6. □

Theorem 4.9. *Let (X, L, τ, K) be a ditopological texture subspace of a ditopological texture space (Y, L', τ', K') and (Y, L', τ', K') be a ditopological texture subspace of a ditopological texture space (Z, L'', τ'', K''). Then (X, L, τ, K) is a subspace of (Z, L'', τ'', K'').*

Proof. We want to prove that $L''_X = L$, $\tau''_X = \tau$ and $K''_X = K$. For $L''_X = L$, let $H \in L''_X$. Then $H = X \cap A$ for some $A \in L''$. Since (Y, L') is a principal texture of (Z, L''), then $Y \cap A \in L'$ for some $A \in L''$, hence $X \cap (Y \cap A) \in L$. This implies that $H = X \cap A \in L$. Also, let $S \in L$. Since (X, L) is a principal texture of (Y, L'), then $S = X \cap K$ for some $K \in L'$, also (Y, L') is a principal texture of (Z, L''), then $K = Y \cap N$ for some $N \in L''$, then $S = X \cap (Y \cap N) = X \cap N$ for some $N \in L''$, hence $S \in L''_X$. The rest of the proof by a similar way. □

Proposition 4.10. *A subset Z of a ditopological texture space (X, L, τ, K) is connected if and only if $\tau_Z \cap K_Z = \{Z, \phi\}$.*

Proof. Immediate by Theorem 3.5. □

Theorem 4.11. *Let (X, L, τ, K) be a ditopological texture space, A be a clopen subset of X and Z be a connected subset of X. Then either $Z \subseteq A$ or $Z \subseteq A'$.*

Proof. Since A is a clopen subset of X, then $Z \cap A$ is a clopen subset in the subspace (Z, L_z, τ_Z, K_Z) by Definition 4.1, hence $Z \cap A \in \tau_Z \cap K_Z$. Since Z is connected, then $Z \cap A$ either empty or equal to Z. This means $Z \subseteq A$ or $Z \subseteq A'$. □

Proposition 4.12. *Let (X, L, τ, K) be a complemented ditopological texture space s.t $K = \tau' = \{X - G : G \in \tau\}$ and Z be a dense subset of X. Then X is connected if Z is connected.*

Proof. Suppose that X is disconnected. Then there exists a proper subset A of X which is clopen. Since Z is connected, then $Z \subseteq A$ or $Z \subseteq A'$ by Theorem 4.11. Since Z is dense subset of X, then $[Z] \subseteq [A]$ or $[Z] \subseteq [A']$. This implies that $X \subseteq A$ or $X \subseteq A'$. Hence $A = X$ or $A = \phi$, which is a contradiction. □

Theorem 4.13. *Every clopen set is a union of components.*

Proof. Let A be a clopen set in a ditopological texture space (X, L, τ, K). For every $a \in A$, $C(A, a) \subseteq A$ by Theorem 4.11. Hence $\bigcup_{a \in A} C(A, a) \subseteq A$. But $C(A, a)$ is the maximal connected set containing a, then $A \subseteq \bigcup_{a \in A} C(A, a)$. Hence $A = \bigcup_{a \in A} C(A, a)$. □

Theorem 4.14. *Let (Y, L_Y, τ_Y, K_Y) be a ditopological texture subspace of a ditopological texture space (X, L, τ, K) and $Z \subseteq Y$. Then Z is connected in the subspace (Y, L_Y, τ_Y, K_Y) if and only if it is connected in the ditopological texture space (X, L, τ, K).*

Proof. Suppose that Z is not connected in the ditopological texture space (X, L, τ, K). Then there exists a partition $\{G, F\}$ of Z with $(G, F) \in \tau \text{x} K$. Hence $G \cap Z \neq \phi, Z \nsubseteq F$ and $Z \cap G = Z \cap F$. Since $Z \subseteq Y$, then $Z \cap (Y \cap G) \neq \phi$, $Z \nsubseteq (F \cap Y)$ and $Z \cap (Y \cap G) = Z \cap (Y \cap F)$. This implies that $(Y \cap G, Y \cap F) \in \tau_Y \text{x} K_Y$ is a partition of Z. Hence Z is not connected in the subspace (Y, L_Y, τ_Y, K_Y), which is a contradiction. Conversely, if Z is not connected in the subspace (Y, L_Y, τ_Y, K_Y). Then there exists a partition $\{A, B\}$ of Z with $(A, B) \in \tau_Y \text{x} K_Y$. Then $A \cap Z \neq \phi, Z \nsubseteq B$ and $Z \cap A = Z \cap B$. Since $(A, B) \in \tau_Y \text{x} K_Y$, then $A = Y \cap G$ for some $G \in \tau$ and $B = Y \cap F$ for some $F \in K$. Hence $G \cap Z \neq \phi$, $Z \nsubseteq F$ and $G \cap Z = F \cap Z$. Hence Z is not connected in the ditopological texture space (X, L, τ, K), which is a contradiction. This completes the proof. □

Theorem 4.15. *The property of disconnectedness is hereditary w.r.t clopen subspaces.*

Proof. It follows from Corollary 4.7 and Corollary 4.8. □

Remark 4.16. A connectedness property is not hereditary property in general, as in the following examples.

(1) Let $X = \{a, b, c\}$, $L = P(X)$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $K = \{X, \phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, then X is connected. Let $Y = \{a, b\} \subseteq X$, then $L_Y = \{Y, \phi, \{a\}, \{b\}\}$, $\tau_Y = \{Y, \phi, \{a\}\}$, $K_Y = \{Y, \phi, \{a\}, \{b\}\}$, then we have (Y, L_Y, τ_Y, K_Y) is disconnected even Y is τ-open.

(2) Let $X = \{a, b, c\}$, $L = P(X)$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $K = \{X, \phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, then X is connected. Let $Y = \{b, c\} \subseteq X$, then $L_Y = \{Y, \phi, \{b\}, \{c\}\}$, $\tau_Y = \{Y, \phi, \{b\}\}$, $K_Y = \{Y, \phi, \{b\}, \{c\}\}$, then we have (Y, L_Y, τ_Y, K_Y) is disconnected even Y is K-closed.

Theorem 4.17. *The property of locally connectedness is hereditary w.r.t open subspaces.*

Proof. Suppose that (Y, L_Y, τ_Y, K_Y) be an open subspace of a locally connected ditopological texture space (X, L, τ, K) and let $x \in Y$. Since X is locally connected, then $\exists\, G \in \tau$ s.t G is τ-connected subset of X and $x \in G \subseteq A$. Since $G \in \tau$ and G is τ-connected set, then $G \cap Y \in \tau_Y$, and $G \cap Y$ is τ_Y-connected subset of Y by Theorem 4.14. So Y is locally connected at x, hence Y is locally connected. □

Theorem 4.18. *The components of every open subspace of a locally connected ditopological texture space are τ-open.*

Proof. Suppose that (Y, L_Y, τ_Y, K_Y) be an open subspace of a locally connected ditopological texture space (X, L, τ, K), let C be a component of Y. Since X is locally connected, then Y is locally connected by Theorem 4.17, hence by Theorem 3.20 we get the proof. □

Definition 4.19. A ditopological texture space (X, L, τ, K) is said to be extremely disconnected if for every open set $G \subseteq X$ we have $[G]$ is open in X.

Proposition 4.20. *Let (X, L, τ, K) be a ditopological texture space s.t $K = \tau' = \{X - G : G \in \tau\}$. Then X is extremely disconnected if and only if for every pair G, H of disjoint open subsets of X we have $[G] \cap [H] = \phi$.*

Proof. Immediate. □

Theorem 4.21. *The extremely disconnectedness is hereditary w.r.t open subspaces.*

Proof. Suppose that (Y, L_Y, τ_Y, K_Y) be an open subspace of an extremely disconnected ditopological texture space (X, L, τ, K), and let $A \in \tau_Y$. Then $A \in \tau$ by Corollary 4.7, and $[A] \in \tau$. Since $Cl_Y(A) = Y \cap [A]$ by Theorem 4.4, then $Cl_Y(A) \in \tau_Y$. Hence Y is extremely disconnected. □

Definition 4.22. Let $(X_\alpha, L_\alpha, \tau_\alpha, K_\alpha)$, $\alpha \in \Lambda$, be a family of pairwise disjoint ditopological texture spaces, i.e $X_\alpha \cap X_{\alpha'} = \phi$ for $\alpha \neq \alpha'$ and let (X, L) be the sum of textures (X_α, L_α), $\alpha \in \Lambda$. Define $\tau = \{A \subseteq X : A \cap X_\alpha \in \tau_\alpha,\ \alpha \in \Lambda\}$, $K = \{A \subseteq X : A \cap X_\alpha \in K_\alpha, \alpha \in \Lambda\}$, where τ, $K \subseteq L$. Then (X, L, τ, K) is called the sum ditopological texture space of the ditopological texture spaces $(X_\alpha, L_\alpha, \tau_\alpha, K_\alpha)$, $\alpha \in \Lambda$, and denoted by $\oplus_{\alpha\in\Lambda} X_\alpha$.

Theorem 4.23. *The sum $\oplus_{\alpha\in\Lambda} X_\alpha$, where $\Lambda \neq \phi$ and $X_\alpha \neq \phi$ for $\alpha \in \Lambda$, is totally disconnected if and only if all spaces X_α are totally disconnected.*

Proof. Suppose that X is totally disconnected, and let $x, y \in X_\alpha$ s.t $x \neq y$. Then $x, y \in X$, since X is totally disconnected, then $\exists$ a non empty disjoint clopen proper subsets A, B of X s.t $x \in A$ and $y \in B$. Hence $A \cap X_\alpha, B \cap X_\alpha$ are non empty disjoint clopen proper subsets of X_α s.t $x \in A \cap X_\alpha$ and $y \in B \cap X_\alpha$. Hence X_α is totally disconnected $\forall\ \alpha \in \Lambda$. Conversely, let X_α is totally disconnected $\forall\ \alpha \in \Lambda$, and let $x, y \in X$ s.t $x \neq y$. Hence $x, y \in X_\alpha$ for some $\alpha \in \Lambda$. Since X_α is totally disconnected, so $\exists$ a non empty disjoint clopen proper subsets $A \cap X_\alpha, B \cap X_\alpha$ of X_α s.t $A, B \in \tau \cap K$ by Definition 4.22, and $x \in A \cap X_\alpha$, $y \in B \cap X_\alpha$. Now $A, B \in \tau \cap K$ s.t $x \in A$ and $y \in B$. Hence X is totally disconnected. □

5. Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the ditopological texture spaces.In this paper some additional properties related connectedness in ditopological texture spaces have obtained. Also, we introduce the notion of ditopological texture subspaces and study some of their properties. Some new types of connectedness in ditopological texture spaces namely, locally connectedness, totally disconnectedness and extremely disconnectedness have investigated. Therefore, we show that the sum of ditopological texture spaces is totally disconnected if and only if each ditopological texture space is totally disconnected.

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