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On fixed points in fuzzy metric spaces

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Abstract. In this paper , we state and prove some fixed point theorems in fuzzy metric spaces in the sense of Kramosil and Michalek. Our results extend the famous fixed point theorems due to Caccioppoli and Edelstein on classical metric spaces. In particular , we deduce the Banach contraction theorem on fuzzy metric spaces due to Grabiec as a corollary. Further we support our results with a suitable example.

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1. Introduction

With the advent of the concept of fuzzy metric spaces, a new era of studying Fixed Point Theory in these spaces sets in. There are several notions of fuzzy metric spaces as introduced by various authors such as [3, 4, 12, 13]. In particular, Kramosil and Michalek [13] introduced a notion of fuzzy metric spaces in the year 1975 by generalizing the concept of probabilistic metric spaces introduced by Menger to fuzzy setting. George and Veeramani [5] modified the notion of fuzzy metric spaces introduced by Kramosil and Michalek with the help of continuous t-norm and obtained a Hausdorff topology for this kind of fuzzy metric spaces. Many authors think that the George and Veeramani's definition is an appropriate notion of metric fuzziness in the sense that it provides rich fuzzy topological structure which can be obtained, in many cases, from classical theorems. In recent years, many mathematicians such as [1, 2, 6, 10, 11, 22] etc., established several fixed point theorems in fuzzy metric spaces. Our present work is in the direction of extending the classical fixed point theorems due to Caccioppoli and Edelstein in fuzzy metric spaces. Recent literatures on fixed point theory in fuzzy metric spaces can also be viewed in [7, 8, 14, 16, 17, 18, 19]. For basic analysis we refer to [9, 15]. The structure of the paper is as follows. After the preliminaries, we prove the fuzzy version of Caccioppoli and Edelstein fixed point theorems in section [3] and deduce fuzzy Banach contraction theorem due to Grabiec [6] as a corollary. We have also incorporated an example to support the fuzzified Caccioppoli fixed point theorem.

2. Preliminaries

In this section, we recall some definitions and known results.

Definition 2.1 ([23]). A fuzzy set A in X is a mapping $A : X \to [0, 1]$. For $x\epsilon X, A(x)$ is called the grade of membership of x .

Definition 2.2 ([21]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm, if $([0, 1], *)$ is an abelian topological monoid with unity 1 such that $a * b \leq c * d$, whenever $a \leq c, b \leq d$, for all $a, b, c, d\epsilon[0, 1]$.

Definition 2.3 ([13]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty[$, satisfying the following conditions:
For all $x, y, z\epsilon X$ and $s, t > 0$,
$M(x, y, 0) = 0,$ (2.3.1)
$M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$, (2.3.2)
$M(x, y, t) = M(y, x, t)$, (2.3.3)
$M(x, z, s + t) \geq M(x, y, s) * M(y, z, t),$ (2.3.4)
$M(x, y, .) : [0, \infty[\to [0, 1]$ is left continuous, and (2.3.5)
$lim_{t\to\infty} M(x, y, t) = 1.$ (2.3.6)

Example 2.4 ([5]). Let (X, d) be a metric space and $a * b = min\{a, b\} or ab$ for every $a, b\epsilon[0, 1]$. Let M_d be a fuzzy set in $X^2 \times [0, \infty[$ given by $M_d(x, y, t) = \frac{t}{(t+d(x,y))}$ if $t > 0$ and $M_d(x, y, 0) = 0$. Then $(X, M_d, *)$ is a fuzzy metric space and M_d is called the **standard fuzzy metric** induced by the metric d. For further examples of fuzzy metric spaces we refer to [20].

Definition 2.5 ([6]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called
(a) a **Cauchy sequence**, if $lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0, p > 0$.
(b) **convergent** to x(in symbols, $\lim_{n\to\infty} x_n = x$ or $x_n \to x$), if $lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$.

Definition 2.6 ([6]). A fuzzy metric space $(X, M, *)$ is said to be **complete**,if every Cauchy sequence in X is convergent.

Definition 2.7 ([1]). Let $(X, M, *)$ be a fuzzy metric space and $\varepsilon > 0$. A finite sequence $x = x_0, x_1, ..., x_{n-1}, x_n = y$ is called an **ε-chain** from x to y if $M(x_{(i-1)}, x_i, t) > 1 - \varepsilon$, for all $t > 0, i = 1, 2, ..., n$. A fuzzy metric space $(X, M, *)$ is said to be **ε-chainable** if for every $x, y\epsilon X$, there is an ε-chain from x to y.

Definition 2.8. Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a mapping.
(a) T is said to be **continuous**, if for every $x\epsilon X, x_n \to x$ implies $Tx_n \to Tx$.

(b) For $\varepsilon > 0, 0 < \lambda < 1, T$ is called (ε, λ) **uniformly locally contractive**, if $M(x, y, t) > 1-\varepsilon$ implies $M(Tx, Ty, t) \geq M(x, y, \frac{t}{\lambda})$, (2.7.1)
for all $x, y \epsilon X, t > 0$. Clearly a uniformly locally contractive mapping T is continuous.

Theorem 2.9. (Caccioppoli) *Let (X, ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose for each positive integer $n, \rho(T^n x, T^n y) \leq a_n \rho(x, y)$ for all $x, y \epsilon X$ and $a_n > 0$ is independent of x, y. If the series Σa_n is convergent, then T has a unique fixed point.*

Theorem 2.10. *Let (X, ρ) be a complete metric space and $T : X \to X$ be a continuous mapping. If for some positive integer m, T^m is a contraction mapping, i.e., $\rho(T^m x, T^m y) \leq \alpha \rho(x, y)$ for all $x, y \epsilon X$, and some $0 < \alpha < 1$, then T has a unique fixed point.*

Theorem 2.11. (Edelstein) *Let (X, ρ) be a complete, ε-chainable metric space and $T : X \to X$ be (ε, λ) uniformly locally contractive. Then there exists a unique fixed point of T.*

Theorem 2.12. (Grabiec [6]) *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a contraction, i.e., a mapping satisfying $M(Tx, Ty, kt) \geq M(x, y, t)$, for all $x, y \epsilon X, t > 0$ and some $0 < k < 1$. Then T has a unique fixed point.*

3. Main results

In this section, we state and prove the main results of our paper. We extend the theorems 2.9, 2.10 and 2.11 to fuzzy metric spaces and deduce theorem 2.12 as a corollary. We further construct an example in support of the fuzzy version of the theorem 2.9

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a mapping satisfying the followings:*
For every positive integer n and $t > 0$,
$M(T^n x, T^n y, k_n t) \geq M(x, y, t)$, (3.1.1)
for all $x, y \epsilon X, k_n > 0$ being independent of x, y. If $k_n \to 0$, then T has a unique fixed point in X.

Proof. Let $x \epsilon X, x_n = T^n x, n \epsilon \mathbb{N}$. Now $\{x_n\}$ is a sequence of points of X such that $x_1 = Tx, x_2 = Tx_1, \dots \dots, x_{n+1} = Tx_n, n \epsilon \mathbb{N}$. We get

$$\begin{aligned} 1 &\geq M(x_n, x_{n+p}, t) \\ &\geq M\left(x_n, x_{n+1}, \tfrac{t}{p}\right) * M\left(x_{n+1}, x_{n+2}, \tfrac{t}{p}\right) * \dots \dots * M\left(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}\right) \\ &\geq M(x, x_1, \tfrac{t}{pk_n}) * M(x, x_1, \tfrac{t}{pk_{n+1}}) * \dots \dots * M(x, x_1, \tfrac{t}{pk_{n+p-1}}) \quad \text{(by 3.1.1)}. \end{aligned}$$

$\to 1 * 1 * \dots \dots * 1 = 1$, as $n \to \infty$, for all $t > 0, p > 0$ by(2.3.6). Therefore $lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0, p > 0$ and so $\{x_n\}$ is a Cauchy sequence in X. As X is complete, $\exists y \epsilon X$ such that $x_n \to y$, as $n \to \infty$. We thus have $1 \geq M(y, Ty, t) \geq M(y, x_{n+1}, \frac{t}{2}) * M(x_{n+1}, Ty, \frac{t}{2}) \geq M(x_{n+1}, y, \frac{t}{2}) * M(x_n, y, \frac{t}{2k_1})$ by (3.1.1). $\to 1 * 1 = 1$, as $n \to \infty$, for all $t > 0$.
Hence $M(y, Ty, t) = 1$, for all $t > 0$, and thus $Ty = y$, a fixed point of T. To show uniqueness , let $z \epsilon X$ such that $Tz = z$. We get $T^n y = y, T^n z = z$, for all $n \epsilon \mathbb{N}$ .Now ,$1 \geq M(y, z, t) = M(T^n y, T^n z, t \geq M(y, z, \frac{t}{k_n}) \to 1$, as $n \to \infty$ for all $t > 0$.

Therefore $M(y,z,t) = 1$, for all $t > 0$. Hence $y = z$ and so the fixed point is unique. □

We now deduce the theorem 2.12 due to Grabiec [6] as a corollary:

Corollary 3.2. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* $T : X \to X$ *be a contraction , i.e., a mapping satisfying*

$$M(Tx, Ty, kt) \geq M(x, y, t), \tag{3.2.1}$$

for all $x, y\epsilon X, t > 0$ *and some* $0 < k < 1$. *Then* T *has a unique fixed point.*

Proof. For any positive integer n, we have

$$\begin{aligned} M(T^n x, T^n y, k^n t) &\geq M(T^{n-1}x, T^{n-1}y, k^{n-1}t), by (3.2.1). \\ &\geq M(T^{(n-2)}x, T^{(n-2)}y, k^{(n-2)}t) \geq ...... \geq M(x, y, t), \end{aligned}$$

for all $x, y\epsilon X, t > 0$ and some $0 < k < 1$. As $k^n \to 0$, as $n \to \infty$, we have by the preceding theorem, T has a unique fixed point. □

Example 3.3. Let $X = [0,1]$ and $d(x,y) = |x-y|$ for every $x, y\epsilon X$. Then (X, d) is a complete metric space. Let M be a fuzzy set in $X^2 \times [0, \infty[$ given by $M(x,y,t) = \frac{t}{t+d(x,y)}$ if $t > 0$ and $M(x,y,0) = 0$. With $a*b = min\{a,b\}$ for every $a, b\epsilon[0,1]$, $(X, M, *)$ is a complete fuzzy metric space. Let $T : X \to X$ be given by $Tx = \frac{x}{5}$, for every $x\epsilon X$. Now

$$\begin{aligned} M(T^n x, T^n y, k^n t) &= \frac{k^n t}{k^n t + d(T^n x, T^n y)} \\ &= \frac{\frac{t}{2^n}}{\frac{t}{2^n} + |T^n x - T^n y|}, \text{with } k = \frac{1}{2}. \\ &= \frac{t}{t + (\frac{2}{5})^n |x-y|} \geq \frac{t}{t+d(x,y)} = M(x,y,t), \end{aligned}$$

for every $x, y\epsilon X, t > 0, n > 0$. Also $k^n = \frac{1}{2^n} \to 0$. Therefore , the conditions of the theorem 3.1 are satisfied. We note that $0\epsilon X$ is the unique fixed point of T.

Theorem 3.4. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* $T : X \to X$ *be a continuous mapping. Let there exist a positive integer* m *such that*

$$M(T^m x, T^m y, kt) \geq M(x,y,t), \textit{ for all } x, y\epsilon X, t > 0 \textit{ and some } 0 < k < 1. \tag{3.4.1}$$

$$\textit{If } x_n \to x, y_n \to y \textit{ in } X \textit{ implies } M(x_n, y_n, t) \to M(x,y,t)\ , \tag{3.4.2}$$

for all $t > 0$, *then* T *has a unique fixed point in* X.

Proof. We put $B = T^m$. Then for any $x_0\epsilon X$ and any positive integer n, we have

$$\begin{aligned} M(B^n T x_0, B^n x_0, k^n t) &\geq M(B^{n-1}Tx_0, B^{n-1}x_0, k^{n-1}t) \textit{ by } (3.4.1). \\ &\geq M(B^{n-2}Tx_0, B^{n-2}x_0, k^{n-2}t) \\ &\geq \ldots\ldots \\ &\geq M(Tx_0, x_0, t). \end{aligned} \tag{3.4.3}$$

As B is a contraction, we have B has a unique fixed point $x\epsilon X$. As in the proof of the fuzzy Banach contraction Theorem, we get $B^n x_0 \to x$ as $n \to \infty$. This gives $TB^n x_0 = B^n T x_0 \to Tx$. By (3.4.2), we have

$$M(B^n T x_0, B^n x_0, t) \to M(Tx, x, t), \text{ as } n \to \infty, \text{ for all } t > 0. \tag{3.4.4}$$

Using (3.4.3), we get $1 \geq M(B^n T x_0, B^n x_0, t) \geq M(Tx_0, x_0, \frac{t}{k^n}) \to 1$, as $n \to \infty$, for all $t > 0$. Therefore , $lim_{n\to\infty} M(B^n T x_0, B^n x_0, t) = 1$, for all $t > 0$. Then by (3.4.4), we obtain ,$M(Tx, x, t) = 1$, for all $t > 0$. Hence $Tx = x$, a fixed point of T. If $Ty = y$, for some $y\epsilon X$ we get $By = T^m y = T^{m-1} y = \ldots\ldots = y$, and so y is a fixed point of $B = T^m$. Therefore, $x = y$ and so the fixed point of T is unique. □

Theorem 3.5. *Let $(X, M, *)$ be a complete, ε-chainable fuzzy metric space and $T : X \to X$ be an (ε, λ) uniformly locally contractive mapping. Then T has a unique fixed point.*

Proof. Let $x \epsilon X$ be arbitrarily fixed. Let $Tx \neq x$ (otherwise a fixed point is assured). Let $x = x_0, x_1, \dots \dots, x_{n-1}, x_n = Tx$ be an ε-chain from x to Tx. We get $M(x_{i-1}, x_i, t) > 1 - \varepsilon$, for all $t > 0, i = 1, 2, \dots \dots, n$. Let us first prove the result:
$M(T^m x_{i-1}, T^m x_i, t) \geq M\left(x_{i-1}, x_i, \frac{t}{\lambda m}\right)$, (3.5.1)
for all $t > 0, m > 0, i = 1, 2, \dots \dots, n$.
By (2.7.1) , we get $M(Tx_{i-1}, Tx_i, t) \geq M\left(x_{i-1}, x_i, \frac{t}{\lambda}\right)$, for all $t > 0, i = 1, 2, \dots \dots, n$. So (3.5.1) holds for $m = 1$. To apply induction , let $m > 1$ and assume (3.5.1) for all $j < m$. We get $1 - \varepsilon < M\left(x_{i-1}, x_i, \frac{t}{\lambda m}\right)$
$\leq M\left(T^{m-1}x_{i-1}, T^{m-1}x_i, \frac{t}{\lambda}\right)$, by induction hypothesis.
$\leq M(T^m x_{i-1}, T^m x_i, t) by (2.7.1)$, for all $t > 0, i = 1, 2, \dots \dots, n$.
Therefore, (3.5.1) holds for m. Hence by induction, (3.5.1) holds for all $m \epsilon \mathbb{N}$. We now get,
$1 \geq M(T^m x, T^{m+1}x, t) = M(T^m x_0, T^m x_n, t)$
$\geq M(T^m x_0, T^m x_1, \frac{t}{n}) * M(T^m x_1, T^m x_2, \frac{t}{n}) * \dots \dots * M(T^m x_{n-1}, T^m x_n, \frac{t}{n})$
$\geq M(x_0, x_1, \frac{t}{n\lambda^m}) * M(x_1, x_2, \frac{t}{n\lambda^m}) * \dots \dots * M(x_{n-1}, x_n, \frac{t}{n\lambda^m}) by (3.5.1)$.
$\to 1 * 1 * \dots \dots * 1 = 1$, as $m \to \infty$, for all $t > 0$.
Therefore, $lim_{m\to\infty} M(T^m x, T^{m+1}x, t) = 1$, for all $t > 0$. (3.5.2)
Then we get, $1 \geq M(T^m x, T^{m+p}x, t) \geq M\left(T^m x, T^{m+1}x, \frac{t}{p}\right) *$
$M\left(T^{m+1}x, T^{m+2}x, \frac{t}{p}\right) * \dots \dots * M\left(T^{m+p-1}x, T^{m+p}x, \frac{t}{p}\right) \to 1 * 1 * \dots \dots * 1 = 1$, as $m \to \infty$, for all $t > 0, p > 0$, by (3.5.2). Hence $lim_{m\to\infty} M(T^m x, T^{m+p}x, t) = 1$, for all $t > 0, p > 0$ and so $\{T^m x\}$ is a Cauchy sequence in X. As X is complete, $\exists y \epsilon X$ such that $lim_{m\to\infty} T^m x = y$. As T is obviously continuous, we get $lim_{m\to\infty} T^{m+1}x = Ty$. Hence $Ty = y$, a fixed point of T. To show uniqueness, let $Tz = z$ for some $z \epsilon X$. Let $y = w_0, w_1, \dots \dots, w_{k-1}, w_k = z$ be an ε-chain. Now for any positive integer l,

$$\begin{aligned} 1 \quad &\geq M(y, z, t) = M(T^l y, T^l z, t) = M(T^l w_0, T^l w_k, t) \\ &\geq M(T^l w_0, T^l w_1, \tfrac{t}{k}) * M(T^l w_1, T^l w_2, \tfrac{t}{k}) * \dots \dots * M(T^l w_{k-1}, T^l w_k, \tfrac{t}{k}) \\ &\geq M(w_0, w_1, \tfrac{t}{k\lambda^l} * M(w_1, w_2, \tfrac{t}{k\lambda^l}) * \dots \dots * M(w_{k-1}, w_k, \tfrac{t}{k\lambda^l})\ by\ (3.5.1). \\ &\to 1 * 1 * \dots \dots * 1 = 1, as\ l \to \infty, \text{for all}\ t > 0 \end{aligned}$$

Therefore, $M(y, z, t) = 1$, for all $t > 0$, and so $y = z$. Hence the fixed point is unique. □

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