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On ideal convergent sequence spaces of fuzzy real numbers associated with multiplier sequences defined by sequence of Orlicz functions

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Abstract. In this article we introduce some new sequence spaces of fuzzy real numbers using ideal convergence and the sequences of Orlicz functions, and study some basic topological and algebraic properties of the spaces. Also we establish the relations between the spaces. Further, we introduce a kind of space with the help of Hukuhara difference property.

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1. Introduction

Zadeh [49] introduced the concept of fuzzy set theory. It has a wide range of applications in almost all the branches in science, where mathematics is used. The notion of fuzzyness are using by many researchers for Cybernetics, Artificial Intelligence, Expert System and Fuzzy Control, Pattern recognition, Operation Research, Decision making, Image Analysis, Projectiles, Probabilty theory, Weather forecasting etc. It attracted many workers on sequence spaces and summability theory to introduce different types of fuzzy sequence spaces and study their different properties. Our studies are based on the linear spaces of sequences of fuzzy numbers which are very important for higher level studies in Quantum Mechanics, Particle Physics and Statistical Mechanics etc. Different classes of sequences of fuzzy numbers have been discussed by Nanda [35], Nuray and Savas [36], Matloka [31], Mursaleen and Basarir [32], Mursaleen [1], Tripathy and Dutta [48], Hazarika [20] and references therein.

Kostyrko et al. [28] introduced the notion of I-convergence with the help of an admissible ideal I denotes the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence (see [8]). It was further studied by Cakalli and Hazarika [2], Esi and Hazarika ([5, 6]), Hazarika([10, 11, 12, 13, 14, 15, 18, 19, 21]), Hazarika and Kumar [24], Hazarika and Savas [22], Kumar and Kumar [29], Mursaleen and Mohiuddine [33], Mursaleen et al., [34], Šalát et al. ([39, 40]), Savas [41], Tripathy and Hazarika ([44, 45, 46], Subramanian et al., [42] and references therein.

Goes and Goes [9] initially introduced the differential sequence space dE and the integrated sequence space $\int E$ for a given sequence space E, by using the multiplier sequences (k^{-1}) and (k) respectively, where $E = c, c_0, \ell_\infty$. A multiplier sequence can be used to accelerate the convergence of the sequences. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of the convergence of a sequence. This it also covers a larger class of sequences for study. Tripathy and Mahanta [47] used a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars for all $k \in \mathbb{N}$.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a given sequence space E, the multiplier sequence space $E(\Lambda)$ associated with multiplier sequence Λ is defined by (for details see [47])

$$E(\Lambda) = \{(x_k) : (\lambda_k x_k) \in E\}.$$

Recall in [27] that an Orlicz function M is continuous, convex , nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle [38]. An Orlicz function M is said to satisfy $\Delta_2 -$ *condition* for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Lemma 1.1. *Let M be an Orlicz function which satisfies $\Delta_2 -$ condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there exist positive constants α, β and x_0 such that

$$M_1(\alpha) \leq M_2(x) \leq M_1(\beta) \text{ for all } x \text{ with } 0 \leq x < x_0.$$

Lindenstrauss and Tzafriri [30] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$|| \ x \ || = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$.

In the later stage Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [37], Esi [3], Esi and Et [4], Hazarika ([16, 17]), Tripathy and Sarma [43], Esi and Hazarika [7], Hazarika et al [23] and references therein.

Throughout the article w^F, c^F, c_0^F and ℓ_∞^F denote the classes of *all, convergent, null* and *bounded* fuzzy real-valued sequence spaces, respectively. Also $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and set of real numbers, respectively. The zero sequence is denoted by θ.

Throughout the paper, we denote I is an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

2. Preliminaries

We now recall some difinitions related to ideal convergence and sequences of fuzzy real numbers.

Definition 2.1 ([28])**.** Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* on X if and only if

(i) $\phi \in I$.
(ii) for each $A, B \in I$, we have $A \cup B \in I$
(iii) for each $A \in I$ and each $B \subset A$, we have $B \in I$.

Definition 2.2 ([28])**.** A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if

(i) $\phi \notin F$
(ii) for each $A, B \in F$, we have $A \cap B \in F$
(iii) each $A \in F$ and each $A \subset B$, we have $B \in F$.

Definition 2.3 ([28])**.** An ideal I is called *non-trivial ideal* if $I \neq \phi$ and $X \notin I$.

Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

Definition 2.4 ([28])**.** A non-trivial ideal $I \subset 2^X$ is called

(i) *admissible* if and only if $\{\{x\} : x \in X\} \subset I$.
(ii) *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset }\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density (see [8]) of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$,

$$d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}, \text{ where } X = [x_1, x_2] \text{ and } Y = [y_1, y_2].$$

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space (see [25]). Also the relation " $\leq$ " is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.5. A fuzzy number X is said to be

(i) *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$.

(ii) *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

(iii) *upper semi-continuous* if for each $\epsilon > 0, X^{-1}([0, a+\epsilon))$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$.

Let $\mathbb{R}(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in \mathbb{R}(J)$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where

$$[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha, \text{ if } \alpha \in [0,1]\},$$

$$[X]^0 = \text{ closure of } (\{t \in \mathbb{R} : X(t) > \alpha, \text{ if } \alpha = 0\}).$$

The set $\mathbb{R}$ of real numbers can be embedded in $\mathbb{R}(J)$ if we define $\overline{r} \in \mathbb{R}(J)$ by

$$\overline{r}(t) = \begin{cases} 1, \text{ if } t = r : \\ 0, \text{ if } t \neq r \end{cases}$$

The absolute value , $|X|$ of $X \in \mathbb{R}(J)$ is defined by (for details see [25])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, \text{ if } t \geq 0 : \\ 0, \text{ if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

It is known that $(\mathbb{R}(J), \bar{d})$ is a complete metric space (for details see [25]).

Definition 2.6 ([32]). A metric on $\mathbb{R}(J)$ is said to be *translation invariant* if

$$\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y), for\ X, Y, Z \in \mathbb{R}(J).$$

Definition 2.7 ([31]). A sequence $X = (X_k)$ of fuzzy numbers is said to be

(i) *convergent* to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_k, X_0) < \epsilon$ for all $n \geq n_0$.

(ii) *bounded* if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

Definition 2.8 ([29]). A sequence $X = (X_k)$ of fuzzy numbers is said to be

(i) *I-convergent* to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \epsilon\} \in I.$$

The fuzzy number X_0 is called *I-limit* of the sequence (X_k) of fuzzy numbers and we write $I - \lim X_k = X_0$.

(ii) *I-bounded* if there exists $M > 0$ such that

$$\{k \in \mathbb{N} : \bar{d}(X_k, \bar{0}) > M\} \in I.$$

Definition 2.9. A sequence space E_F of fuzzy numbers is said to be

(i) *solid* (or *normal*) if $(Y_k) \in E_F$ whenever $(X_k) \in E_F$ and $\bar{d}(Y_k, \bar{0}) \leq \bar{d}(X_k, \bar{0})$ for all $k \in \mathbb{N}$.

(ii) *symmetric* if $(X_k) \in E_F$ implies $(X_{\pi(k)}) \in E_F$ where π is a permutation of $\mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A *K-step space* of E is a sequence space

$$\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}.$$

A canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is a sequence $\{y_k\} \in w$ defined as

$$y_k = \begin{cases} x_k, & \text{if } k \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E , i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.10. A sequence space E_F is said to be *monotone* if E_F contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = G, D = \max\{1, 2^{G-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \text{ for all } k \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{C}$$

Also $|a_k|^{p_k} \leq \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

Lemma 2.11. *A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [26], page 53).*

Lemma 2.12. *(Kostyrko et al., [28], Lemma 5.1). If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.*

3. Some new multiplier sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_k)$ be a sequence of positive real numbers for all $k \in \mathbb{N}$. Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions and $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars and $X = (X_k)$ be a sequence of fuzzy numbers, we define the following sequence spaces as foolows:

$$W^{I(F)}(\mathbf{M}, \Lambda, p) = \{(X_k) \in w^F : \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} \left[M_k\left(\frac{\bar{d}(\lambda_k X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I \text{ for some } \rho > 0 \text{ and } X_0 \in R(J)\},$$

$$W_0^{I(F)}(\mathbf{M}, \Lambda, p) = \{(X_k) \in w^F : \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} \left[M_k\left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I \text{ for some } \rho > 0\},$$

$$W_\infty^F(\mathbf{M}, \Lambda, p) = \{(X_k) \in w^F : \sup \frac{1}{n}\sum_{k=1}^{n} \left[M_k\left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho}\right)\right]^{p_k} < \infty \text{ for some } \rho > 0\},$$

$$W_\infty^{I(F)}(\mathbf{M}, \Lambda, p) = \{(X_k) \in w^F : \exists K > 0 \; s.t. \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} \left[M_k\left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho}\right)\right]^{p_k} \geq K\right\} \in I \text{ for some } \rho > 0\}.$$

It is not possible in general to find some fuzzy number $X - Y$ such that $X = Y + (X - Y)$ (called the Hukuhara difference when it exists). Since, every real number is a fuzzy number, we can assume that $Tw^F \subset w^F$ be such a set of sequences of fuzzy numbers with Hukuhara difference property.

Now, we examine the basic topological and algebraic properties of these spaces and obtain the inclusion relation between these spaces.

Theorem 3.1. *$W^{I(F)}(\mathbf{M}, \Lambda, p)$, $W_0^{I(F)}(\mathbf{M}, \Lambda, p)$, and $W_\infty^{I(F)}(\mathbf{M}, \Lambda, p)$ are linear spaces.*

Proof. We prove the result only for the space $W_0^{I(F)}(\mathbf{M}, \Lambda, p)$. The other spaces can be treated, similarly. Let $X = (X_k)$ and $Y = (Y_k)$ be two elements of $W_0^{I(F)}(\mathbf{M}, \Lambda, p)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k Y_k, \bar{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I$$

Let α, β be two scalars. By the continuity of the function $\mathbf{M} = (M_k)$ the following inequality holds:

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k(\alpha X_k + \beta Y_k, \bar{0}))}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{p_k} \\ &\leq D \frac{1}{n} \sum_{k=1}^{n} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho_1} \right) \right]^{p_k} \\ &+ D \frac{1}{n} \sum_{k=1}^{n} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{\bar{d}(\lambda_k Y_k, \bar{0})}{\rho_2} \right) \right]^{p_k} \\ &\leq D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho_1} \right) \right]^{p_k} + D \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k Y_k, \bar{0})}{\rho_2} \right) \right]^{p_k}. \end{aligned}$$

From the above relation we obtain the following:

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k(\alpha X_k + \beta Y_k, \bar{0}))}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq \\ &\left\{ n \in \mathbb{N} : DK \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \\ &\cup \left\{ n \in \mathbb{N} : DK \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{\bar{d}(\lambda_k Y_k, \bar{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I. \end{aligned}$$

This completes the proof. □

Remark 3.2. It is easy to verify that $W_\infty^F(\mathbf{M}, \Lambda, p)$ is a linear space.

For the next result, we consider $TW_\infty^F(\mathbf{M},\Lambda,p)\subset W_\infty^F(\mathbf{M},\Lambda,p)$ be the space of sequences of fuzzy numbers with Hukuhara difference property.

Theorem 3.3. *The space $TW_\infty^F(\mathbf{M},\Lambda,p)$ is a paranormed space (not totally paranormed) with the paranorm g_Λ defined by*

$$g_\Lambda(X)=\inf\left\{\rho^{\frac{p_k}{H}}:\sup_k M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\le 1,\ for\quad some\quad \rho>0\right\},$$

where $H=\max\{1,\sup_k p_k\}$.

Proof. Clearly $g_\Lambda(-X)=g_\Lambda(X)$ and $g_\Lambda(\theta)=0$. Let $X=(X_k)$ and $Y=(Y_k)$ be two elements of $TW_\infty^F(\mathbf{M},\Lambda,p)$. Then for $\rho>0$ we put

$$A_1=\left\{\rho>0:\sup_k M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\le 1\right\}$$

and

$$A_2=\left\{\rho>0:\sup_k M_k\left(\frac{\bar{d}(\lambda_k Y_k,\bar{0})}{\rho}\right)\le 1\right\}.$$

Let $\rho_1\in A_1$ and $\rho_2\in A_2$. If $\rho=\rho_1+\rho_2$ then we obtain the following

$$M_k\left(\frac{\bar{d}(\lambda_k(X_k+Y_k),\bar{0})}{\rho}\right)\le\frac{\rho_1}{\rho_1+\rho_2}M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho_1}\right)+\frac{\rho_2}{\rho_1+\rho_2}M_k\left(\frac{\bar{d}(\lambda_k Y_k,\bar{0})}{\rho_2}\right).$$

Thus we have

$$\sup_k\left[M_k\left(\frac{\bar{d}(\lambda_k(X_k+Y_k),\bar{0})}{\rho}\right)\right]^{p_k}\le 1$$

and

$$\begin{aligned}g_\Lambda(X+Y)&=\inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}}:\rho_1\in A_1,\rho_2\in A_2\right\}\\&\le\inf\left\{\rho_1^{\frac{p_k}{H}}:\rho_1\in A_1\right\}+\inf\left\{\rho_2^{\frac{p_k}{H}}:\rho_2\in A_2\right\}\\&=g_\Lambda(X)+g_\Lambda(Y).\end{aligned}$$

Let $t_k\to t$ where $t_k,t\in C$ and let $g_\Lambda(X_k-X)\to 0$ as $k\to\infty$. To prove that $g_\Lambda(t_kX_k-tX)\to 0$ as $k\to\infty$.

We put

$$A_3=\left\{\rho_k>0:\sup_k\left[M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho_k}\right)\right]^{p_k}\le 1\right\}$$

and

$$A_4=\left\{\rho_l>0:\sup_k\left[M_k\left(\frac{\bar{d}(\lambda_k Y_k,\bar{0})}{\rho_l}\right)\right]^{p_k}\le 1\right\}.$$

By the continuity of the function $\mathbf{M}=(M_k)$ we observe that

$$\begin{aligned}&M_k\left(\frac{\bar{d}(\lambda_k(t_kX_k-tX),\bar{0})}{|t_k-t|\rho_k+|t|\rho_l}\right)\\&\le M_k\left(\frac{\bar{d}(\lambda_k(t_kX_k-tX_k),\bar{0})}{|t_k-t|\rho_k+|t|\rho_l}\right)+M_k\left(\frac{\bar{d}(\lambda_k(tX_k-tX),\bar{0})}{|t_k-t|\rho_k+|t|\rho_l}\right)\\&\le\frac{|t_k-t|\rho_k}{|t_k-t|\rho_k+|t|\rho_l}M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho_k}\right)+\frac{|t|\rho_l}{|t_k-t|\rho_k+|t|\rho_l}M_k\left(\frac{\bar{d}(\lambda_k Y_k,\bar{0})}{\rho_l}\right)\end{aligned}$$

From the above inequality it follows that

$$\sup_k \left[M_k \left(\frac{\bar{d}(\lambda_k(t_k X_k - tX), \bar{0})}{|t_k - t|\rho_k + |t|\rho_l} \right) \right]^{p_k} \leq 1$$

and consequently

$$g_\Lambda(t_k X_k - tX) = \inf \left\{ (|t_k - t|\rho_k + |t|\rho_l)^{\frac{p_k}{H}} : \rho_k \in A_3, \rho_l \in A_4 \right\}$$

$$\leq |t_k - t|^{\frac{p_k}{H}} \inf \left\{ (\rho_k)^{\frac{p_k}{H}} : \rho_k \in A_3 \right\} + |t|^{\frac{p_k}{H}} \inf \left\{ (\rho_l)^{\frac{p_k}{H}} : \rho_l \in A_4 \right\}$$

$$\leq \max \left\{ 1, |t_k - t|^{\frac{p_k}{H}} \right\} g_\Lambda(X_k) + \max \left\{ 1, |t|^{\frac{p_k}{H}} \right\} g_\Lambda(X_k - X). \tag{3.1}$$

Note that $g_\Lambda(X_k) \leq g_\Lambda(X) + g_\Lambda(X_k + X)$ for all $k \in \mathbb{N}$.

Hence by our assumption the right hand side of the relation (3.1) tends to 0 as $k \to \infty$ and the result follows. This completes the proof. □

Theorem 3.4. *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{S} = (S_k)$ *be sequences of Orlicz functions. Then the following hold:*

(i) $W_0^{I(F)}(\mathbf{S}, \Lambda, p) \subseteq W_0^{I(F)}(\mathbf{M}.\mathbf{S}, \Lambda, p)$, *provided* $p = (p_k)$ *be such that* $G_0 = \inf p_k > 0$.

(ii) $W_0^{I(F)}(\mathbf{M}, \Lambda, p) \cap W_0^{I(F)}(\mathbf{S}, \Lambda, p) \subseteq W_0^{I(F)}(\mathbf{M} + \mathbf{S}, \Lambda, p)$.

Proof. (i) Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0$ such that $\max \left\{ \varepsilon_1^G, \varepsilon_1^{G_0} \right\} < \varepsilon$. Choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M_k(t) < \varepsilon_1$ for each $k \in \mathbb{N}$. Let $X = (X_k)$ be any element in $W_0^{I(F)}(\mathbf{S}, \Lambda, p)$. Put

$$A_\delta = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) \right]^{p_k} \geq \delta^G \right\}.$$

Then by the definition of ideal we have $A_\delta \in I$. If $n \notin A_\delta$ we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) \right]^{p_k} < \delta^G$$

$$\Rightarrow \sum_{k=1}^{n} \left[S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) \right]^{p_k} < n\delta^G$$

$$\Rightarrow \left[S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) \right]^{p_k} < \delta^G, \text{ for } \ k = 1, 2, 3, ..., n$$

$$\Rightarrow S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) < \delta^G, for\ k = 1, 2, 3, ..., n. \tag{3.2}$$

Using the continuity of the function $\mathbf{M} = (M_k)$ from the relation (3.2) we have

$$M_k \left(S_k \left(\frac{\bar{d}(\lambda_k X_k, \bar{0})}{\rho} \right) \right) < \varepsilon_1, \text{ for } \ k = 1, 2, 3, ..., n.$$

Consequently we get

$$\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right)\right]^{p_k} < n.\max\left\{\varepsilon_1^{G},\varepsilon_1^{G_0}\right\} < n\varepsilon$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right)\right]^{p_k} < \varepsilon.$$

This implies that

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(S_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right)\right]^{p_k} \geq \varepsilon\right\} \subseteq A_\delta \in I.$$

This completes the proof.

(ii) Let $X=(X_k)\in W_0^{I(F)}(\mathbf{M},\Lambda,p)\cap W_0^{I(F)}(\mathbf{S},\Lambda,p)$. Then by the following inequality the result follows:

$$\frac{1}{n}\sum_{k=1}^{n}\left[(M_k+S_k)\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right]^{p_k}$$

$$\leq D\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right]^{p_k} + D\frac{1}{n}\sum_{k=1}^{n}\left[S_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right]^{p_k}.$$

□

The proof of the following theorems are easy and so omitted.

Theorem 3.5. *Let* $0<p_k\leq q_k$ *and* $\left(\frac{q_k}{p_k}\right)$ *is bounded, then*

$$W_0^{I(F)}(\mathbf{M},\Lambda,q)\subseteq W_0^{I(F)}(\mathbf{M},\Lambda,p).$$

Theorem 3.6. *For any two sequences* $p=(p_k)$ *and* $q=(q_k)$ *of positive real numbers, then the following holds:*

$Z(\mathbf{M},\Lambda,p)\cap Z(\mathbf{M},\Lambda,q)\neq\phi,$*, for* $Z=W^{I(F)}, W_0^{I(F)}, W_\infty^{I(F)}$ *and* W_∞^{F}.

Proposition 3.7. *The sequence spaces* $Z(\mathbf{M},\Lambda,p)$ *are normal as well as monotone, for* $Z=W_0^{I(F)}$ *and* $W_\infty^{I(F)}$.

Proof. We shall give the prove of the theorem for $W_0^{I(F)}(\mathbf{M},\Lambda,p)$ only. Let $X=(X_k)\in W_0^{I(F)}(\mathbf{M},\Lambda,p)$ and $Y=(Y_k)$ be such that $\bar{d}(Y_k,\bar{0})\leq\bar{d}(X_k,\bar{0})$ for all $k\in\mathbb{N}$. Then for given $\varepsilon>0$ we have

$$B=\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{\bar{d}(\lambda_k X_k,\bar{0})}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I.$$

Again the set $E=\left\{n\in\mathbb{N}: \frac{1}{n}\sum\limits_{k=1}^{n}\left[M_k\left(\frac{\bar{d}(\lambda_k Y_k,\bar{0})}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq B.$

Hence $E\in I$ and so $Y=(Y_k)\in W_0^{I(F)}(\mathbf{M},\Lambda,p)$. Thus the space $W_0^{I(F)}(\mathbf{M},\Lambda,p)$ is normal. Also from the Lemma 2.11, it follows that $W_0^{I(F)}(\mathbf{M},\Lambda,p)$ is monotone. □

Proposition 3.8. *If I is not a maximal ideal, then the space $W^{I(F)}(\mathbf{M},\Lambda,p)$ is neither normal nor monotone.*

Proof. We first prove that the space $W^{I(F)}(\mathbf{M},\Lambda,p)$ is not monotone. Let us consider a sequence $X=(X_k)$ of fuzzy numbers defined by

$$X_k(t)=\begin{cases}2^{-1}(1+t), & \text{if } t\in[-1,1];\\ 2^{-1}(-t+3), & \text{if } t\in[1,3];\\ 0, & \text{otherwise}\end{cases}$$

Then $(X_k)\in W^{I(F)}(\mathbf{M},\Lambda,p)$.

Since I is not maximal, so by Lemma 2.12, there exists a subset K in $\mathbb{N}$ such that $K\notin I$ and $\mathbb{N}-K\notin I$.

Let us define a sequence $Y=(Y_k)$ by

$$Y_k=\begin{cases}X_k, & \text{if } k\in K;\\ \bar{1}, & \text{otherwise}\end{cases}$$

Then $Y=(Y_k)$ belongs to the canonical pre-image of the K-step space of $(X_k)\in W^{I(F)}(\mathbf{M},\Lambda,p)$. But $(Y_k)\notin W^{I(F)}(\mathbf{M},\Lambda,p)$. Hence $W^{I(F)}(\mathbf{M},\Lambda,p)$ is not monotone. Therefore by Lemma 2.11, it follows that the space $W^{I(F)}(\mathbf{M},\Lambda,p)$ is not normal. □

Proposition 3.9. *If I is neither maximal nor $I=I_f$ then the spaces $W^{I(F)}(\mathbf{M},\Lambda,p)$ and $W_0^{I(F)}(\mathbf{M},\Lambda,p)$ are not symmetric.*

Proof. Let us consider a sequence $X=(X_k)$ of fuzzy real numbers defined by

$$X_k(t)=\begin{cases}1+t-2k, & \text{if } t\in[2k-1,2k];\\ 1-t+2k, & \text{if } t\in[2k,2k+1];\\ 0, & \text{otherwise}\end{cases}$$

for $k\in A\subset I$ an infinite set. Then $(X_k)\in W_0^{I(F)}(\mathbf{M},\Lambda,p)\subseteq W^{I(F)}(\mathbf{M},\Lambda,p)$. Let $K\subseteq\mathbb{N}$ be such that $K\notin I$ and $\mathbb{N}-K\notin I$ (the set K exists by Lemma 2.12, as I is not maximal).

Consider a sequence $Y=(Y_k)$ a rearrangement of the sequence (X_k) defined as follows:

$$Y_k=\begin{cases}X_k, & \text{if } k\in K;\\ \bar{1}, & \text{otherwise}\end{cases}$$

Then $(Y_k)\notin W_0^{I(F)}(\mathbf{M},\Lambda,p)$. Also $(Y_k)\notin W^{I(F)}(\mathbf{M},\Lambda,p)$. Hence $W^{I(F)}(\mathbf{M},\Lambda,p)$ and $W_0^{I(F)}(\mathbf{M},\Lambda,p)$ are not symmetric. □

Proposition 3.10. *If I is neither maximal nor $I=I_f$ then the space $W_\infty^{I(F)}(\mathbf{M},\Lambda,p)$ is not symmetric.*

Proof. Let us consider a sequence $X=(X_k)$ of $W_\infty^{I(F)}(\mathbf{M},\Lambda,p)$ defined by

$$X_k(t)=\begin{cases}1+t-3k, & \text{if } t\in[3k-1,3k];\\ 1-t+3k, & \text{if } t\in[3k,3k+1];\\ 0, & \text{otherwise}\end{cases}$$

for $k\in A\subset I$ an infinite set. Otherwise $X_k=\bar{1}$.

Since I is not maximal, so by Lemma 2.12, there exists a subset K in $\mathbb{N}$ such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $f : K \to A$ and $h : \mathbb{N} - K \to \mathbb{N} - A$ be bijections. Consider a sequence $Y = (Y_k)$ a rearrangement of the sequence (X_k) defined as follows:

$$Y_k = \begin{cases} X_{f(k)}, & \text{if } k \in K; \\ X_{h(k)}, & \text{if } k \in \mathbb{N} - K \end{cases}$$

Then $(Y_k) \notin W_\infty^{I(F)}(\mathbf{M}, \Lambda, p)$. Hence $W_\infty^{I(F)}(\mathbf{M}, \Lambda, p)$ is not symmetric. □

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