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On induced fuzzy supra-topological spaces

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Abstract. The concept of induced fuzzy topological space, introduced by Weiss [J. Math. Anal. Appl. 50(1975), 142-150], was defined with the notion of a lower semi-continuous function. The aim of this paper is to introduce and study the concepts of induced fuzzy supra-topological spaces and s-lower β-continuous functions. s-Lower β-continuous functions turn out to be the natural tool for studying the induced fuzzy supra-topological spaces.

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1. Introduction

In 1997 [6], Bhaumik and Mukherjee introduced and studied the concepts of induced fuzzy supra-topological spaces and s-Lower semi-continuous functions. In 1998 [10], Mukherjee defined and studied a new class of fuzzy supra topological space under the name of α-induced fuzzy supra topological space. Moreover, Mukherjee in 2003 [11], introduced the concept of $S-$induced $L-$ fuzzy supra topological space and Scott s-continuity. After the introduction of β-open subsets by Abd El-Monsef et al. [1], various concepts in topological space were introduced with the help of β-open subsets instead of open subsets. In Section 2, the concept of s-lower β-continuous function is introduced by using β-open subsets. Some characterizations and properties of these functions are examined. In Section 3, these functions are used to define a new class of fuzzy supra-topological spaces, called induced fuzzy supra-topological spaces. The fuzzy supra-continuous functions and initial supra-topological spaces are also investigated. The supra-interior and supra-closure of a fuzzy subset μ are denoted, respectively by μ^{si} and μ^{sc} [2, 9].

2. s-Lower β-continuous functions

Definition 2.1. A function $f : (X, \tau) \to (\mathbb{R}, \sigma)$ from a topological space (X, τ) to usual topology $(\mathbb{R}, \sigma)$ is said to be s-lower β-continuous (resp. s-upper β-continuous) at $x_0 \in X$ iff for each $\epsilon > 0$, there exists a β-open neighbourhood $N(x_0)$ such that $x \in N(x_0)$ implies $f(x) > f(x_0) - \epsilon$ (resp. $f(x) < f(x_0) + \epsilon$)

The following results can easily be proved analogous to the Theorem 2 in [3].

Result 2.2. *The necessary and sufficient condition for a real-valued function f to be s-lower β-continuous is that for all $r \in \mathbb{R}$, the set $\{x \in X : f(x) > r\}$is β-open (or equivalently $\{x \in X : f(x) \leq r\}$ is β-closed).*

Result 2.3. *The characteristic function of a β-open subset is s-lower β-continuous.*

Result 2.4. *The sum and product of two s-lower β-continuous functions are not necessarily s-lower β-continuous functions.*

Result 2.5. *If $\{f_i : j \in J\}$ is an arbitrary family of s-lower β-continuous functions, then the function g, defined by $g(x) = sup_j f_j(x)$ is s-lower β-continuous.*

Remark 2.6. If $f_1, f_2, f_3, ..., f_n$ are s-lower β-continuous functions, then the function h, defined by $h(x) = inf_i(f_i(x))$, where $i = 1, 2, ..., n$ is not s-lower β-continuous.

Result 2.7. *A function f from a topological space (X, τ) into a space $(\mathbb{R}, \sigma_1)$, where $\sigma_1 = \{(r, \infty) : r \in \mathbb{R}\}$ is s-lower β-continuous iff the inverse image of any open subset of $(\mathbb{R}, \sigma_1)$ is β-open in (X, τ).*

Definition 2.8 ([7])**.** Recall that a function $f : (X, \tau) \longrightarrow [0, 1]$ is called Scott continuous or (lower β-continuous) at $a \in X$ iff for every $\alpha \in [0, 1)$ with $\alpha < f(a)$ there is a neighborhood U of a such that $\alpha < f(x)$ for every $x \in U$. f is called Scott continuous (or lower semi continuous) on X iff f is Scott continuous (or lower semi continuous) at every point of X.

Since every open subset is β-open, we have the following result.

Result 2.9. *Every lower β-continuous function is s-lower β-continuous.*

The converse of the Result 2.9 is not true which can be seen from the following example.

Example 2.10. Let $X = \{a, b, c, d\}$ and $Y = \{0, 1, 2\}$. Let

$$\tau = \{X, \varnothing, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\} \text{ and } \tau_1 = \{Y, \varnothing, \{2\}\}$$

be the topologies on X and Y, respectively. A function $g : X \to Y$ is defined by $g(a) = g(d) = 2, g(b) = 1$ and $g(c) = 0$. Now $g^{-1}(0) = \{c\}$, $g^{-1}(1) = \{b\}$ and $g^{-1}(2) = \{a, d\}$. We observe that $\{a, d\}$ is β-open in (X, τ), since $\{d\} \subseteq \{a, d\} \subseteq \{a, b, d\} = C_1\{d\}$. For all $r \in Y$, by Result 2.2, g is s-lower β-continuous. Since inverse image of the open subset $\{2\}$ of Y is β-open, g is not lower β-continuous.

2.1. **Initial supra-topology.** Finally we shall define an initial supra-topology on X.

Definition 2.11. Let $(X, \wp(\tau))$ be an induced fuzzy supra-topological space. The family $\{\sigma_r(\alpha) : \alpha \in \wp(\tau), r \in I\}$ of all β-open subsets of X form a supra-topology on X, called the initial supra-topology on X and is denoted by $i(\wp)$. $(X, i(\wp))$ is called the initial supra-topological space. Thus the relation between the initial supra-topology and the corresponding topology τ of $\wp(\tau)$ is $\tau \subseteq i(\wp)$.

Example 2.12. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$ be a topology on X. Besides the members of τ, $\{a, d\}, \{b, d\}$ and $\{b, c, d\}$ are also β-open subsets in (X, τ).
Now $1_\varnothing$, 1_X, $1_{\{c\}}$, $1_{\{d\}}$, $1_{\{c,d\}}$, $1_{\{a,c,d\}}$, $1_{\{a,d\}}$, $1_{\{b,d\}}$ and $1_{\{b,c,d\}}$ are s-lower β-continuous, since the characteristic function of a β-open subset is s-lower β-continuous. Thus the collection of all these functions forms an induced fuzzy supra-topology on X. Here $i(\wp) = \{\varnothing, X, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{b, c, d\}, \{a, c, d\}$. Thus $\tau \subseteq i(\wp)$.

Note. If we take $\beta O(X)$, the family of all β-open subsets of X, then $\beta O(X) = i(\wp)$.

3. Induced fuzzy supra-topological space

The notion of supra-topology or pre-topology was due to Garg and Naimpally [8] and that of induced fuzzy topology was due to Weiss [13]. Abd E1-Monsef and Ramadan [2] introduced the concept of fuzzy supra-topology as follows: A family $\mathcal{F}' \subseteq 1^X$ is called a fuzzy supra-topology on X if $0, 1 \in \mathcal{F}'$ and $\mathcal{F}'$ is closed under arbitrary union. In this section the notion of induced fuzzy supra-topology is introduced as a generalization of induced fuzzy topology. Its properties and the concepts of fuzzy supra-continuity in induced supra-topological spaces and initial supra-topology are also studied.

3.1. **Induced fuzzy supra-topology and its properties.**

Theorem 3.1. *Let (X, τ) be a topological space. The family of all s-lower β-continuous functions from the topological space (X, τ) to the closed unit interval I forms a fuzzy supra-topology on X.*

Proof. Let $\wp$ be the collection of all s-lower β-continuous functions from the topological space (X, τ) to I. We will now prove that $\wp$ is a fuzzy supra-topology on X.

(i) Since X is open, it is β-open and by Result 2.3, 1_x is s-lower β-continuous. Thus $1_x \in \wp$.

(ii) $\varnothing$ is β-open since it is open in X. Thus $1_\varnothing$is s-lower β-continuous, i.e. $1_\varnothing \in \wp$.

(iii) Let $\{n_j\}$ be an arbitrary family of s-lower β-continuous functions. By Result 2.5, $Sup\{n_j\}$ is also s-lower β-continuous. Hence $\bigvee n_j \in \wp$.

Thus $\wp$ satisfies conditions $(i)-(iii)$ of supra-topology.Hence $\wp$ forms a fuzzy supra-topology. □

Definition 3.2. The fuzzy supra-topology, obtained as above, is called induced fuzzy supra-topology and the space $(X, \wp(\tau))$ is called the induced fuzzy supra-topological space. The members of $\wp(\tau)$ are called fuzzy supra-open subsets.

Theorem 3.3. *A fuzzy subset α in an induced fuzzy supra-topological space $(X, \wp(\tau))$ is fuzzy supra-open iff for each $r \in I$, the strong r-cut $\sigma_r(\alpha)$ is β-open in the topological space (X, τ).*

Proof. A fuzzy subset α is fuzzy supra-open in $(X, \wp(\tau))$ if $\alpha \in \wp(\tau)$ iff α is s-lower β-continuous (by Theorem 3.1). Thus for each $r \in I$, $\{x \in X : \alpha(X) > r\}$ is β-open in (X, τ) (by Result 2.2). That is, $\sigma_r(\alpha)$ is β-open in (X, τ). □

Corollary 3.4. *A fuzzy subset α in an induced fuzzy supra-topological space $(X, \wp(\tau))$ is fuzzy supra-closed iff for each $r \in I$, the weak r-out $W_r(\alpha)$ is β-closed in the topological space (X, τ).*

Theorem 3.5. *If A is β-open in (X, τ), then 1_A is fuzzy supra β-open in $(X, \wp(\tau))$.*

Proof. Let A be β-open in (X, τ), then $A \subseteq cl(int(cl(A)))$, i.e. $1_A \subseteq 1_{clintclA} = cl1_{intclA} = clint1_{clA} = clintcl1_A$. Then 1_A is fuzzy supra β-open in $(X, \wp(\tau))$. □

The following theorem follows immediately from Result 2.9.

Theorem 3.6. *If $\mathcal{F}$ is an induced fuzzy topology and $\wp$ is an induced supra-topology on X, then $\mathcal{F} \subseteq \wp$.*

In [4] the completely induced fuzzy topology was introduced and by Lemma 2.4 of [5] we have the following corollary.

Corollary 3.7. *$\Im \subseteq \mathcal{F} \subseteq \wp$, where $\Im$ is a completely induced fuzzy topology.*

Let $(X, \mathcal{F})$ be a fuzzy topological space and $\mathcal{F}'$ be a fuzzy supra-topology on X. We call $\mathcal{F}'$ a fuzzy supra-topology associated with $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$. The family $F\beta O(X)$ of all fuzzy β-open subsets in $(X, \mathcal{F})$ is fuzzy supra-topology associated with $\mathcal{F}$.

Theorem 3.8. *Let $(X, \mathcal{F}')$ be a fuzzy supra-topological space where $\mathcal{F}'$ is associated with the fuzzy topology $\mathcal{F}$ on X and $\tau = \mathcal{F} \cap 2^X$. Then the induced fuzzy supra-topology $\wp(\tau)$ on X is equivalent to the fuzzy supra-topology $\mathcal{F}'$ if for any fuzzy subset μ and $r \in I$, $W_r(\mu^{sc}) = \bigcap\{Cl_\tau(W_t(\mu)) : t < r\}$ (resp.$\sigma_r(\mu^{si}) = \bigcup\{Int_\tau(\sigma_t(\mu)) : t > r\}$) is β-closed (resp. β-open) in (X, τ).*

Proof. Let μ be $\mathcal{F}'$ $-$closed subset of X and $r \in I$, then by the given condition, $W_r(\mu)W_r(\mu^{sc}) = \bigcap\{Cl_\tau(W_t(\mu)) : t < r\}$ is β-closed in (X, τ) . By Theorem 3.3, μ is $\wp(\tau)$-closed. Now if α is $\wp(\tau)$-closed and $r \in I$, then

$$W_r(\alpha^{sc}) = \bigcap\{Cl_\tau(W_t(\alpha)) : t < r\} = \bigcap\{W_t(\alpha) : t < r\} = W_r(\alpha).$$

Thus $\alpha^{Sc} = \alpha$, i.e. α is $\mathcal{F}'$ −closed. This completes the proof. □

Analogous to Theorem 3.18 of [12], we have the following theorem.

Theorem 3.9. *If $\wp(\tau)$ is induced fuzzy supra-topology on X, then*

$$\mathcal{F}_{\wp(\tau)} = \{\alpha \subseteq X : \mu \in \wp(\tau) \Rightarrow \alpha \cap \mu \in \wp(\tau)\}$$

is a fuzzy topology on X and $\mathcal{F}_{\wp(\tau)} \subseteq \wp(\tau)$.

3.2. **Fuzzy supra-continuity in induced fuzzy supra-topological spaces.**

Definition 3.10. Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}_1)$ be fuzzy topological spaces and $\mathcal{F}'$ be an associated fuzzy supra-topology with $\mathcal{F}$. A function $f : X \to Y$ is a fuzzy S-continuous if the inverse image of each fuzzy open subset in Y is $\mathcal{F}'$ -supra-open in X.

Theorem 3.11. *Let $f : (X, \wp(\tau)) \to (Y, \mathcal{F})$ be a function from an induced fuzzy supra-topological space $(X, \wp(\tau))$ into a fuzzy topological space $(Y, \mathcal{F})$. Then the following statements are equivalent:*

(i) *f is fuzzy S-continuous..*
(ii) *The inverse image of each fuzzy closed subset in Y is $\wp(\tau)$-closed.*
(iii) *$(f^{-1}(\gamma))^{Sc} \subseteq f^{-1}(Cl\gamma)$ for any fuzzy subset γ in Y.*
(iv) *$f(\alpha^{Sc}) \subseteq Cl(f(\alpha))$ for any fuzzy subset α in X,*
(v) *For any fuzzy point x_p in X and fuzzy open subset γ in Y containing $f(x_p)$, there exists $\alpha \in \wp(\tau)$ such that $x_p \in \alpha$ and $f(\alpha) \subseteq \gamma$.*

Proof. It is straightforward and hence omitted. □

In [2] fuzzy supra-continuity was defined as follows: Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be two fuzzy topological spaces, $(X, \mathcal{F}_1^{'})$ and $(X, \mathcal{F}_2^{'})$ be two associated fuzzy supra-topological spaces with $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. A function $f : X \to Y$ is a fuzzy supra-continuous if the inverse image of $\mathcal{F}_2^{'}$-supra-open subset is $\mathcal{F}_1^{'}$-supra-open. Also we know that a function $f : (X, \tau) \to (Y, \tau_1)$ is β-irresolute if the inverse of β-open subset is β-open.

Theorem 3.12. *Let $\wp(\tau)$ and $\wp(\tau_1)$ be two induced fuzzy supra-topological associated with $\mathcal{F}$ and $\mathcal{F}_1$. Then a function $f : (X, \mathcal{F}) \to (Y, \mathcal{F}_1)$ is fuzzy supra-continuous iff $f : (X, \tau) \to (Y, \tau_1)$ is β-irresolute function.*

Proof. Let f be a fuzzy supra-continuous function and A be a β-open subset in (Y, τ_1). Then $1_A \in \wp(\tau)$ and

$$\begin{aligned} f^{-1}(A) &= \{x \in X : 1_A(f(X)) = 1\} \\ &= \{x \in X : f^{-1}(1_A(x)) > r \text{ and } 0 < r < 1\} \\ &= \sigma_r(f^{-1}(1_A)). \end{aligned}$$

Thus $f^{-1}(1_A)$ is fuzzy supra-open. Since f is fuzzy supra-continuous. By Theorem 3.3, $\sigma_r(f^{-1}(1_A))$ is β-open in the topological space (X, τ). Thus f is β-irresolute function.

Conversely, let $f : (X, \tau) \to (Y, \tau_1)$ be β-irresolute function and α is a fuzzy supra-open subset in $(Y, \wp(\tau_1))$. Now for $r > 0$,

$$\sigma_r(f^{-1}(\alpha)) = \{x \in X : f^{-1}(\alpha(x)) > r\} = (\alpha f)^{-1}(r, \infty) = f^{-1}(\alpha^{-1}(r, \infty)).$$

Since $\alpha \in \wp(\tau_1)$, α is s-lower β-continuous and then $(\alpha)^{-1}(r, \infty)$ is β-open in (Y, τ_1). Also by hypothesis, $f^{-1}(\alpha^{-1}(r, \infty))$ is β-open in (X, τ),i.e. $\sigma_r(f^{-1}(\alpha))$ is β-open in (X, τ) which implies $f^{-1}(\alpha) \in \wp(\tau)$. Hence the theorem. □

Fuzzy supra-open function is defined in [2] as follows:
A function f from a fuzzy supra-topological space $(X, \mathcal{F}'_1)$ into a fuzzy supra-topological space $(Y, \mathcal{F}'_2)$ is called fuzzy supra-open if $f(\alpha) \in \mathcal{F}'_2$ for each $\alpha \in \mathcal{F}'_1$. We have the following theorem.

Theorem 3.13. *Let $(X, \wp(\tau_1))$ and $(Y, \wp(\tau_2))$ be two induced fuzzy supra-topological spaces. If $f : (X, \wp(\tau_1)) \to (Y, \wp(\tau_2))$ is an injective fuzzy supra-continuous and fuzzy supra-open, then $f : \left(X, \mathcal{F}'_{\wp(\tau_1)}\right) \to \left(Y, \mathcal{F}'_{\wp(\tau_2)}\right)$ is fuzzy continuous.*

Proof. Let $f : (X, \wp(\tau_1)) \to (Y, \wp(\tau_2))$ be an injective fuzzy supra-continuous supra-open function. If $\mu \in \wp(\tau_1)$ then $f(\mu) \in \wp(\tau_2)$ by supra-open function. Now for each $\alpha \in \mathcal{F}_{\wp}(\tau_2)$, $\alpha \cap f(\mu) \in \wp(\tau_2)$ by Theorem 3.9. Then $f^{-1}(\alpha \cap f(\mu)) = f^{-1}(\alpha) \cap \mu \in \wp(\tau_1)$ by injective supra-continuity of f. Thus $f^{-1}(\alpha) \in \mathcal{F}_{\wp(\tau_1)}$ for each $\alpha \in \mathcal{F}_{\wp(\tau_2)}$ which proves that $f : \left(X, \mathcal{F}_{\wp(\tau_1)}\right) \to \left(Y, \mathcal{F}_{\wp(\tau_2)}\right)$ is fuzzy continuous. □

4. Conclusion

In this paper We studied the concepts of induced fuzzy supra-topological spaces and s-lower β-continuous functions. We deduced the properties of induced fuzzy supra-topological spaces. Fuzzy supra-continuity in induced fuzzy supra-topological spaces are defined. Finally, we defined the Initial supra-topology.

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