Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 7, No. 2, (February 2014), pp. 275–279
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Some improved results on the convexity of fuzzy nets

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Received 6 June 2013; Accepted 7 July 2013

Abstract. In this paper, we give some improved results on the ϑ-convex hull of fuzzy net in the literature.

2010 AMS Classification: 03E72, 08A72

Keywords: Fuzzy sets, Fuzzy nets, Convexity, Pseudo-partial order.

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1. Introduction

As a suitable mathematical model to handle vagueness and uncertainty, fuzzy set theory is emerging as a powerful theory and has attracted the attention of many researchers and practitioners who contributed to its development and applications (see, for instance, [2, 3, 4, 5, 7, 8, 10, 13, 14, 15, 17, 18, 19] and the references therein). Convexity plays a most useful role in the theory and applications of fuzzy sets. Therefore, the research on convexity and generalized convexity is one of the most important aspects of fuzzy set theory ([1, 6, 11, 12, 16]). In the paper [9], a natural generalization of the concept of fuzzy sets under the name of fuzzy nets is given and then their convexity properties is investigated. In this paper, we give some improved results on the ϑ-convex hull of fuzzy net in [9].

2. Preliminaries

Let $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. A fuzzy set in $\mathbb{R}^d$ is a function with domain $\mathbb{R}^d$ and values in the closed interval $[0,1]$. For a fuzzy set μ, the subset of $\mathbb{R}^d$ in which μ assumes nonzero values, is known as the support of A (see [20]).

The definition of a convex fuzzy set can be rewritten as follows: the fuzzy set $\mu : \mathbb{R}^d \to [0,1]$ is said to be convex if

$$\mu(tx + (1-t)y) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in \mathbb{R}^d$, and $t \in [0,1]$.

Let J be an index set. A J-tuple of elements of $[0,1]$ is a function $t : J \to [0,1]$. If $\beta \in J$, then denote the value of t at β by t_β which is βth coordinate of t. Denote the set of all J-tuples of elements of $[0,1]$ by $[0,1]^J$.

Remark 2.1. In mathematical terms, the J-tuple of elements of $[0,1]$ are the same as the fuzzy sets of J.

Definition 2.2 ([9])**.** For an index set J, a fuzzy net $\mu : \mathbb{R}^d \to [0,1]^J$ of $\mathbb{R}^d$ is defined by $\mu(x) = (\mu_i(x))_{i\in J}$, where each μ_i is a fuzzy set of $\mathbb{R}^d$, and $(\mu_i(x))_i \in J$ is a J-tuple.

Remark 2.3. If $J \subseteq [0,1]$, then indeed the fuzzy nets are tpye-2 fuzzy sets [21]. So the concept of "fuzzy net" can be regard as a generalization of type-2 fuzzy set.

Denote the set of all fuzzy nets of $\mathbb{R}^d$ by $\mathscr{F}\mathscr{N}(\mathbb{R}^d)$ and let ϑ be a fuzzy set valued function $\vartheta : \mathscr{F}\mathscr{N}(\mathbb{R}^d) \to [0,1]^{\mathbb{R}^d}$ which maps each element μ in $\mathscr{F}\mathscr{N}(\mathbb{R}^d)$ to a fuzzy set $\vartheta(\mu)$ of $\mathbb{R}^d$.

Definition 2.4 ([9])**.** A fuzzy net $\mu : \mathbb{R}^d \to [0,1]^J$ is called ϑ-convex if $\vartheta(\mu)$ is a convex fuzzy set of $\mathbb{R}^d$.

Definition 2.5 ([9])**.** A fuzzy net λ is said to be ϑ-dominated by the fuzzy net μ and denoted by $\mu \geq_\vartheta \lambda$ if $\vartheta(\mu)(x) \geq \vartheta(\lambda)(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.6. For the pseudo-partial order $\geq_\vartheta$ induced by ϑ on $\mathscr{F}\mathscr{N}(\mathbb{R}^d)$, if $\mu \geq_\vartheta \lambda$ and $\lambda \geq_\vartheta \mu$, then $\mu =_\vartheta \lambda$, but there may not be $\mu = \lambda$, where $\mu = \lambda$ is in the sense of that $\mu(x) = \lambda(x)$ for all $x \in \mathbb{R}^d$. In other words, if ϑ is not injective, then there exist fuzzy nets $\mu, \lambda \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$ such that $\mu \neq \lambda$ but $\mu =_\vartheta \lambda$.

Definition 2.7 ([9])**.** The ϑ-convex hull of a fuzzy net λ is defined to be

$$inf\{\mu \in \mathscr{F}\mathscr{N}(\mathbb{R}^d) : \mu \geq_\vartheta \lambda,\ \mu \text{ is } \vartheta\text{-convex}\}$$

and will be denoted by $co_\vartheta(\lambda)$.

3. Major section

For any fuzzy net $\lambda \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, let

$$A = \{\mu \in \mathscr{F}\mathscr{N}(\mathbb{R}^d) : \mu \geq_\vartheta \lambda,\ \mu \text{ is } \vartheta\text{-convex}\}.$$

In the pseudo-partially ordered set $(\mathscr{F}\mathscr{N}(\mathbb{R}^d), \leq_\vartheta)$, the infimum of subset A is an element ν of $\mathscr{F}\mathscr{N}(\mathbb{R}^d)$ which must meet the following conditions

(1) $\nu \leq_\vartheta \mu$, for all $\mu \in A$ and

(2) for all $\omega \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, if $\omega \leq_\vartheta \mu$, for all $\mu \in A$, then $\omega \leq_\vartheta \nu$.

Now, we show that Definition 2.4 is not proper by the following counterexamples.

Example 3.1. Let $d = 1$ and $J = [0,1]$. First, we arbitrarily fix an element λ and a denumerable subset of $\mathscr{F}\mathscr{N}(\mathbb{R})$ as $B = \{\mu^i \in \mathscr{F}\mathscr{N}(\mathbb{R}) : i = 1,2,3,\cdots\}$ such that $\lambda \notin B$. Then define the fuzzy set valued function ϑ as

$$\vartheta(\mu) = \begin{cases} \omega^i, & if\ \mu = \mu^i \in B, \\ \gamma, & if\ \mu = \lambda, \\ \tau, & otherwise, \end{cases}$$

where ω^i, γ and τ are fuzzy sets of $\mathbb{R}$ defined as

$$\omega^i(x) = \begin{cases} \dfrac{i}{i+1}x + \dfrac{1}{i+1}, & if\ x \in [0,1] \\ 0, & otherwise, \end{cases}$$

and

$$\gamma(x) = \begin{cases} x, \ if\ x \in [0, \frac{1}{2}), \\ 1-x, \ if\ x \in (\frac{1}{2}, 1], \\ 0, \quad otherwise, \end{cases} \quad \tau(x) = \begin{cases} 1-2x, & if\ x \in [0, \frac{1}{2}], \\ 0, & otherwise. \end{cases}$$

For the fuzzy net λ, by a simple calculation we have that $A = B$. However, the set B does not have the infimum. Thus λ does not have the ϑ-convex hull.

Example 3.2. Let $d = 1$ and $J = [0, 1]$. First, we arbitrarily fix two elements λ, ν and a denumerable subset of $\mathscr{F}\mathscr{N}(\mathbb{R})$ as $B = \{\mu^i \in \mathscr{F}\mathscr{N}(\mathbb{R}) : i = 1, 2, 3, \cdots\}$ such that $\lambda, \nu \notin B$. Then define the fuzzy set valued function ϑ as

$$\vartheta(\mu) = \begin{cases} \omega^i, \ if\ \mu = \mu^i \in B, \\ \gamma, \ \ if\ \mu = \lambda, \\ \delta, \ \ if\ \mu = \nu, \\ \tau, \ \ otherwise, \end{cases}$$

where ω^i, γ, δ and τ are fuzzy sets of $\mathbb{R}$ defined as

$$\omega^i(x) = \begin{cases} \dfrac{i}{i+1}x + \dfrac{1}{i+1}, & if\ x \in [0,1] \\ 0, & otherwise, \end{cases} \quad \gamma(x) = \begin{cases} x, & if\ x \in [0, \frac{1}{2}), \\ 1-x, & if\ x \in (\frac{1}{2}, 1], \\ 0, & otherwise, \end{cases}$$

$$\delta(x) = \begin{cases} x, & if\ x \in [0, \frac{1}{2}), \\ \frac{1}{3}, & if\ x = \frac{1}{2}, \\ 1-x, & if\ x \in (\frac{1}{2}, 1], \\ 0, & otherwise, \end{cases} \quad \tau(x) = \begin{cases} 1-2x, & if\ x \in [0, \frac{1}{2}], \\ 0, & otherwise, \end{cases}$$

For the fuzzy net λ, by a simple calculation we have that $A = B$ and $infA = \nu$ which implies $\nu = co_\vartheta(\lambda)$. However, ν is not a ϑ-convex fuzzy net because δ is not a convex fuzzy set.

Example 3.3. In Example 3.2, if we define δ as

$$\delta(x) = \begin{cases} \frac{1}{2}x, & if\ x \in [0,1], \\ 0, & otherwise, \end{cases}$$

then we have $A = B$ and $infA = \nu$ which implies $\nu = co_\vartheta(\lambda)$. However, λ is not ϑ-dominated by ν since the inequality $\delta(x) \geq \gamma(x)$ does not hold for all $x \in \mathbb{R}$.

Theorem 3 in [9] states that if ϑ is surjective, then $\vartheta(co_\vartheta(\lambda)) = co_\vartheta(\vartheta(\lambda))$ (indeed, it should be $co(\vartheta(\lambda))$. However, from the above examples, the first thing has to do is to prove the existence of the ϑ-convex hull. Therefore, in order to define the ϑ-convex hull of a fuzzy net, we must add some assumptions to Definition 2.4.

Definition 3.4. For a fuzzy net λ, if the infimum of the set

$$A = \{\mu \in \mathscr{F}\mathscr{N}(\mathbb{R}^d) : \mu \geq_\vartheta \lambda,\ \mu \text{ is } \vartheta\text{-convex}\}$$

exists and belongs to A, then $infA$ is called a ϑ-convex hull of the fuzzy net λ and is denoted by $co_\vartheta(\lambda)$.

Remark 3.5. Form Remark 2.3, it follows that if the ϑ-convex hull of the fuzzy net λ exists, it does not have to be unique since the infimum of set A does not have to be unique.

Theorem 3.6. *For a fuzzy net λ, if it has a ϑ-convex hull, then $\vartheta(co_\vartheta(\lambda)) \geq co(\vartheta(\lambda))$.*

Proof. Form Definition 2.2 and Definition 3.4, it follows that $co_\vartheta(\lambda) \geq_\vartheta \lambda$ and $\vartheta(co_\vartheta(\lambda))$ is a convex fuzzy set. Consequently, we have $\vartheta(co_\vartheta(\lambda)) \geq co(\vartheta(\lambda))$. □

Theorem 3.7. *If the range of ϑ contains all of the convex fuzzy sets of $\mathbb{R}^d$, then for any fuzzy net $\lambda \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, it has a ϑ-convex hull and $\vartheta(co_\vartheta(\lambda)) = co(\vartheta(\lambda))$.*

Proof. For any fuzzy net $\lambda \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, let

$$A = \{\mu \in \mathscr{F}\mathscr{N}(\mathbb{R}^d) : \mu \geq_\vartheta \lambda,\ \mu \text{ is } \vartheta\text{-convex}\},$$

$$B = \{\nu \in [0,1]^{\mathbb{R}^d} : \text{ there exists a fuzzy net } \mu \in A \text{ such that } \vartheta(\mu) = \nu\}.$$

Now we have that the fuzzy set defined as $\cap_{\nu\in B}\nu$ is a convex fuzzy set and $\nu \geq \cap_{\nu\in B}\nu \geq \vartheta(\lambda)$ for all $\nu \in B$. By the hypothesis, there exists a fuzzy net ξ such that $\vartheta(\xi) = \cap_{\nu\in B}\nu$. Thus $\xi \in A$ and $\xi \leq_\vartheta \mu$ for all $\mu \in A$. On the other hand, for any $\omega \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, if $\omega \leq_\vartheta \mu$, for all $\mu \in A$, then $\vartheta(\omega) \leq \nu$, for all $\nu \in B$. Thus $\vartheta(\omega) \leq \cap_{\nu\in B}\nu$ which implies $\omega \leq_\vartheta \xi$. Therefore, ξ is a ϑ-convex hull of the fuzzy net λ. By Theorem 3.6, we have $\vartheta(co_\vartheta(\lambda)) \geq co(\vartheta(\lambda))$. Conversely, let ν be a convex fuzzy set with $\nu \geq \vartheta(\lambda)$. By the hypothesis, there exists a fuzzy net μ such that $\vartheta(\mu) = \nu$. It is clear that μ is a ϑ-convex fuzzy net and $\mu \geq_\vartheta \lambda$. Thus $\nu \in B$ and $\nu \geq \vartheta(co_\vartheta(\lambda))$. Because ν is arbitrary, we have $co(\vartheta(\lambda)) \geq \vartheta(co_\vartheta(\lambda))$. We complete the whole proof here. □

Corollary 3.8. *If ϑ is surjective, then for any fuzzy net $\lambda \in \mathscr{F}\mathscr{N}(\mathbb{R}^d)$, it has a ϑ-convex hull and $\vartheta(co_\vartheta(\lambda)) = co(\vartheta(\lambda))$.*

Proof. Since the range of ϑ contains all of the convex fuzzy sets of $\mathbb{R}^d$, it follows quickly from Theorem 3.7. □

Acknowledgements. The authors thank the anonymous reviewers for their valuable comments. This work was supported by The National Natural Science Foundation of China (Grant No. 11201512) and The Natural Science Foundation Project of CQ CSTC (cstc2012jjA00001).

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