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Positive implicative filters of BE-algebras

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Abstract. The concept of positive implicative filters is introduced in BE-algebras. A set of equivalent conditions is established for every filter of a BE-algebra to become a positive implicative filter. Fuzzification of positive implicative filters of BE-algebras is considered. Some properties of fuzzy positive implicative filters are studied with respect to fuzzy relations and cartesian products of fuzzy sets. Normalization is also considered for fuzzy positive implicative filters of BE-algebras.

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1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [8]. Some properties of filters of BE-algebras were studied by S.S. Ahn and K.S. So in [1] and then by H.S. Kim and Y.H. Kim in [8]. The concepts of a fuzzy set and a fuzzy relation on a set was initially defined by L.A. Zadeh [12]. Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee [3]. In 1996, Y.B. Jun and S.M. Hong [5] discussed the fuzzy deductive systems of Hilbert algebras. Y. B. Jun [6] also studied the properties of fuzzy positive implicative filters in lattice implication algebras. Some properties of sub algebras of BE-algebras were studied by A. Rezaei and A. B. Saeid in [9]. Later, W.A. Dudek and Y.B. Jun [4] considered the fuzzification of ideals in Hilbert algebras and discussed the relation between fuzzy ideals and fuzzy deductive systems. Properties of anti fuzzy ideals and fuzzy quasi-ideals are studied in [2] and [7]. In [10], the author introduced the notion of fuzzy filters in BE-algebras and discussed some related properties. Recently, the concept of fuzzy implicative filters [11] is introduced and studied the properties of these filters in BE-algebras.

In this paper, the notion of positive implicative filters is introduced in BE-algebras as a generalization of fuzzy positive implicative filters of lattice implication algebras. A set of equivalent conditions is established for every filter of a BE-algebra to become a positive implicative filter. The concept of fuzzy positive implicative filters is also introduced in a BE-algebra. Some properties of fuzzy positive implicative filters are studied. A necessary and sufficient condition is derived for every fuzzy filter of a BE-algebra to become a fuzzy positive implicative filter. Properties of fuzzy positive implicative filters are studied in terms of fuzzy relations and cartesian products. Finally, the concept of normality is introduced for fuzzy positive implicative filters of BE-algebras. An extension property is also derived for normal fuzzy positive implicative filters of BE-algebras. The notion of maximal fuzzy positive implicative filters is introduced and obtained a relation between a maximal fuzzy positive implicative filter and normal fuzzy positive implicative filter of a BE-algebra.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [8], [10], [11] and [12] for the ready reference of the reader.

Definition 2.1 ([8])**.** An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$
(2) $x * 1 = 1$
(3) $1 * x = x$
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$

Theorem 2.2 ([8])**.** *Let $(X, *, 1)$ be a BE-algebra. Then we have the following:*

(1) $x * (y * x) = 1$
(2) $x * ((x * y) * y)) = 1$

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ implies $x * y = 1$. A BE-algebra X is called self-distributive if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$.

Definition 2.3 ([8])**.** Let $(X, *, 1)$ be a BE-algebra. A non-empty subset F of X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(a) $1 \in F$
(b) $x \in F$ and $x * y \in F$ imply that $y \in F$

Definition 2.4 ([12])**.** Let X be a set. Then a fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.5 ([10])**.** Let X be a BE-algebra. A fuzzy set μ of X is called a fuzzy filter if it satisfies the following properties, for all $x, y \in X$:

(F_1) $\mu(1) \geq \mu(x)$
(F_2) $\mu(y) \geq min\{\mu(x), \mu(x * y)\}$

Definition 2.6 ([10])**.** Let μ be a fuzzy set in a BE-algebra X. For any $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of X.

Definition 2.7 ([11])**.** A fuzzy relation on a set S is a fuzzy set $\mu : S \times S \longrightarrow [0, 1]$.

Definition 2.8 ([11])**.** Let μ be a fuzzy relation on a set S and ν a fuzzy set in S. Then μ is a fuzzy relation on ν if for all $x, y \in S$, it satisfies

$$\mu(x, y) \leq min\{\nu(x), \nu(y)\}$$

Definition 2.9 ([11])**.** Let μ and ν be two fuzzy sets in a BE-algebra X. Then the cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\}$$

for all $x, y \in X$.

3. Positive implicative filters

In this section, the notion of positive implicative filters is introduced in BE-algebras. Some properties of positive implicative filters of BE-algebras are studied. A necessary and sufficient conditions is derived for any filter of a BE-algebra to become a positive implicative filter.

Definition 3.1. A subset F of a BE-algebra X is called a positive implicative filter if, for all $x, y, z \in X$, it satisfies the following properties.

$(F1)$ $1 \in F$

$(F2)$ $x * ((y * z) * y) \in F$ and $x \in F$ imply that $y \in F$

Example 3.2. Let $X = \{1, a, b, c\}$ be a non-empty set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. It is easy to check that $F = \{b, c, 1\}$ is a positive implicative filter in X.

Proposition 3.3. *Every positive implicative filter of a BE-algebra is a filter.*

Proof. Let F be a positive implicative filter of a BE-algebra X. Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then we get $x * ((y * y) * y) = x * y \in F$ and $x \in F$. Hence by condition $(F2)$, it yields that $y \in F$. Therefore F is a filter in X. □

Since every positive implicative filter of a BE-algebra is a filter, the following corollary is a direct consequence of the above proposition.

Corollary 3.4. *Let F be a positive implicative filter of X. Then the following hold.*

(1) $x \in F$ *and* $x \leq y$ *imply that* $y \in F$

(2) $x \leq y * z$ *implies* $z \in F$

The converse of the above Proposition 3.3 is not true. It can be seen in the following example.

Example 3.5. Let $X = \{1, a, b, c\}$ be a non-empty set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	1
c	1	b	c	1

Then $(X, *, 1)$ is a BE-algebra. It is clear that $F = \{1\}$ is a filter in X but it is not a positive implicative filter because of $1 * ((c * a) * c) = 1 \in F$ and $1 \in F$. We can see that $c \notin F$. Hence $F = \{1\}$ is a filter but not a positive implicative filter in X.

However, we now derive a set of equivalent conditions for every filter of a BE-algebra to become a positive implicative filter.

Theorem 3.6. *Let F be a filter of a BE-algebra X. Then the following conditions are equivalent.*

(1) F is a positive implicative filter
(2) for all $x, y \in X$, $(x * y) * x \in F$ implies $x \in F$
(3) for all $a, x, y \in X$, $a \in F, (x * y) * (a * x) \in F$ implies $x \in F$

Proof. (1) $\Rightarrow$ (2): Assume that F is a positive implicative filter of X. Let $x, y \in X$ and $(x * y) * x \in F$. Then clearly $1 * ((x * y) * x) = (x * y) * x \in F$. Since $1 \in F$ and F is positive implicative, we get that $x \in F$.
(2) $\Rightarrow$ (3): Assume that the condition (2) holds in X. Let $a, x, y \in X$. Suppose $a \in F$ and $(x * y) * (a * x) \in F$. Then we have

$$a * ((x * y) * x) = (x * y) * (a * x) \in F$$

Since $a \in F$ and F is a filter in X, we get that $(x * y) * x \in F$. Therefore by the condition (2), it concludes that $x \in F$.
(3) $\Rightarrow$ (1): Assume that the condition (3) holds in X. Let $x, y \in X$. Suppose $x * ((y * z) * y) \in F$ and $x \in F$. Then we have

$$(y * z) * (x * y) = x * ((y * z) * y) \in F$$

Since $x \in F$, by the assumed condition (3), we get that $y \in F$. Therefore F is a positive implicative filter in X. ☐

4. Fuzzification of positive implicative filters

In this section, the concept of fuzzy positive implicative filters is introduced in a BE-algebra. Some properties of these classes of filters are then studied with respect to fuzzy relations and cartesian products.

Definition 4.1. A fuzzy set μ of a BE-algebra X is called a fuzzy positive implicative filter of X if it satisfies the following properties.

(1) $\mu(1) \geq \mu(x)$
(2) $\mu(y) \geq min\{\mu(x), \mu(x * ((y * z) * y))\}$ for all $x, y, z \in X$

If we replace z in the condition (2) of the above definition by 1, then it can be easily observed that every fuzzy positive implicative filter of a BE-algebra is a fuzzy filter. However every fuzzy filter of a BE-algebra is not a fuzzy positive implicative filter as shown in the following example.

Example 4.2. Let $X = \{1, a, b, c, d\}$ be a non-empty set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. Define a fuzzy set μ on X as follows:

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = a, 1 \\ 0.3 & \text{otherwise} \end{cases}$$

for all $x \in X$. Then clearly μ is a fuzzy filter of X, but μ is not a fuzzy positive implicative filter of X since $\mu(c) \ngeq min\{\mu(a), \mu(a * ((b * c) * b))\}$.

We now derive a necessary and sufficient condition for every fuzzy filter of a BE-algebra to become a fuzzy positive implicative filter.

Theorem 4.3. *A fuzzy filter F of a BE-algebra X is a fuzzy positive implicative filter if and only if it satisfies the following condition.*

$$\mu(x) = \mu((x * y) * x) \text{ for all } x, y \in X$$

Proof. Let F be a fuzzy filter of X. Assume that F is a fuzzy positive implicative filter in X. Let $x, y \in X$. Since $x \leq (x * y) * x$ and μ is a fuzzy filter, we get that $\mu(x) \leq \mu((x * y) * x)$. Since F is a fuzzy positive implicative filter, we get

$$\begin{aligned} \mu(x) &\geq min\{\mu(1), \mu(1 * ((x * y) * x))\} \\ &= \mu(1 * ((x * y) * x)) \\ &= \mu((x * y) * x) \end{aligned}$$

Hence $\mu(x) = \mu((x * y) * x)$. Conversely, assume that the condition holds in X. Clearly $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Since F is a fuzzy filter, by the assumed condition, we get the following:

$$\begin{aligned} \mu(y) &= \mu((y * z) * y) \\ &\geq min\{\mu(x), \mu(x * ((y * z) * y))\} \end{aligned}$$

Therefore μ is a fuzzy positive implicative filter in X. □

Theorem 4.4. *Let F be a non-empty subset of a BE-algebra X. Define a fuzzy set $\mu_F : X \longrightarrow [0, 1]$ as follows:*

$$\mu_F(x) = \begin{cases} \alpha & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases}$$

where $0 < \alpha < 1$ is fixed. Then μ_F is a fuzzy positive implicative filter in X if and only if F is a positive implicative filter in X.

Proof. Assume that μ_F is a fuzzy positive implicative filter in X. Since $\mu_F(1) \geq \mu_F(x)$ for all $x \in X$, we get $\mu_F(1) = \alpha$ and hence $1 \in F$. Let $x, y, z \in X$ be such that $x, x * ((y * z) * y) \in F$. Then $\mu_F(x) = \mu_F(x * ((y * z) * y)) = \alpha$. Since μ_F is a fuzzy positive implicative filter, we get

$$\mu_F(y) \geq min\{\mu_F(x), \mu_F(x * ((y * z) * y))\} = min\{\alpha, \alpha\} = \alpha$$

Hence $\mu_F(y) = \alpha$ and so $y \in F$. Therefore F is a positive implicative filter in X.

Conversely, assume that F is a positive implicative filter of X. Since $1 \in F$, we get that $\mu_F(1) = \alpha$. Hence $\mu_F(1) \geq \mu_F(x)$ for all $x \in X$. Let $x, y, z \in X$. Suppose $x, x * ((y * z) * y) \in F$. Since F is a positive implicative filter, we get $y \in F$. Then $\mu_F(x) = \mu_F(x*((y*z)*y) = \mu_F(y) = \alpha$. Hence $\mu_F(y) \geq min\{\mu_F(x), \mu_F(x*((y*z)*y))\}$. Suppose $x*((y*z)*y)) \notin F$ and $x \notin F$. Then $\mu_F(x) = \mu_F(x*((y*z)*y)) = 0$. Hence $\mu_F(y) \geq min\{\mu_F(x), \mu_F(x*((y*z)*y))\}$. If exactly one of x and $x*((y*z)*y)$ is in F, then exactly one of $\mu_F(x)$ and $\mu_F(x * ((y * z) * y))$ is equal to 0. Hence $\mu_F(y) \geq min\{\mu_F(x), \mu_F(x * ((y * z) * y))\}$. By summarizing the above results, we get $\mu_F(y) \geq min\{\mu_F(x), \mu_F(x * ((y * z) * y))\}$ for all $x, y, z \in X$. Therefore μ_F is a fuzzy positive implicative filter of X. □

Proposition 4.5. *Let μ be a fuzzy set in a BE-algebra X. Then μ is a fuzzy positive implicative filter in X if and only if for each $\alpha \in [0, 1]$, the level subset μ_α is a positive implicative filter in X, when $\mu_\alpha \neq \varnothing$.*

Proof. Assume that μ is a fuzzy positive implicative filter of X. Then $\mu(1) \geq \mu(x)$ for all $x \in X$. In particular, $\mu(1) \geq \mu(x) \geq \alpha$ for all $x \in \mu_\alpha$. Hence $1 \in \mu_\alpha$. Let $x, x * ((y * z) * y) \in \mu_\alpha$. Then $\mu(x) \geq \alpha$ and $\mu(x * ((y * z) * y)) \geq \alpha$. Since μ is a fuzzy positive implicative filter, we get $\mu(y) \geq min\{\mu(x), \mu(x * ((y * z) * y))\} \geq \alpha$. Thus $y \in \mu_\alpha$. Therefore μ_α is a positive implicative filter in X.

Conversely, assume that μ_α is a positive implicative filter of X for each $\alpha \in [0, 1]$ with $\mu_\alpha \neq \varnothing$. Suppose there exists $x_0 \in X$ such that $\mu(1) < \mu(x_0)$. Let $\alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0))$. Then $\mu(1) < \alpha_0$ and $0 \leq \alpha_0 < \mu(x_0) \leq 1$. Hence $x_0 \in \mu_{\alpha_0}$ and $\mu_{\alpha_0} \neq \varnothing$. Since μ_{α_0} is a positive implicative filter in X, we get $1 \in \mu_{\alpha_0}$ and hence $\mu(1) \geq \alpha_0$, which is a contradiction. Therefore $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu(x * ((y * z) * y)) = \alpha_1$ and $\mu(x) = \alpha_2$. Then $x * ((y * z) * y) \in \mu_{\alpha_1}$ and $x \in \mu_{\alpha_2}$. Without loss of generality, assume that $\alpha_1 \leq \alpha_2$. Then clearly $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. Hence $x \in \mu_{\alpha_1}$. Since μ_{α_1} is a positive implicative filter in X, we get $y \in \mu_{\alpha_1}$. Thus $\mu(y) \geq \alpha_1 = min\{\alpha_1, \alpha_2\} = \min\{\mu(x * ((y * z) * y)), \mu(x)\}$. Therefore μ is a fuzzy positive implicative filter of X. □

For any $\alpha \in [0, 1]$, the above positive implicative filter μ_α of a BE-algebra is called a level positive implicative filter. Now, in the following, we derive a necessary and sufficient condition for two level filters two become equal.

Theorem 4.6. *Let μ be a fuzzy positive implicative filter of a BE-algebra X. Then two level positive implicative filters μ_{α_1} and μ_{α_2} (with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ for $\alpha_1 < \alpha_2$. Suppose there exists some $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Then μ_{α_2} is a proper subset of μ_{α_1}, which is impossible. Conversely, assume that there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Since $\alpha_1 < \alpha_2$, we get that $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \geq \alpha_1$. Hence by the assume condition, we get $\mu(x) \geq \alpha_2$. Hence $x \in \mu_{\alpha_2}$ and so $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Therefore $\mu_{\alpha_1} = \mu_{\alpha_2}$. □

The following Lemma is a direct consequence of the above definitions.

Lemma 4.7. *Let μ, ν be two fuzzy sets in a BE-algebra X. Then the following hold.*

(1) $\mu \times \nu$ *is a fuzzy relation on* X

(2) $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ *for all* $\alpha \in [0, 1]$

For any two BE-algebras X and Y, define an operation $*$ on $X \times Y$ as follows:

$$(x, y) * (x', y') = (x * x', y * y') \text{ for all } x, x' \in X \text{ and } y, y' \in Y$$

Then it can be easily observed that $(X \times Y, *, (1, 1))$ is a BE-algebra.

Theorem 4.8. *Let μ and ν be two fuzzy positive implicative filters of a BE-algebra X. Then $\mu \times \nu$ is a fuzzy positive implicative filter in $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Since μ, ν are fuzzy positive implicative filters in X, we get

$$\begin{aligned} (\mu \times \nu)(1, 1) &= min\{\mu(1), \nu(1)\} \\ &\geq min\{\mu(x), \nu(y)\} \qquad \text{for all } x, y \in X \\ &= (\mu \times \nu)(x, y) \end{aligned}$$

Let $(x, x'), (y, y'), (z, z') \in X \times X$. Put $t = x * ((y * z) * y)$ and $t' = x' * ((y' * z') * y')$. Clearly $(t, t') = (x, x') * (((y, y') * (z, z')) * (y, y'))$. Since μ and ν are fuzzy positive implicative filters in X, we can obtain the following consequence.

$$\begin{aligned} (\mu \times \nu)(y, y') &= min\{\mu(y), \nu(y')\} \\ &\geq min\{min\{\mu(x), \mu(t)\}, min\{\nu(x'), \nu(t')\}\} \\ &= min\{min\{\mu(x), \nu(x')\}, min\{\mu(t), \nu(t')\}\} \\ &= min\{(\mu \times \nu)(x, x'), (\mu \times \nu)(t, t')\} \\ &= min\{(\mu \times \nu)(x, x'), (\mu \times \nu)((x, x') * (((y, y') * (z, z')) * (y, y')))\} \end{aligned}$$

Therefore $\mu \times \nu$ is a fuzzy positive implicative filter in $X \times X$. □

Definition 4.9. Let ν be a fuzzy set in a BE-algebra X. Then the strongest fuzzy relation μ_ν is a fuzzy relation on X defined by

$$\mu_\nu(x, y) = min\{\nu(x), \nu(y)\}$$

for all $x, y \in X$.

Theorem 4.10. *Let ν be a fuzzy set in a BE-algebra X and μ_ν the strongest fuzzy relation on X. Then ν is a fuzzy positive implicative filter in X if and only if μ_ν is a fuzzy positive implicative filter of $X \times X$.*

Proof. Assume that ν is a fuzzy positive implicative filter of X. Then for any $(x, y) \in X \times X$, we have

$$\mu_\nu(x, y) = min\{\nu(x), \nu(y)\} \leq min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1)$$

Let $(x, x'), (y, y')$ and $(z, z') \in X \times X$. Then we have the following:

$$\begin{aligned} \mu_\nu(y, y')) &= min\{\nu(y), \nu(y')\} \\ &\geq min\{min\{\nu(x), \nu(t)\}, min\{\nu(x'), \nu(t')\}\} \\ &\qquad \text{where } t = x * ((y * z) * y) \text{ and } t' = x' * ((y' * z') * y') \\ &= min\{min\{\nu(x), \nu(x')\}, min\{\nu(t), \nu(t')\}\} \\ &= min\{\mu_\nu(x, x'), \mu_\nu(t, t')\} \\ &= min\{\mu_\nu(x, x'), \mu_\nu(x * ((y * z) * y)), x' * ((y' * z') * y'))\} \\ &= min\{\mu_\nu(x, x'), \mu_\nu((x, x') * (((y, y') * (z, z')) * (y, y'))\} \end{aligned}$$

Therefore μ_ν is a fuzzy positive implicative filter in $X \times X$. Conversely, assume that μ_ν is a fuzzy positive implicative filter in $X \times X$. Then

$$\nu(1) = min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1) \geq \mu_\nu(x, x) = min\{\nu(x), \nu(x)\} = \nu(x)$$

for all $x \in X$. Hence it yields that $\nu(x) \leq \nu(1)$ for all $x \in X$. Let $x, y, z \in X$. Then we have the following consequence.

$$\begin{aligned} \nu(y) &= min\{\nu(y), \nu(1)\} \\ &= \mu_\nu(y, 1) \\ &\geq min\{\mu_\nu(x, 1), \mu_\nu((x, 1) * (((y, 1) * (z, z')) * (y, 1)))\} \\ &= min\{\mu_\nu(x, 1), \mu_\nu(x * ((y * z) * y), 1 * (1 * z') * 1))\} \\ &= min\{\mu_\nu(x, 1), \mu_\nu(x * ((y * z) * y), 1)\} \\ &= min\{min\{\nu(x), \nu(1)\}, min\{\nu(x * ((y * z) * y), \nu(1)\}\} \\ &= min\{\nu(x), \nu(x * ((y * z) * y))\} \end{aligned}$$

Therefore ν is a fuzzy positive implicative filter in X. □

5. Normalization of fuzzy positive implicative filters

In this section, the notion of normal fuzzy positive implicative filters is introduces in BE-algebra. The concept of maximal fuzzy positive implicative filters is introduced and then studied some properties of these two classes of fuzzy filters.

Definition 5.1. A fuzzy positive implicative filter μ of a BE-algebra X is called a normal if there exists $x \in X$ such that $\mu(x) = 1$.

For any normal fuzzy positive implicative filter μ, we obviously have $\mu(1) = 1$.

Proposition 5.2. *For any fuzzy positive implicative filter μ of a BE-algebra X, define a fuzzy set μ^+ in X as $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$. Then μ^+ is a normal fuzzy positive implicative filter of X such that $\mu \subseteq \mu_+$.*

Proof. Let $x \in X$. Then we have $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1 \geq \mu^+(x)$, which proves (F_1). To prove (F_2), let $x, y, z \in X$. Then

$$\begin{aligned} \mu^+(y) &= \mu(y) + 1 - \mu(1) \\ &\geq min\{\mu(x), \mu(x * ((y * z) * y))\} + 1 - \mu(1) \\ &= min\{\mu(x) + 1 - \mu(1), \mu(x * ((y * z) * y)) + 1 - \mu(1)\} \\ &= min\{\mu^+(x), \mu^+(x * ((y * z) * y))\} \end{aligned}$$

Therefore μ^+ is a normal fuzzy positive implicative filter in X. Now, for any $x \in X$, it is clear that $\mu(x) \leq \mu^+(x)$. Therefore it concludes that $\mu \subseteq \mu^+$. □

It is clear that a fuzzy positive implicative filter μ of X is normal if and only if $\mu^+ = \mu$, and for any fuzzy positive implicative filter μ of X, we have $(\mu^+)^+ = \mu^+$. Hence if μ is a normal fuzzy positive implicative filter of L, then $(\mu^+)^+ = \mu$.

Theorem 5.3. *Let μ be a fuzzy positive implicative filter. If there exists a fuzzy positive implicative filter ν of X satisfying $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Let ν be a fuzzy positive implicative filter such that $\nu^+ \subseteq \mu$. Then we get $1 = \nu^+(1) \leq \mu$. Hence $\mu(1)$, which yields that μ is normal. □

Corollary 5.4. *Let ν and μ be two fuzzy positive implicative filters of X such that $\nu \subseteq \mu$. If ν is normal, then μ is also normal.*

Definition 5.5. Let μ be a fuzzy set in a BE-algebra X. Then define the set X_μ by $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$.

Proposition 5.6. *If μ is a normal fuzzy positive implicative filter of a BE-algebra X, then X_μ is a positive implicative filter in X*

Proof. Clearly $1 \in X_\mu$. Let $x, y, z \in X$ be such that $x, x * ((y * z) * y) \in X_\mu$. Then $\mu(x) = \mu(x * ((y * z) * y)) = \mu(1)$. Since μ is a fuzzy positive implicative filter, we get

$$\mu(y) \geq min\{\mu(x), \mu(x * ((y * z) * y))\} = \mu(1)$$

Since μ is normal, we get $\mu(y) \geq \mu(1) = 1$. Hence $\mu(y) = 1 = \mu(1)$. Thus $y \in X_\mu$, which yields that X_μ is a positive implicative filter in X. □

Proposition 5.7. *Let μ and ν be two normal fuzzy positive implicative filters of X such that $\mu \subseteq \nu$. Then $X_\mu \subseteq X_\nu$.*

Proof. Let $x \in X_\mu$. Then $\nu(x) \geq \mu(x) = \mu(1) = 1$. Hence $\nu(x) = 1 = \nu(1)$, which concludes that $x \in X_\nu$. Therefore $X_\mu \subseteq X_\nu$. □

Theorem 5.8. *Let μ be a fuzzy positive implicative filter of X and $\phi : [0, \mu(1)] \longrightarrow [0, 1]$ an increasing function. Let μ_ϕ be a fuzzy set in X defined as $\mu_\phi(x) = \phi(\mu(x))$ for all $x \in X$. Then μ_ϕ is a fuzzy positive implicative filter in X. Moreover, if μ is normal, then μ_ϕ is also normal.*

Proof. Since ϕ is increasing and $\mu(1) \geq \mu(x)$ for all $x \in X$, we get

$$\mu_\phi(1) = \phi(\mu(1)) \geq \phi(\mu(x)) = \mu_\phi(x) \text{ for all } x \in X$$

Let $x, y, z \in X$. Then

$$\begin{aligned} \mu_\phi(y) &= \phi(\mu(y)) \\ &\geq \phi(min\{\mu(x), \mu(x * ((y * z) * y))\}) \\ &= min\{\phi(\mu(x)), \phi(\mu(x * ((y * z) * y)))\} \\ &= min\{\mu_\phi(x), \mu_\phi(x * ((y * z) * y))\} \end{aligned}$$

Therefore μ_ϕ is a positive implicative filter in X. Moreover, if μ is normal, then clearly $\phi(\mu(1)) = \phi(1)$. Since ϕ is increasing in $[0, \mu(1)]$, it is clear that $\phi(1) = 1$. Hence $\mu_\phi(1) = \phi(\mu(1)) = 1$. Therefore, it concludes that μ_ϕ is normal. □

Let $\mathcal{NF}(X)$ be the class of all normal fuzzy positive implicative filters of a BE-algebra X. Then clearly $\mathcal{NF}(X)$ is a partially ordered set under the set inclusion.

Definition 5.9. A non-constant fuzzy positive implicative filter μ of X is called maximal if there exists no non-constant fuzzy positive implicative filter ν such that $\mu \subseteq \nu$.

Lemma 5.10. *Every maximal fuzzy positive implicative filter is normal.*

Proof. Let μ be a maximal fuzzy positive implicative filter of X. Then μ is non-constant and hence μ^+ is non-constant. Otherwise, suppose $\mu^+(x) = c$ for all $x \in X$, where c is a constant. Then for all $x \in X, c = \mu^+(x) = \mu(x) + 1 - \mu(1)$ and so μ is constant. Since $\mu \subseteq \mu^+$ and μ is maximal, we get $\mu = \mu^+$. Hence μ is normal. □

Theorem 5.11. *Let μ be a maximal fuzzy positive implicative filter of X. Then μ takes only the values* 0 *and* 1.

Proof. Since μ is maximal, by above lemma, μ is normal and hence $\mu(1) = 1$. Let $x \in X$ be such that $\mu(x) \neq 0$. Suppose $\mu(x) \neq 1$. Then there exists some $x_0 \in X$ such that $0 < x_0 < 1$. Then define a fuzzy set ν in X as follows:

$$\nu(x) = \tfrac{1}{2}(\mu(x) + \mu(x_0)) \quad \text{for all } x \in X$$

Clearly ν is well-defined. Let $x \in X$. Then $\nu(1) = \frac{1}{2}(\mu(1)+\mu(x_0)) = \frac{1}{2}(1+\mu(x_0)) \geq \frac{1}{2}(\mu(x) + \mu(x_0)) = \nu(x)$. Let $x, y, z \in X$. Then we have

$$\begin{aligned} \nu(y) &= \frac{1}{2}\{\mu(y) + \mu(x_0)\} \\ &\geq \frac{1}{2}\{min\{\mu(x), \mu(x * ((y * z) * y))\} + \mu(x_0)\} \\ &= \frac{1}{2}\{min\{\mu(x) + \mu(x_0), \mu(x * ((y * z) * y)) + \mu(x_0)\} \\ &= min\ \{\frac{1}{2}(\mu(x) + \mu(x_0)), \frac{1}{2}(\mu(x * ((y * z) * y)) + \mu(x_0))\} \\ &= min\{\nu(x), \nu(x * ((y * z) * y))\} \end{aligned}$$

Therefore ν is a fuzzy positive implicative filter of X. Hence by Proposition 5.2, we get that ν^+ is a normal fuzzy positive implicative filter of X. Clearly $\nu^+(x) \geq \mu(x)$ for all $x \in X$. Now

$$\begin{aligned} \nu^+(x_0) &= \nu(x_0) + 1 - \nu(1) \\ &= \frac{1}{2}\{\mu(x_0) + \mu(x_0)\} + 1 - \frac{1}{2}\{\mu(1) + \mu(x_0)\} \\ &= \frac{1}{2}\{\mu(x_0) + 1\} \\ &> \mu(x_0) \end{aligned}$$

and also $\nu^+(x_0) < 1 = \nu^+(1)$. Hence ν^+ is non-constant such that $\mu \subseteq \nu^+$. Therefore μ is not maximal, which is a contradiction. Hence $\mu(x) = 1$. □

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