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Fuzzy compactification of fuzzy frames

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Abstract. The notion of fuzzy frames was introduced by T. Rajesh[16]. In this paper we attempt to discuss fuzzy compactification of fuzzy frames. Regularity and compactness properties are also studied in the fuzzy background. We further investigate fuzzy approach to Stone-Cech compactification of fuzzy frames.

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1. Introduction

A frame is a complete lattice L which satisfies the distributive law. Frame theory is the lattice theory applied to topological space and thus frame is an extension of a topological space. In topological spaces, compactification is a process of embedding a given space as a dense subset of some compact Hausdorff space. The first study of compactifications was made by Tychonoff in 1929. Later E. Cech and M.H. Stone in 1937 defined and discussed the compactification of a completely regular space. The frame analogue of compactification was introduced by M.H. Stone in 1929. Compactification of a frame is a compact regular frame which contains a dense onto homomorphic image of a given regular frame. The further study of compactification of frames progressed rapidly, since last three decades. In 1980, B. Banaschewski and C.J. Mulvey[4] established the Stone-Čech compactification of locales and the concept was further studied in 1984[5], while, P.T. Johnstone[13] in 1982, presented an alternative construction of compactification of locales. In 1990, B. Banaschewski[3] provided a comprehensive view of all compactifications of a given frame which are determined by strong inclusions. In 1991, D. Baboolal and B. Banaschewski[2] generalized a classical result of Stone–Čech compactification of topological spaces to frames and established the frame counterpart of an analogous result of Wallace in terms of a certain property of covers. Again, in 2004, D. Baboolal[1] discussed the necessary and sufficient conditions for a completely regular frame to have a locally connected compactification. T. Dube[8] discussed the notions of extension of closed and nearly closed sublocales of frames and characterised the compact frames in terms of closedness, in 2004. In 2008, G. Curi [6] considered the relations of strong inclusions on pseudocomplemented distributive lattices to refine existing constructions of compactifications of frames. Many works have been done on fuzzy topology, since the time of C. L. Chang. R. Lowen[14],in (1976) discussed Fuzzy topological spaces and Fuzzy compactness. Later, S. Ganguly and S. Saha[10] described the fuzzy aspect of compactness in terms of fuzzy open covers. Fuzzy compactness can also be discussed in view of gradation of closedness. K. El-Saady and A. Ghareeb [9] introduced the notion of (r,s)-fuzzy regular semiopen sets and then discussed RS-fuzzy compactness in 2012, while, A. Haydar Es [12], discussed several types of degrees of fuzzy compactness and discussed RS-fuzzy compactness in terms of S-closedness. Further, T. Rajesh [16]defined a fuzzy frame in a different fashion. Following this style, here we introduce regularity and compactness of frames in the fuzzy context and we study its properties. We prove that the product of fuzzy compact fuzzy frames is fuzzy compact. We also define fuzzy compactification of fuzzy frames and deduce the fuzzy analogue of the Stone-Cech compactification of frames. We establish that any fuzzy compactification of a fuzzy frame is a quotient of fuzzy Stone-Cech Compactification of the fuzzy frame.

2. Preliminaries

This section contains some basic definitions and results of frames and compactification of locales which we require in the following discussions. Some background materials of fuzzy frames and fuzzy binary relations are also mentioned.

Definition 2.1 ([15])**.** Let X be a set. Then a frame L on X is a complete lattice, satisfying the distributive law

$$a \wedge (\bigvee S) = \bigvee\{a \wedge b \mid b \in S\} \text{ for all } a \in L \text{ and } S \subseteq L.$$

Definition 2.2 ([15])**.** Let L, M be two frames. A frame homomorphism f from L to M is a mapping preserving all suprema including the bottom 0 and all finite infima, including the top 1.

$$\text{i.e., if } f : L \to M, \text{ with } f(a \wedge b) = f(a) \wedge f(\text{b}) \text{ and}$$
$$f(\bigvee B) = \bigvee \{f(\text{b}) | \ b \in \text{B}\}, \text{ for } a, b \in L \text{ and any subset } B \subseteq L.$$

S.Vickers gives the presentation of a frame in terms of generators and relations. Accordingly, the tensor product of two frames is also defined.

Definition 2.3 ([17])**.** Let A and B be two frames. The tensor product $A \otimes B$ of A and B is a frame and is represented by

$$\begin{aligned}\mathrm{Fr}\langle(a \otimes b) : a \in A,\ b \in B \mid & \wedge(a_i \otimes b_i) = (\wedge a_i) \otimes (\wedge b_i)\\ & \vee(a_i \otimes b) = (\vee a_i) \otimes b\\ & \vee(a \otimes b_i) = a \otimes (\vee b_i)\rangle.\end{aligned}$$

Definition 2.4 ([17])**.** Let X and Y be two sets and let L and M be frames on X and Y respectively. Then the tensor product $L \otimes M$ is a frame on $X \times Y$ with respect to a relation $\models$, given by

$$(x, y) \models \vee_i(a_i \otimes b_i) \Leftrightarrow x \models a_i \text{ and } y \models b_i, \text{ for some } i.$$

Definition 2.5 ([13])**.** Let L be a frame and let $a, b \in L$. Then b is said to be rather below, written as $b \prec a$, if there exists $c \in L$ with $c \wedge b = 0$ and $c \vee a = 1$.

Definition 2.6 ([15])**.** A frame L is said to be regular if for each $a \in L$,

$$a = \vee\{b \mid b \prec a\}.$$

Definition 2.7 ([15])**.** A frame homomorphism $f : L \to M$ is said to be dense if

$$f(a) = 0 \Rightarrow a = 0.$$

Definition 2.8 ([15])**.** A cover of a frame L is a subset $A \subseteq L$ such that $\vee A = 1$ and a subcover of L is subset $B \subseteq A$, which is still a cover.

Definition 2.9 ([15])**.** A frame L is said to be compact if each cover of L has a finite subcover.
i.e., if for any $A \subseteq L$ such that $\bigvee A = 1$, there is a $B \subseteq A$, B finite, such that

$$\vee \text{ B} = 1.$$

Definition 2.10 ([13])**.** A compactification of a frame L is a dense onto homomorphism $h : M \to L$ with compact regular M.

Locales and continuous maps between them are determined by their corresponding frames and frame homomorphisms between them, the homomorphisms being taken in opposite directions.

Theorem 2.11 ([13])**.** *(Tychonoff's embedding theorem for locales) The following conditions on a locale are equivalent:*

i. *A is completely regular.*
ii. *A is isomorphic to a sublocale of a compact (completely) regular locale.*
iii. *A is isomorphic to a sublocale of a product of copies of L[0,1].*

Theorem 2.12 ([13])**.** *(Stone-Cech compactification for locales) The inclusion functor from the category **K(C)RegLoc** of compact (completely) regular locales to **Loc** has a left adjoint β. Moreover, the unit $A \to \beta A$ of the adjunction is a regular mono in **Loc** iff A is completely regular.*

Definition 2.13 ([16])**.** Let F be a frame; then a fuzzy subset $\mu : F \to [0, 1]$ of F is said to be a fuzzy frame if

1. $\mu(\vee S) \geq \inf\{\mu(a) \mid a \in S\}, \forall\ S \subseteq F$
2. $\mu(a \wedge b) \geq \min\{\mu(a), \mu(b)\}, \forall\ a, b \in F$
3. $\mu(e_F) = \mu(0_F) \geq \mu(a), \forall\ a \in F$, where e_F and 0_F are respectively the unit and zero element of the frame F.

Definition 2.14 ([16])**.** Let μ and λ be fuzzy frames of L and M respectively. Then the product of fuzzy frames $\mu \times \lambda$ of the product $L \times M$ is defined as

$$\mu \times \lambda\ (a \times b) = \min\{\mu(a),\ \lambda(b)\}, \text{ for } a \in L, b \in M.$$

Definition 2.15 ([11])**.** Let X be a set. A fuzzy binary relation R on X is a map $R : X \times X \to [0, 1]$.

Definition 2.16 ([11]). Let R be a fuzzy binary relation on a set X. Then R is said to be a fuzzy equivalence relation (or fuzzy similarity relation) if the following conditions are satisfied.
i. $R(a, a) = 1,\ a \in X$
ii. $R(a, b) = R(b, a),\ a, b \in X$
iii. $R(a, c)\ \geq\ \min\{R(a, b), R(b, c)\},\ a, b, c \in X.$

Definition 2.17 ([7]). Let L be a frame on X. A congruence μ in a frame L is an equivalence relation such that

i. if $(a_\alpha, b_\alpha) \in \mu$, for any set of indices α, then $(\vee a_\alpha, \vee b_\alpha) \in \mu$ and
ii. if $(a_i, b_i) \in \mu$ for $i = 1, 2$, then $(a_1 \wedge a_2, b_1 \wedge b_2) \in \mu$.

3. Fuzzy compactness

A topological space is said to be compact if for every open cover of it, there exists a finite subcover. The frame analogue of compactness follows in the same style. Here we discuss regularity, optimality and compactness of frames in the fuzzy environment. We establish that the product of fuzzy compact fuzzy frames is a fuzzy compact fuzzy frame.
We start with the definitions of fuzzy regular and fuzzy compact fuzzy frames.

Definition 3.1. Let L be a frame on X. A fuzzy frame $F = (L, \mu)$ is said to be fuzzy regular, if for each $a \in L$,

$$\mu(a) \geq \sup\{\mu(b) \mid b \prec a\}$$

Let X be a set and L be a frame on it with respect to a relation $\models$. For $a \in L$, we denote U_a as $U_a = \{x \in X \mid x \models a\}$. Then

$$T_L = \{U_a \mid a \in A\}.$$

A frame L is defined to be an optimal frame if

$$U_a = \varnothing \ \Leftrightarrow\ a = 0 \ \text{ and}$$
$$U_a = X \ \Leftrightarrow\ a = 1.$$

Definition 3.2. Let $F = (L, \mu)$ be a fuzzy frame. Then F is said to be an optimal fuzzy frame, if

$$\mu(a) = 1 \Leftrightarrow a = e_L \ \ or \ \ a = o_L$$

Definition 3.3. Let $F = (L, \mu)$ be an optimal fuzzy frame on L. If $S \subseteq L$, then (S, μ) is said to be a fuzzy cover of L, if $\mu(\vee S) \geq \mu(a)$, $\forall a \in L$, provided S contains nonzero elements.
F is said to be fuzzy compact, if for any fuzzy cover (S, μ) of F, $\exists$ a finite $T \subseteq S$ such that $\mu(\vee T) \geq \mu(a)$, $\forall a \in L$ and $\mu(\vee T) = \mu(\vee S)$, provided T contains nonzero elements.

It is proved that the product of two fuzzy frames is again a fuzzy frame. We establish the result for fuzzy compact fuzzy frames.

Proposition 3.4. *Product of any two fuzzy compact fuzzy frames is fuzzy compact.*

Proof. Let (L,μ) and (M,λ) be two fuzzy compact fuzzy frames on L and M respectively. We claim that $(L\otimes M,\mu\times\lambda)$ is a fuzzy compact fuzzy frame.
Let U be a subframe of L and W be a subframe of M. Since (L,μ) is fuzzy compact, $\exists$ a finite $T\subseteq U$ such that $\mu(\vee T)\geq\mu(a)$, $\forall a\in L$ and $\mu(\vee T)=\mu(\vee U)$. Similarly, since (M,λ) is fuzzy compact, $\exists$ a finite $S\subseteq W$ such that $\lambda(\vee S)\geq\lambda(b)$, $\forall b\in M$ and $\lambda(\vee S)=\lambda(\vee W)$.
Now,

$$\begin{aligned}\mu\times\lambda(\vee T\otimes b) &= \min\{\mu(\vee T),\lambda(b)\},\ \ b\in M, \text{ by Definition 2.14}\\ &\geq \mu\times\lambda(a_i\otimes b),\ \ \forall a_i\in T,\ b\in M\\ \mu\times\lambda(a\otimes\vee S) &= \min\{\mu(a),\lambda(\vee S)\},\ \ a\in L\\ &\geq \mu\times\lambda(a\otimes b_j),\ \ a\in L,\ \forall b_j\in S\end{aligned}$$

Since $\mu(\vee T)=\mu(\vee U)$, $\min\{\mu(\vee T),\lambda(b)\}=\min\{\mu(\vee U),\lambda(b)\}$
i.e., $\mu\times\lambda(\vee T\otimes b)=\mu\times\lambda(\vee U\otimes b)$, for $b\in M$
Similarly, since $\lambda(\vee S)=\lambda(\vee W)$, $\mu\times\lambda(a\otimes\vee S)=\mu\times\lambda(a\otimes\vee W)$, $a\in L$.
Hence $\mu\times\lambda$ is fuzzy compact. □

Theorem 3.5. *Product of any collection of fuzzy compact fuzzy frames is a fuzzy compact fuzzy frame.*

Proof. Let $\{A_\alpha\mid\alpha\in\wedge\}$ be a collection of frames and $\{F_\alpha\mid\alpha\in\wedge\}$ where $F_\alpha=(A_\alpha,\mu_\alpha)$ be a collection of fuzzy frames. Let $A=\otimes_\alpha A_\alpha$, $\mu=\prod\mu_\alpha$. Let $F=\prod F_\alpha=(\otimes_\alpha A_\alpha,\prod\mu_\alpha)$. We show that F is fuzzy compact.
Let $\mathscr{C}=(C,\mu)$ be a fuzzy cover of F. Then

$$\mu(\vee C)\ \geq\ \mu(\underline{a}),\ \forall\ \underline{a}\in A,\text{ where } \underline{a}\ =\ (\underline{a_k}),$$

is the finite tensor product in which all the components are corresponding e_α's, except the k^{th} one, which is a_k. Each F_α is fuzzy compact and hence for any fuzzy cover $\mathscr{C}_\alpha=(C_\alpha,\mu_\alpha)$ of F_α, $\exists$ a fuzzy sub cover $\mathscr{S}_\alpha\subseteq\mathscr{C}_\alpha$, where $\mathscr{S}_\alpha=(S_\alpha,\mu_\alpha)$, $\mathscr{S}_\alpha$ finite, such that

$$\mu_\alpha(\vee S_\alpha)\ \geq\ \mu_\alpha(a_\alpha),\ \forall\ a_\alpha\in A_\alpha.$$

We prove that $\exists$ a fuzzy subcover $\mathscr{S}\subseteq\mathscr{C}$, $\mathscr{S}$ finite, such that

$$\mu(\vee\underline{S_k})\ \geq\ \mu(\underline{a}),\ \forall\ \underline{a}\in A\text{ and }\mu(\vee\underline{S_k})\ =\ \mu(\vee\underline{C_k}),$$

where $\vee S_k$ occurs at the k^{th} place of the product and all other components are corresponding e_α's.

$$\mu(\vee\underline{S_k})\ =\ \prod\mu_\alpha(\vee\underline{S_k})\ =\ \inf\{\underline{\langle\mu_k(\vee S_k)\rangle}\},$$

where $\underline{\langle\mu_k(\vee S_k)\rangle}$ is the infinite tuple in which k^{th} component is $\mu_k(\vee S_k)$ and all others are corresponding $\mu_\alpha(e_\alpha)$'s.

$$\begin{aligned}&\geq\ \inf\{\underline{\langle\mu_k(a_k)\rangle}\},\ \underline{a_k}\in S_k\\ &\geq\ \prod\mu_\alpha(\underline{a})\\ \text{i.e., }\mu(\vee\underline{S_k})\ &\geq\ \mu(\underline{a}),\ \forall\ \underline{a}\in A.\end{aligned}$$

Also, $\mu(\vee\underline{S_k})\ =\ \mu(\vee\underline{C_k})$; or,
since each F_α is fuzzy compact, $\mu_\alpha(\vee S_\alpha)\ =\ \mu_\alpha(\vee C_\alpha)$, $\forall\ \alpha$.
Now, $\inf\{\underline{\langle\mu(\vee S_k)\rangle}\}\ =\ \inf\{\underline{\langle\mu(\vee C_k)\rangle}\}$

$$\mu(\vee\underline{S_k})\ =\ \mu(\vee\underline{C_k}).$$

□

4. Fuzzy congruence

A binary relation defined on a set concerns about the two extremes, "completely related" and "not related," whereas the fuzzy relation takes into account of an infinite number of degrees of relationship between these two. A.J. Ross defines a fuzzy relation as a mapping from the cartesian product $X \times Y$ to the interval [0,1]. The congruence on a frame L is an equivalence relation $R(\phi) = \{(a, b) \in L \times L \mid \phi(a) = \phi(b)\}$ which is closed under arbitrary join and finite meet. In this section, we define fuzzy congruence of fuzzy frames and discuss the fuzzy quotient of fuzzy frames.

Definition 4.1. Let $F = (L, \mu)$ be a fuzzy frame. Let R be a fuzzy binary relation on L. The associated fuzzy binary relation R_F on F is a map $R_F : L \times L \to [0, 1]$, such that

$$R_F(a, b) = R(a, b)\mu(a)\mu(b), \quad \forall \quad a, b \in L.$$

Definition 4.2. Let R_F be a fuzzy binary relation on a fuzzy frame F. Then R_F is said to be a fuzzy equivalence relation (or fuzzy similarity relation) on F if the following conditions are satisfied.

(1) $R_F(a, a) = (\mu(a))^2$, $\forall\ a \in L$
(2) $R_F(a, b) = R_F(b, a)$, $\forall\ a, b \in L$
(3) $R_F(a, c) \geq \min\{R_F(a, b), R_F(b, c)\}$, $\forall\ a, b, c \in L$.

Lemma 4.3. *Let L and M be frames and ϕ be a frame homomorphism from M to L. Define a relation $R(\phi)$ on L as $R(\phi) = R\{(a, b) \in L \times L \mid \phi(a) = \phi(b)\}$. Then $R(\phi)$ is a congruence on L.*

Proof. The proof is obvious, since ϕ a frame homomorphism. For,
let $(a_\alpha, b_\alpha) \in R(\phi)$, $\forall\ \alpha \in \wedge$. Then $(\vee a_\alpha, \vee b_\alpha) \in R(\phi)$, since

$$\phi(\vee a_\alpha) = \vee\phi(a_\alpha) \text{ and } \phi(\vee b_\alpha) = \vee\phi(b_\alpha).$$

For $(a_1, b_1), (a_2, b_2) \in R(\phi)$, $(a_1 \wedge a_2, b_1 \wedge b_2) \in R(\phi)$,since

$$\phi(a_1 \wedge a_2) = \wedge[\phi(a_1), \phi(a_2)] \text{ and } \phi(b_1 \wedge b_2) = \wedge[\phi(b_1), \phi(b_2)]. \qquad \square$$

Definition 4.4. Let L be a frame and R be a fuzzy equivalence relation on L. Then R is said to be a fuzzy congruence on L if

(1) $R(\vee a_\alpha, \vee b_\alpha) \ \geq\ \sup\{R(a_\alpha, b_\alpha) \mid \alpha \in \wedge\}$, $(a_\alpha, b_\alpha) \in L \times L$, $\forall\ \alpha \in \wedge$ and
(2) $R(a_1 \wedge a_2, b_1 \wedge b_2) \ \geq\ \min\{R(a_1, b_1), R(a_2, b_2)\}$, $(a_i, b_i) \in L \times L$, for $i = 1, 2$.

Lemma 4.5. *Let L and M be frames and ϕ be a frame homomorphism from M to L. Define a fuzzy binary relation $R(\phi)$ on L as*

$$R(\phi) \ =\ R\{(a, b) \in L \times L \mid \phi(a) \ =\ \phi(b)\}.$$

Then $R(\phi)$ is a fuzzy congruence on L.

Proof.
$$\begin{aligned} R(\phi)(\vee a_\alpha, \vee b_\alpha) &= \{(\vee a_\alpha, \vee b_\alpha) \in L \times L \mid \phi(\vee a_\alpha) = \phi(\vee b_\alpha)\} \\ &= \{(\vee a_\alpha, \vee b_\alpha) \in L \times L \mid \vee\phi(a_\alpha) = \vee\phi(b_\alpha)\} \\ &\geq \vee_\alpha\{(a_\alpha, b_\alpha) \in L \times L \mid \phi(a_\alpha) = \phi(b_\alpha)\} \end{aligned}$$

i.e., $R(\phi)(\vee a_\alpha, \vee b_\alpha) \geq \sup\{R(\phi)(a_\alpha, b_\alpha) \mid \alpha \in \wedge\}$
For $(a_1, b_1), (a_2, b_2) \in R(\phi)$,

$$\begin{aligned} R(\phi)(a_1 \wedge a_2, b_1 \wedge b_2) &= \{(a_1 \wedge a_2, b_1 \wedge b_2) \in L \times L \mid \phi(a_1 \wedge a_2) = \phi(b_1 \wedge b_2)\} \\ &= \{\wedge[(a_1, b_1), (a_2, b_2)] \in L \times L \mid \wedge[\phi(a_1), \phi(a_2)] = \wedge[\phi(b_1), \phi(b_2)]\} \end{aligned}$$

$\geq \min\{(a_i, b_i) \in L \times L \mid \phi(a_i) = \phi(b_i)\}$, for $i = 1, 2$.
$\geq \min\{R(\phi)(a_i, b_i) \mid i = 1, 2\}$.

Hence $R(\phi)$ is a fuzzy congruence on L. □

Definition 4.6. Let $F = (L, \mu)$ be a fuzzy frame and R_F be a fuzzy equivalence relation on F. Then R_F is said to be a fuzzy congruence on F if

(1) $R_F(\vee a_\alpha, \vee b_\alpha) \geq \sup\{R_F(a_\alpha, b_\alpha) \mid \alpha \in \wedge\}$, $(a_\alpha, b_\alpha) \in L \times L$, $\forall\ \alpha \in \wedge$
(2) $R_F(a_1 \wedge a_2, b_1 \wedge b_2) \geq \min\{R_F(a_1, b_1), R_F(a_2, b_2)\}$, $(a_i, b_i) \in L \times L$, for $i = 1, 2$,

where R_F is defined as in Definition 4.1

Lemma 4.7. *Let (L, μ) and (M, λ) be two fuzzy frames and $\phi : (L, \mu) \to (M, \lambda)$ be a fuzzy frame homomorphism such that $\mu = \lambda \circ \phi$. Define a fuzzy relation $R_F(\phi)$ on L as*

$$R_F(\phi) \ = \ R\{(a, b) \in L \times L \mid \phi(a) = \phi(b), \mu(a) = \mu(b)\}\mu(a)^2$$

Then $R_F(\phi)$ is a fuzzy congruence on F.

Proof. $R_F(\phi) = R\{(a, b) \in L \times L \mid \phi(a) = \phi(b), \mu(a) = \mu(b)\}\mu(a)^2$

(i) Let $(a_\alpha, b_\alpha) \in R_F(\phi)$, $\forall\ \alpha \in \wedge$

$$\begin{aligned} \text{Then } R_F(\vee a_\alpha, \vee b_\alpha) &= R\{(\vee a_\alpha, \vee b_\alpha) \in L \times L \mid \phi(\vee a_\alpha) = \phi(\vee b_\alpha), \\ &\qquad \mu(\vee a_\alpha) = \mu(\vee b_\alpha)\}\mu(\vee a_\alpha)^2 \\ &= R\{(\vee a_\alpha, \vee b_\alpha) \in L \times L \mid \vee\phi(a_\alpha) = \vee\phi(b_\alpha), \\ &\qquad \mu(\vee a_\alpha) = \mu(\vee b_\alpha)\}\mu(\vee a_\alpha)^2 \\ &\geq \vee R\{(a_\alpha, b_\alpha) \in L \times L \mid \phi(a_\alpha) = \phi(b_\alpha), \\ &\qquad \mu(a_\alpha) = \mu(b_\alpha)\}\mu(a_\alpha)^2 \end{aligned}$$

i.e., $R_F(\vee a_\alpha, \vee b_\alpha) \geq \sup\{R_F(a_\alpha, b_\alpha) \mid \alpha \in \wedge\}$

(ii) For (a_1, b_1), $(a_2, b_2) \in R_F(\phi)$,

$$\begin{aligned} R_F(a_1 \wedge a_2, b_1 \wedge b_2) &= R\{(a_1 \wedge a_2, b_1 \wedge b_2) \in L \times L \mid \phi(a_1 \wedge a_2) = \phi(b_1 \wedge b_2) \\ &\qquad \mu(a_1 \wedge a_2) = \mu(b_1 \wedge b_2)\}\mu(a_1 \wedge a_2)^2 \\ &= R\{\wedge[(a_1, b_1), (a_2, b_2)] \in L \times L \mid \wedge[\phi(a_1), \phi(a_2)] = \wedge[\phi(b_1), \phi(b_2)] \\ &\qquad \mu(a_1 \wedge a_2) = \mu(b_1 \wedge b_2)\}\mu(a_1 \wedge a_2)^2 \\ &\geq \min R\{(a_i, b_i) \in L \times L \mid \phi(a_i) = \phi(b_i) \\ &\qquad \mu(a_i) = \mu(b_i)\}\mu(a_i)^2,\ i = 1, 2 \\ &\geq \min\{R_F(a_i, b_i) \mid i = 1, 2\}. \end{aligned}$$

Hence R_F is a fuzzy congruence on F. □

The fuzzy congruence R_F on F partitions F into equivalence classes, with top and bottom elements and hence the set of equivalence classes form a fuzzy quotient frame M_{R_F} of F. Then there exists a fuzzy quotient map $R_F : F \to M_{R_F}$, mapping each element of F into its equivalence class. The fuzzy congruence determined by this fuzzy quotient map is again R_F and $R_F : F \to M_{R_F}$ is the fuzzy quotient corresponding to the fuzzy congruence R_F.

Proposition 4.8. *Let $F = (L, \mu)$ be a fuzzy frame. Let R_F be a fuzzy congruence on F. We call $F|R_F$ as a fuzzy quotient fuzzy frame of F. If F is fuzzy compact, $F|R_F$ is also fuzzy compact.*

Proof. R_F is a fuzzy congruence on F and the quotient $R_F : F \to F|R_F$ maps each element of F to the equivalence class determined by that element. $F|R_F = (\overline{L}, \overline{\mu})$, where $\overline{L}$ is a quotient frame of L and $\bar{\mu}$ is the fuzzy quotient frame of $\bar{L}$.

$$\therefore \quad \overline{\mu}(\overline{a}) = \mu(a), \quad \forall\ \overline{a} \in \overline{L}\ \ and\ \ \forall\ \ a \in L, \text{ by definition of quotient frame } \overline{L}.$$

We claim that $F|R_F$ is fuzzy compact. Let F be fuzzy compact. Then, for $S \subseteq L$, $\exists$ a $T \subseteq S$, finite, such that $\mu(\vee T) = \mu(\vee S)$
Let $\overline{S} \subseteq \overline{T}$ and $\overline{T} \subseteq \overline{S}$, $\overline{T}$ finite.

$$\begin{aligned} \overline{\mu}(\vee\overline{T}) &= \mu(\vee T) \\ &\geq \mu(a) = \overline{\mu}(\overline{a}), \forall\ a \in L, \overline{a} \in \overline{L}. \end{aligned}$$

$$\overline{\mu}(\vee\overline{T}) = \mu(\vee T) = \mu(\vee S) = \overline{\mu}(\vee\overline{S})$$

i.e., $\overline{\mu}$ is fuzzy compact. □

Corollary 4.9. *The fuzzy quotient fuzzy frame $F|R_F = (\overline{L}, \overline{\mu})$, is the fuzzy homomorphic image of $F = (L, \mu)$.*
i.e., if $\alpha : F \to F|R_F$, is a fuzzy frame homomorphism from F to $F|R_F$, then $\alpha(\mu) = \overline{\mu}$.

Proof. Proof is obvious. □

Now we proceed to define the compactification of fuzzy frames.

5. Fuzzy compactification of fuzzy frames

A compactification of a frame L is a dense onto homomorphism from M to L with compact regular M. Here we deal with fuzzy frames and their fuzzy compactifications. We start with the definition of fuzzy compactification of fuzzy frames and then we discuss the fuzzy version of Stone-Cech Compactification of frames, in view of 'Stone-Cech Compactification of Locales' given by P.T.Johnstone[13].

Definition 5.1. Let $F = (L, \mu)$ be a fuzzy regular fuzzy frame and $G = (M, \mu^*)$ be a fuzzy compact fuzzy regular fuzzy frame. A fuzzy frame compactification of F is a dense onto fuzzy frame homomorphism $\alpha : M \to L$, such that $\mu^* = \mu \circ \alpha$ and is denoted by αG.

Definition 5.2. (Fuzzy Stone-Cech Compactification of Fuzzy Frames) Let $F = (L, \mu)$ be a fuzzy regular fuzzy frame. Let $G = (M, \mu^*)$ be a fuzzy compact fuzzy regular fuzzy frame and $\beta : M \to L$ be a dense onto fuzzy frame homomorphism from M to L, with $\mu^* = \mu \circ \beta$. If $T = (Y, \lambda^*)$ is any other fuzzy compact fuzzy regular fuzzy frame and $\gamma : Y \to L$ is a dense onto fuzzy frame homomorphism such that $\lambda^* = \mu \circ \lambda$, then $\exists$ a unique dense onto fuzzy isomorphism f from Y to M with $\lambda^* = \mu^* \circ f$ such that the diagram commutes. Then βG is the fuzzy Stone-Cech compactification of F.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\beta"'] & & Y \arrow[dl, "\gamma"] \\ & L & \end{tikzcd}$$

Lemma 5.3. *Let $F^* = (L, \lambda)$ be a fuzzy regular fuzzy frame with L, completely regular. Let S be the set of all frame homomorphisms from [0, 1] to L. Let $I_\alpha = [0, 1]$ be a closed unit interval for each α and $(I_\alpha)^* = (I_\alpha, \mu_\alpha)$ be a unit fuzzy frame which is fuzzy compact, then $\exists$ a dense onto fuzzy isomorphism f from the product of copies of I_α to L.*
i.e., $f : \prod I_\alpha \to L$ is a dense onto fuzzy isomorphism such that $\prod \mu_\alpha = \lambda \circ f$.

Proof. Proof follows from Tychonoff embedding theorem for locales.
By Tychonoff's theorem, if $f_\alpha : I_\alpha \to L$ is an isomorphism, $\exists$ an isomorphism f from the product of copies of I_α to L. Define $f : \prod I_\alpha \to L$ by

$$f(\langle a_\alpha \rangle) = \wedge f_\alpha(a_\alpha),\ a_\alpha \in I_\alpha,$$

where $\prod I_\alpha$ is the $S-$indexed product of copies of I_α. Since f is an isomorphism, f is dense, onto.
Clearly, $\prod \mu_\alpha = \lambda \circ f$; for,

$$f : \prod I_\alpha \to L \text{ and } \lambda : L \to [0, 1].$$

Hence f is a dense onto fuzzy isomorphism. □

Theorem 5.4 (**Existence and uniqueness**). *Let $F^* = (F, \mu)$ be a fuzzy regular fuzzy frame. Let $G^* = (G, \mu^*)$ be a fuzzy compact fuzzy frame and $\beta : G \to F$ be a dense, onto, fuzzy frame map. If $B^* = (B, \lambda)$ is another fuzzy compact fuzzy regular fuzzy frame and $\eta : H \to B$ is a dense onto fuzzy frame map, where H^* is a fuzzy compact fuzzy regular fuzzy frame of frame H, then the fuzzy isomorphism $h : B \to F$ induces a fuzzy frame map $\overline{h} : H \to G$, such that h has a unique factorisation through β.*

Proof. Let $F^* = (L, \mu)$ be a fuzzy regular fuzzy frame. Let $I^*_\alpha = (I_\alpha, \mu_\alpha)$ be a unit fuzzy frame, which is fuzzy compact.

$$\begin{tikzcd} H \arrow[r, "\overline{h}"] \arrow[d, "\eta"'] & G \arrow[d, "\beta"] \\ B \arrow[r, "h"'] & F \end{tikzcd}$$

Let $G = \prod I_\alpha$, $G^* = (\prod I_\alpha, \prod \mu_\alpha)$ be a fuzzy compact, fuzzy regular fuzzy frame.
Similarly, $H^* = (\prod I_\beta, \prod \mu_\beta)$, is fuzzy compact fuzzy regular fuzzy frame.
Then by Lemma 5.3 $\exists$ a dense, onto fuzzy isomorphism $\beta : G \to F$.
Clearly βG^* is a fuzzy compactification of F^*. If B^* is another fuzzy compact fuzzy regular fuzzy frame and $H^*$is a fuzzy compact fuzzy regular fuzzy frame, $\exists$ a dense onto fuzzy isomorphism $\eta : H \to B$, such that $\prod \mu_\beta = \lambda \circ \eta$.
i.e., ηH^* is a fuzzy compactification of B^*.
If there is a fuzzy isomorphism $h : B \to F$, then η, in composition with h, induces a dense, onto fuzzy homomorphism $\overline{h} : H \to G$, which also maps H to F, along β.

$$\text{i.e. } h\eta = \beta\overline{h}.$$

Hence $h = \beta\overline{h}\eta^{-1}$.
If k_1 and k_2 are two such factorizations of h, then the equaliser would be a closed subframe containing the image of H. i.e., the equaliser (K, γ), where

$$K = \{b \in B \mid k_1(b) = k_2(b)\}$$

is the whole of B, since η is dense onto fuzzy isomorphism.

$\therefore\ k_1 = k_2.$

Thus βG^* is the unique fuzzy Stone-Cech compactification of F^*. □

Theorem 5.5. *Let $F = (L, \mu)$ be a fuzzy regular fuzzy frame with fuzzy Stone-Cech compactification, βF. If αF is any other fuzzy compactification of F, then αF is a fuzzy quotient of βF. Conversely, if αF is a fuzzy quotient of βF, then αF is a fuzzy compactification of F.*

Proof.

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & F & \end{tikzcd}$$

By definition of fuzzy compactification, $\exists$ a dense onto fuzzy frame isomorphism $\phi : A \to B$ such that $\beta\phi = \alpha$ and the diagram commutes. Let $\theta : B \to A$ be a fuzzy homomorphism from B to A. Then θ defines a fuzzy congruence $R_F(\theta)$ on B and with respect to this congruence $B|R_F(\theta)$ is a fuzzy quotient of B. Also, $B|R_F(\theta)$ is fuzzy compact fuzzy regular also. Letting $A = B|R_F(\theta)$, αF is the fuzzy quotient of βF.

Conversely, let αF be a fuzzy quotient of βF. A fuzzy frame homomorphism ϕ is dense if $\phi(a) = 0 \Rightarrow a = 0$. Here $\beta\phi = \alpha$.

Let $\alpha(a) = 0$

$\Rightarrow \beta\phi(a) = 0$

$\Rightarrow \phi(a) = 0$, since β is fuzzy Stone-Cech compactification

$\Rightarrow a = 0$, since ϕ is dense.

$\therefore \alpha$ is a dense fuzzy homomorphism from A to F.

Clearly, α is onto. For, Let $c \in F$.

$\Rightarrow \exists\ b \in B$ such that $\beta(b) = c$

$\Rightarrow \exists\ a \in A$ such that $\beta(\phi(a)) = c$

$\Rightarrow \exists\ a \in A$ such that $\alpha(a) = c$

Hence αF is a fuzzy compactification of F. □

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